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Non-Linear Entropy Functionals and a Characteristic Invariant of Symmetry Group Actions on Infinite Quantum Systems

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Abstract

We review the development of the non-Abelian generalization of the Kolmogorov-Sinai (KS) entropy invariant, as initiated by Connes and Størmer and completed by Connes, Narnhofer and Thirring only recently. As an introduction and motivation, the classical KS theory is reformulated in terms of Abelian W^*-algebras. Finally, we describe simple physical applications of the developed characteristic invariant to space-time symmetry group actions on infinite quantum systems.

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1 Introduction: The classical theory

A. N. Kolmogorov [1] and Ya. G. Sinai [2] introduced the concept of *entropy* into classical (measure-theoretic) ergodic theory in order to define a *characteristic invariant* (see below) of measure preserving transformations on probability spaces. Exactly speaking (cf. [3]), they applied to measure theory the *information-theoretic* notion of entropy as introduced by Shannon [4], in so far as they defined the entropy $H_\mu(\xi)$ of a finite measurable partition (μ-mod 0) $\xi = \{A_1, \ldots, A_n\}$ of the measure space X, with probability measure μ (such that $\mu(A_i \cap A_j) = 0$ for $i \neq j$ and $\sum_{i=1}^n \mu(A_i) = 1$), by the *Shannon entropy* of the probability distribution $\{\mu(A_i)\}$:

$$H_\mu(\xi) = \sum_{i=1}^{n} \eta(\mu(A_i)), \tag{1.1}$$

where $\eta(x) \equiv -x \ln x \ \forall x \in [0,1]$ with $\eta(0) \equiv 0$ (and with natural logarithms; cf. [5] for the classical theory). The name "entropy" (of the distribution $\{\mu(A_i)\}$) for the expression (1.1) had been suggested to Shannon by J. von Neumann verbally "for two reasons: In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, no one knows what entropy really is, so in a debate you will always have the advantage!" (cited from [5]). Regarding the *first* reason as the more important one, however, we should add that this "uncertainty function" had even been used by von Neumann *himself* in his definition (cf. [7]) of the *quantum mechanical* entropy $S(\omega)$ of a normal state ω on the (again!) *von Neumann* algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a (separable) Hilbert space $\mathcal{H}$:

$$S(\omega) = -\mathrm{Tr}\rho_\omega \ln \rho_\omega \equiv \mathrm{Tr}\, \eta(\rho_\omega), \tag{1.2}$$

where ρ_ω is the density operator determined by $\omega(A) = \mathrm{Tr}(\rho_\omega A) \ \forall A \in \mathcal{B}(\mathcal{H})$, with the trace Tr in $\mathcal{H}$ (cf. [8], e.g.).

In a basis of $\mathcal{H}$ in which ρ_ω is diagonal, (1.2) in fact takes a form completely analogous to (1.1):

$$S(\omega) = \sum_i \eta(\omega(P_i)), \tag{1.3}$$

where $P_i = P_i^* = P_i^2$ are the projectors onto the eigenspaces of $\rho_\omega = \sum_i \omega_i P_i$ (with $P_i P_j = 0 \ \forall i \neq j$ such that $\omega(P_j) = \omega_j$). This close analogy is then seen to be even a (formal) *identity* by formulating (1.1) in algebraic terms: To each finite (μ-measurable) partition $\xi = \{A_1, \ldots, A_n\}$ of X as above there corresponds a finite decomposition of unity $1 = \sum_{i=1}^n P_i$ in $L^\infty(X, \mu)$ given by the (μ-equivalence classes of) characteristic functions $\chi_{A_i} \equiv P_i$; and we denote by $\mathcal{A}(\xi) \subset L^\infty(X,\mu)$ the n-dimensional $*$-subalgebra generated by the P_i ($i = 1, \ldots, n$), which in turn give the uniquely determined minimal projectors $P_i \in \mathcal{A}(\xi)$. This clearly establishes a bijective mapping $\xi \mapsto \mathcal{A}(\xi)$ from the set $\mathcal{P}$ of all finite (measurable) partitions of X onto the set $\mathcal{F}$ of all finite-dimensional $*$-subalgebras (with unity 1) of the *W^*-algebra* $\mathcal{M}_\mu \equiv L^\infty(X, \mu)$ (cf. [8], e.g.). An invertible measure preserving transformation $T : X \to X$ (as studied by Kolmogorov and Sinai, see *still* below), respectively the probability measure μ on X, induces a $*$–automorphism τ_μ of $\mathcal{M}_\mu$, resp. a faithful normal state ω_μ on $\mathcal{M}_\mu$, by

$$\tau_\mu(f) = f \circ T, \quad \text{resp. } \omega_\mu(f) = \int_X f(x)\, d\mu(x) \quad \forall f \in \mathcal{M}_\mu, \tag{1.4}$$

such that ω_μ is an invariant state: $\omega_\mu \circ \tau_\mu = \omega_\mu$.

Inverting our line of reasoning, we now consider an *arbitrary Abelian* W^*-algebra $\mathcal{M} \ni 1$ with *faithful* normal state ω on $\mathcal{M}$, and we *define* the entropy $H_\omega(\mathcal{A})$ of the n–dimensional $*$–subalgebra $\mathcal{A} \subset \mathcal{M}$ with minimal projectors $P_i \in \mathcal{A}$ $(i = 1, \ldots, n)$ in the form *identical* to (1.3) above:

$$H_\omega(\mathcal{A}) = \sum_{i=1}^{n} \eta(\omega(P_i)); \tag{1.5}$$

and in fact, we obviously get for the entropy (1.1) of a partition ξ: $H_\mu(\xi) = H_{\omega_\mu}(\mathcal{A}(\xi))$ as announced, with the definitions from above and with the identification $\mathcal{M}_\mu \equiv L^\infty(X, \mu) = \mathcal{M}$.

This latter identification is in turn possible also for an *arbitrary* Abelian W^*–algebra $\mathcal{M} \ni 1$ to *start* with, since according to the Riesz' representation theorem (cf. [9], e.g.) the *faithful* normal state ω on $\mathcal{M}$ uniquely determines a regular Borel probability measure μ_ω supported on the Gelfand space $X_\mathcal{M}$ of $\mathcal{M}$ (as an Abelian C^*–algebra, cf. [8]), such that the W^**-algebra* $\mathcal{M}$ is $*$–isomorphic to $L^\infty(X_\mathcal{M}, \mu_\omega)$ (cf. [10]). Starting in particular with $\mathcal{M} = L^\infty(X, \mu)$ and $\omega = \omega_\mu$ as above, however, this "Gelfand"-probability space $(X_\mathcal{M}, \mu_\omega)$ does in general *not* coincide with the original measure space (X, μ) from above (only if (X, μ) is a hyperstonean space with Borel measure, cf. [10]). In spite of this "irreversibility" of the algebraic procedure, the classical Kolmogorov-Sinai theory can be *equivalently* formulated in these algebraic terms because of the $*$–isomorphism $L^\infty(X, \mu) \equiv \mathcal{M} \cong L^\infty(X_\mathcal{M}, \mu_\omega)$ (cf. also [11]) and of the one–to–one correspondence between the finite partitions $\mathcal{P}$ of X (resp. of $X_\mathcal{M}$) and the finite–dimensional unital subalgebras $\mathcal{F}$ of $\mathcal{M}$ (as mentioned above). This algebraic formulation of the classical theory and of physical applications (as a first step towards "quantization", see section 2) has been carried out in [12]; but here we have to restrict ourselves to the following

Definitions: Let $\mathcal{M} \ni 1$ be an Abelian W^*-algebra with faithful normal state ω on $\mathcal{M}$ and with a $*$–automorphism τ of $\mathcal{M}$ such that $\omega \circ \tau = \omega$. We denote by $\mathcal{F}$ the set of all finite-dimensional unital $*$-subalgebras of $\mathcal{M}$, and $\mathcal{A} \vee \mathcal{B} (\in \mathcal{F})$ denotes the subalgebra generated by $\mathcal{A}, \mathcal{B} \in \mathcal{F}$.

(i) For $\mathcal{A}, \mathcal{B} \in \mathcal{F}$ with (linear) dimensions $\dim \mathcal{A} = m$, $\dim \mathcal{B} = n$ and with the minimal projectors $P_i \in \mathcal{A}$, $Q_j \in \mathcal{B}$ as above $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ the *conditional entropy* of $\mathcal{A}$ w. r. t. $\mathcal{B}$ is given by

$$H_\omega(\mathcal{A}|\mathcal{B}) = -\sum_{i=1}^{m} \sum_{j=1}^{n} \omega(P_i Q_j) \ln \frac{\omega(P_i Q_j)}{\omega(Q_j)}. \tag{1.6}$$

(ii) The *"entropy" of τ w. r. t. $\mathcal{A} \in \mathcal{F}$* is given by the (existing) limit

$$h_\omega(\tau, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H_\omega(\mathcal{A} \vee \tau\mathcal{A} \vee \ldots \vee \tau^{n-1}\mathcal{A}) \tag{1.7}$$

with the functional $H_\omega : \mathcal{F} \to \mathbf{R}^+$ from (1.5).

(iii) The (characteristic) *Kolmogorov–Sinai invariant* of τ is given by

$$h_\omega(\tau) = \sup_{\mathcal{A}\in\mathcal{F}} h_\omega(\tau, \mathcal{A}). \tag{1.8}$$

Remarks:

1. In physical applications the probability space (X, μ) in (1.1) is given by the configuration (or phase –) space X of a classical system together with a "physical" probability measure μ on X (representing the "state" of the system). Then the finite partition ξ of X (resp. the finite–dimensional algebra $\mathcal{A}(\xi)$) corresponds to the possible outcomes of a "finite–precision" measurement (of finitely many observables) performed on the system, and the entropy $H_\mu(\xi)$ resp. $H_{\omega_\mu}(\mathcal{A}(\xi))$ from (1.1) resp. (1.5) represents the information (about the whole system) to be gained from this measurement.

2. For $\mathcal{A}, \mathcal{B} \in \mathcal{F}$ as in Def. (i) the minimal projectors of $\mathcal{A} \vee \mathcal{B}$ are given by P_iQ_j ($i = 1, \ldots, m$; $j = 1, \ldots, n$), such that (1.5) and (1.6) are interrelated by

$$H_\omega(\mathcal{A}|\mathcal{B}) = H_\omega(\mathcal{A} \vee \mathcal{B}) - H_\omega(\mathcal{B}), \qquad H_\omega(\mathcal{A}) = H_\omega(\mathcal{A}|\mathbf{C1}), \tag{1.9}$$

where $\mathbf{C1} = \mathcal{A}(\{\emptyset, X\})$ is the trivial (one–dimensional) subalgebra on the trivial partition of X. – Correspondingly, the *conditional* entropy (1.6) represents the information to be gained from the measurement of $\mathcal{A} \in \mathcal{F}$ *in addition* to the (information from the) measurement of $\mathcal{B}$.

3. The (reversible) unit time evolution of a classical system in an "*equilibrium* state" with measure μ is given by a *measure preserving* (invertible) transformation $T : X \to X$; and then the (dynamical) "entropy" $h_{\omega_\mu}(\tau_\mu, \mathcal{A})$ from (1.7), with $\omega_\mu \circ \tau_\mu = \omega_\mu$ from (1.4), represents the mean information gain about the equilibrium system (per measurement) in an "infinite" series of measurements of the observables $\mathcal{A} \in \mathcal{F}$ at unit time intervals. – The existence of the limit in (1.7) is guaranteed by the *subadditivity* (resp. invariance) property of the entropy functional (1.5): For $\mathcal{A}, \mathcal{B} \in \mathcal{F}$ and ω, τ as above,

$$H_\omega(\mathcal{A} \vee \mathcal{B}) \leq H_\omega(\mathcal{A}) + H_\omega(\mathcal{B}), \quad \text{resp. } H_\omega(\tau\mathcal{A}) = H_\omega(\mathcal{A}). \tag{1.10}$$

But the *concavity* of the function $\eta(x) = -x \ln x$ employed in Def. (1.5) even implies the following *strong* subadditivity property of this functional (cf. [13]):

$$H_\omega(\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}) \leq H_\omega(\mathcal{A} \vee \mathcal{C}) + H_\omega(\mathcal{B} \vee \mathcal{C}) - H_\omega(\mathcal{C}) \quad \forall \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{F}, \tag{1.11}$$

and this property in view of (1.9) also guarantees the existence of the following limit, which in turn coincides with its Cesàro limit (1.7):

$$\lim_{n\to\infty} H_\omega(\tau^n\mathcal{A}|\bigvee_{i=0}^{n-1}\tau^i\mathcal{A}) = h_\omega(\tau,\mathcal{A}). \tag{1.12}$$

Thus the mean information gain (1.7) coincides with this "asymptotic" additional information gain (per measurement) after a sufficiently long series of measurements.

4. The Kolmogorov–Sinai (KS) invariant (1.8) then represents the *optimal* information gain possible from a finite measurement (in an "infinite" series of such measurements); and therefore the *dynamical entropy* $h_{\omega_\mu}(\tau_\mu)$, with $\omega_\mu \circ \tau_\mu = \omega_\mu$ from (1.4), is a characteristic measure of the "mixing properties" of the (unit) time evolution $T : X \to X$ of the classical "equilibrium" system (cf. [14], e.g.), what in turn makes it also a measure of *time*: Generally for $n \in \mathbf{Z}$ in (1.8) we have

$$h_\omega(\tau^n) = |n| \cdot h_\omega(\tau). \tag{1.13}$$

Furthermore, by construction (1.8) is a conjugacy invariant of the $\mathbf{Z}$–action on $\mathcal{M}$ by the (iterated) $*$–automorphism τ (leaving ω invariant); that is, for a $*$–isomorphism $\sigma : \mathcal{M} \to \mathcal{N}$ onto another Abelian W^*–algebra $\mathcal{N}$ we have

$$h_{\omega\circ\sigma^{-1}}(\sigma\circ\tau\circ\sigma^{-1}) = h_\omega(\tau), \tag{1.14}$$

where the left hand side has to be computed in $\mathcal{N}$. (1.8) is by far no *complete* invariant, however, since non–conjugate $\mathbf{Z}$–actions can have the same value of $h_\omega(\tau)$.

Of course, the physical applications of the KS invariant (1.8) are not limited to the *dynamical* entropy of an equilibrium time evolution as in 3.), 4.) above: τ in (1.8) can be given by any *symmetry transformation* of a classical system ($\mathcal{M}$) leaving also its state (ω) invariant. To be applicable as entropy of *space translations* in translationally invariant (hence infinite) classical systems with spatial dimension $\nu > 1$, however, the definition (1.8) has to be extended from $\mathbf{Z}$–actions to actions of $\mathbf{Z}^\nu$. Robinson and Ruelle [15] presupposed this generalization "without further discussion" to show that for classical *lattice* systems (on the lattice $\mathbf{Z}^\nu$) the KS invariant of lattice translations coincides with the (spatial) *entropy density* of the (translationally invariant) state in the sense of classical statistical mechanics; the reason being that both the "physical" space and the symmetry group acting on it are given by the same lattice $\mathbf{Z}^\nu$. The tacit presupposition mentioned above was "justified" later on by Conze [16] (among others), who extended the KS entropy theory formulated for a *single* measure space transformation to finitely generated Abelian transformation groups (i.e. $\mathbf{Z}^\nu$–actions). His results were then used by Goldstein [17] for investigating the mixing properties of (translationally invariant) classical *continous* systems in the thermodynamic limit, for which both the time evolution and the group of space translations seperately have typically *infinite* KS invariant. Goldstein showed that the mixing properties with respect to the spacetime group (the Abelian group generated by space translations and time evolution together) provide a much sharper tool of investigation and, in particular, that the systems considered by him have physically significant *spacetime entropy* (KS entropy of the spacetime group), which as a conjugacy invariant serves for distiguishing these systems and their mixing properties, respectively.

Nevertheless, the physical applicability of this classical theory is strictly limited by the *quantum mechanical* nature of microphysics, which in the algebraic approach to quantum theory (cf. [8, 18], e.g.) amounts to replacing the Abelian W^*-algebra $\mathcal{M}$ (first appearing in (1.5) above) by a *non-Abelian* C^*-algebra $\mathcal{M}$, together with its GNS-representations constructed from states ω on $\mathcal{M}$. Thus Robinson and Ruelle stated already in the abovementioned classical work [15]:"Many of the recent results have consisted in extending ergodic theory to the case of a non-Abelian algebra $\mathcal{M}$. It would thus be natural to obtain a non-Abelian extension of the mean entropy introduced by Kolmogorov and Sinai (KS invariant)." - In the following, we shall briefly review the development of this long-desired quantum entropy invariant, as initiated by Connes and Størmer [19] and completed by Connes, Narnhofer and Thirring [20] only recently. Finally, we shall describe simple physical applications of this characteristic invariant to symmetry group actions on infinite quantum systems, in analogy to the classical results [15, 17] cited above.

2 The entropy functionals in quantum theory

For extending the definition (1.8) to the case of a non-Abelian W^*-algebra (or even general C^*-algebra) $\mathcal{M} \ni 1$ with a $*$-automorphism θ of $\mathcal{M}$ and invariant state $\omega = \omega \circ \theta$ on $\mathcal{M}$, it seems natural to maintain the form of (1.8) and (1.7), and thus to look for a non-Abelian generalization of the entropy functional (1.5) (used in (1.7) as the information gain from a finite sequence of repeated measurements). Then the formal identity of this classical entropy (1.5) with the von Neumann entropy (1.3) makes it tempting at first sight to consider the latter as the natural non-Abelian extension of the former. This attempt could be supported by the *strong subadditivity* property [21] of the quantum mechanical entropy (1.2), in (formal) analogy to the property (1.11) of the classical functional (1.5). But as the definition (1.2) refers to a normal state ω on *all* of $\mathcal{B}(\mathcal{H})$, the strong subadditivity property of $S(\omega)$ only makes sense for a triple *tensor product* space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, with the restrictions $\omega|_{\mathcal{B}(\mathcal{H}_i \otimes \mathcal{H}_j)} \equiv \omega_{ij}$ and $\omega|_{\mathcal{B}(\mathcal{H}_k)} \equiv \omega_k$ of $\omega \equiv \omega_{123}$ (given by the partial traces of ρ_ω, cf. [14, 22]):

$$S(\omega_{123}) \leq S(\omega_{13}) + S(\omega_{23}) - S(\omega_3); \tag{2.1}$$

and in particular for trivial $\mathcal{H}_3 = \mathbf{C}$ we get again the (weak) subadditivity, in analogy to the property (1.10): $S(\omega_{12}) \leq S(\omega_1) + S(\omega_2)$. And even in this simple tensor product situation ($\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$) the quantum mechanical entropy is not *monotonic* any more, whereas for the classical functional (1.5) we have monotonicity in the sense that $H_\omega(\mathcal{A} \vee \mathcal{B}) \geq H_\omega(\mathcal{B})$ ($\forall \mathcal{A}, \mathcal{B} \in \mathcal{F}$; since the *conditional* entropy (1.6) as in (1.9) is positive): For a *pure* state ω_{12} on $\mathcal{B}(\mathcal{H})$ (with $\rho_{\omega_{12}}$ given by a single one-dimensional projector P on $\mathcal{H}$) we see from (1.3) that $S(\omega_{12}) = 0$, while then in general $S(\omega_1) = S(\omega_2) > 0$ (cf. [14, 22]). - This first attempt to define the non–Abelian extension of the entropy functional (1.5) is thus doomed to failure; or as E. Lieb put it in [22]: "...an unsolved problem is to define the analogue of the Kolmogorov–Sinai entropy. In other words, it is not clear how to give an unambiguous definition of the density matrix to use in" (1.2) (i.e., in the definition of quantum mechanical entropy)

On the other hand, *every* attempt to define an entropy functional on the set $\mathcal{F}$ of all *finite*–dimensional (unital) $*$–subalgebras of the non–Abelian W^*–algebra $\mathcal{M} \ni 1$ would be doomed to failure, if we tried to define the joint entropy of two algebras $\mathcal{A}, \mathcal{B} \in \mathcal{F}$ (also *without* tensor product structure) by the entropy of the $*$–algebra $\mathcal{A} \vee \mathcal{B} \subset \mathcal{M}$ generated by them; since for non–commuting $\mathcal{A}, \mathcal{B} \in \mathcal{F}$ in general $\mathcal{A} \vee \mathcal{B}$ will be *infinite*–dimensional (cf. [23]).

This dilemma was "circumvented" by A. Connes and E. Størmer [19], who looked for an entropy functional with arbitrarily (but finitely) many arguments from $\mathcal{F}$ and depending only on the *set* of these finite–dimensional $*$–subalgebras. Furthermore, this functional should have the desired properties of monotonicity and subadditivity, and it should coincide with the classical expression (1.5) in the case of an *Abelian* W^*–algebra $\mathcal{M}$. Connes and Størmer succeeded in constructing this entropy functional in the special case of a (non–Abelian) von Neumann algebra $\mathcal{M} \ni 1$ with faithful normal *trace state* τ on $\mathcal{M}$ (i.e. $\tau(AB) = \tau(BA)\ \forall A, B \in \mathcal{M}$), since then for all $\mathcal{A} \in \mathcal{F}$ there exists a unique (faithful, normal) *conditional expectation* map $E_{\mathcal{A}}$ from $\mathcal{M}$ onto $\mathcal{A}$ which is τ*–preserving* (i.e. $E_{\mathcal{A}}$ is a norm one projection from $\mathcal{M}$ onto $\mathcal{A}$ such that $\tau \circ E_{\mathcal{A}} = \tau$, cf. [10]). The definition [19] of this functional $H_\tau(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ for $\mathcal{A}_i \in \mathcal{F}$ $(i = 1, \ldots, n)$ is given by a *variational expression*, as the supremum of a functional (involving the trace τ, the function η and the expectations $E_{\mathcal{A}_i}$) defined on the set S_n of all finite decompositions of unity $1 = \sum_{I_n} x_{I_n} \in \mathcal{M}$ by positive elements $x_{I_n} \in \mathcal{M}$ (with multiindex $I_n \in \mathbf{N}^n$). In this sense the definition is even *intrinsically nonlinear*; in addition to the nonlinear *properties* of the entropy functionals (such as subadditivity), which in turn are partly "inherited" from the nonlinearities of the definition. Connes and Størmer used a similar variational expression to define also a non–Abelian extension of the *conditional* entropy functional (1.6); but for non–Abelian $\mathcal{M}$ this *two–argument* functional $H_\tau(\mathcal{A}|\mathcal{B})$ of $\mathcal{A}, \mathcal{B} \in \mathcal{F}$ does in general *not* coincide with the "additional information gain" $[H_\tau(\mathcal{A}, \mathcal{B}) - H_\tau(\mathcal{B})]$ as in the classical case (1.9), such that also the classical relation (1.12) would not make sense any more. Nevertheless, the *multi–argument* entropy functional $H_\tau(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ makes it possible now to define the characteristic invariant $h_\tau(\theta)$ of a $*$–automorphism θ of the von Neumann algebra $\mathcal{M} \ni 1$, with invariant trace state $\tau = \tau \circ \theta$, in the classical form of (1.7) and (1.8):

$$h_\tau(\theta, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} H_\tau(\mathcal{A}, \theta\mathcal{A}, \ldots, \theta^{n-1}\mathcal{A}) \quad \forall \mathcal{A} \in \mathcal{F}, \tag{2.2}$$

$$h_\tau(\theta) = \sup_{\mathcal{A} \in \mathcal{F}} h_\tau(\theta, \mathcal{A}). \tag{2.3}$$

The existence of the limit in (2.2) is again ensured by the *subadditivity* of $H_\tau(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ (now w. r. t. the set of arguments $\{\mathcal{A}_i\}$, by construction) and by its θ–invariance (w. r. t. θ acting on all arguments $\mathcal{A}_i$ simultaneously). Furthermore, (2.3) is again a conjugacy invariant of the Z-action on $\mathcal{M}$ by θ (with $\tau \circ \theta = \tau$), in analogy to (1.14). The analogue of the "scaling" property (1.13), however, has been proven only in the special case of an AFD (approximately finite-dimensional) von Neumann algebra $\mathcal{M} = (\bigcup_{n\in\mathbb{N}} \mathcal{M}_n)''$, which is the weak closure (or bicommutant, cf. [8, 10]) of an increasing sequence of finite-dimensional (von Neumann) subalgebras $\mathcal{M}_n \subseteq \mathcal{M}_{n+1} \in \mathcal{F}$ ($\forall n \in \mathbb{N}$). For this case, and with the help of the (strongly continous) *conditional* entropy functional $H_\tau(A|B)$ mentioned above, also the non-Abelian "Kolmogorov-Sinai" theorem (generalizing a *weak* version of the classical KS theorem, cf. [5]) was proven in [19]:

$$h_\tau(\theta) = \lim_{n\to\infty} h_\tau(\theta, \mathcal{M}_n). \tag{2.4}$$

For the existence of this increasing (possibly infinite) limit, replacing the supremum in (2.3), now the required *monotonicity* of the functional $H_\tau(A_1, \ldots, A_n)$ (w. r. t. algebraic inclusions, for each of the A_i seperately) is really necessary; and only (2.4) makes it possible to *calculate* the characteristic invariant $h_\tau(\theta)$. This was done first by Connes and Størmer [19] for shift automorphisms θ of the hyperfinite (hence AFD) type II_1-factor $\mathcal{M}$ (cf. [8, 10]) with trace τ, and their results were extended by Besson [24] and Quasthoff [25] later on (cf. also [26]). - In physical applications, the type II_1 representation $\mathcal{M}$ with unit time evolution automorphism θ and tracial "equilibrium" state $\tau = \tau \circ \theta$ corresponds to an infinite quantum system at *infinite* temperature $T = +\infty$ (such that τ is the KMS state determined by θ and T, cf. [8, 18]). To make the notion (2.3) of dynamical entropy $h_\tau(\theta)$ applicable also for "realistic" quantum systems at *finite* temperature T, the construction of the entropy functional $H_\omega(A_1, \ldots, A_n)$ on the sets $\{A_i \in \mathcal{F}\}$ (as used in (2.2) with $\omega = \tau$) has to be extended further to *non-tracial* states ω on $\mathcal{M}$. But then in general [27] for $A \in \mathcal{F}$ there exists no *norm one projection* E_A from $\mathcal{M}$ onto A such that $\omega \circ E_A = \omega$, as it was used in the definition [19] of the H_ω-functional for *tracial* state $\omega = \tau$. Thus Connes and Størmer themselves resignedly stated in [28]: "However, the definition (...) depended very strongly on the trace and thus seemed impossible to extend to more general von Neumann algebras (...)".

Paradoxically, this extension of the H_ω-entropy functional to non-tracial states ω is in fact *possible* with the help of the (quantum) *relative entropy* functional, which had already been discussed by Connes and Størmer in [19] and had even been used by them in the proof of a property of the H_τ-functional (for trace state τ). In the situation of (1.2), the *relative entropy* $S(\omega|\phi)$ of two normal states ω resp. ϕ on $\mathcal{B}(\mathcal{H})$ with density operators ρ_ω resp. ρ_ϕ is defined [29] by

$$S(\omega|\phi) = \mathrm{Tr}\rho_\phi(\ln\rho_\phi - \ln\rho_\omega). \tag{2.5}$$

Although also this definition seems to depend very strongly on the trace Tr, it can be extended to normal states ω resp. ϕ on a general von Neumann algebra $\mathcal{M}$, and even more generally to states ω resp. ϕ on a C^*-algebra $\mathcal{M}$; and this in turn makes possible even the generalization of the *entropy* $S(\omega)$ from normal states ω on $\mathcal{B}(\mathcal{H})$ as in (1.2) to states ω on these algebras $\mathcal{M}$:

Definitions: Let $\mathcal{M} \ni 1$ be a C^*-algebra with positive linear functionals ω, ϕ on $\mathcal{M}$.

(i) The *relative entropy* of ω w. r. t. ϕ is defined by

$$S_{\mathcal{M}}(\omega|\phi) = \sup_{\{x:\mathbf{R}^+\to\mathcal{M}\}} \int_0^\infty \frac{dt}{t}\left[\frac{\phi(1)}{1-t} - \phi(y(t)y^*(t)) - \frac{1}{t}\omega(x(t)x^*(t))\right], \tag{2.6}$$

where the supremum is taken over all step functions $x : \mathbf{R}^+ \to \mathcal{M}$ with finitely many steps and such that $x(t) \equiv 0\ \forall t < \epsilon(> 0)$, and where $y(t) \equiv 1 - x(t)$.

(ii) The *entropy* of the state ω is defined by

$$S_{\mathcal{M}}(\omega) = \sup_{\{\sum_i \lambda_i\omega_i = \omega\}} \sum_i \lambda_i S_{\mathcal{M}}(\omega|\omega_i). \tag{2.7}$$

where the supremum is taken over all *finite* decompositions of ω into *states* ω_i *on* $\mathcal{M}$ with coefficients $\lambda_i > 0$ (such that $\sum_i \lambda_i = 1$).

Remarks:

1. The definition (2.5) of the relative entropy had been first extended by Araki to normal states ω resp. ϕ on a general von Neumann algebra $\mathcal{M}$ (in terms of the relative modular operator), and it was further extended to the variational expression (2.6) by Pusz and Woronowicz and by Kosaki (see [30] and the references therein; cf. also [20, 23]). Again, the main properties of the relative entropy functional follov from this "intrinsically" nonlinear definition (2.6), as already pointed out for the H_τ-entropy functional of [19] used in (2.2). In particular, since the supremum in (2.6) is taken over stepfunctions *with values in* $\mathcal{M}$ (respectively, of a functional which is *jointly affine* in ω and ϕ), the following *monotonicity* (2.8) w. r. t. algebraic inclusions (resp. *joint convexity* (2.9) in ω and ϕ) can be read off from (2.6):

$$\mathcal{A} \subset \mathcal{M} \Longrightarrow S_{\mathcal{A}}(\omega|\phi) \leq S_{\mathcal{M}}(\omega|\phi), \quad \text{resp} \tag{2.8}$$

$$S_{\mathcal{M}}(\sum_{i=1}^n \lambda_i\omega_i | \sum_{j=1}^n \lambda_j\phi_j) \leq \sum_{i=1}^n \lambda_i S_{\mathcal{M}}(\omega_i|\phi_i), \tag{2.9}$$

where $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{M}$, resp. $\lambda_i > 0$ with $\sum_{i=1}^n \lambda_i = 1$.

2. The definition (2.7) was given by Narnhofer and Thirring [31] for (AFD) von Neumann algebras $\mathcal{M}$ using Araki's definition of the relative entropy, and it naturally extends to general C^*-algebras $\mathcal{M}$ with the definition (2.6) for the relative entropy (cf. [20, 23]). In the simplest non-Abelian situation $\mathcal{M} = \mathcal{B}(\mathcal{H})$ with *finite-dimensional* Hilbert space $\mathcal{H}$, by construction (2.6) coincides with (2.5); and because of the joint convexity (2.9), the supremum over $\sum_i \lambda_i S_{\mathcal{M}}(\omega|\omega_i) = \sum_i \mathrm{Tr}\lambda_i\rho_{\omega_i}(\ln\rho_{\omega_i} - \ln\rho_\omega)$ is attained for the one-dimensional eigenprojectors $\rho_{\omega_i} = P_i$ and eigenvalues λ_i of $\rho_\omega = \sum_i \lambda_i P_i$, such that also $S_{\mathcal{M}}(\omega) = -\mathrm{Tr}\rho_\omega \ln\rho_\omega$ again coincides with (1.2). Thus the essential properties of the entropy functional $S(\omega)$ follow from the properties of the *relative* entropy; and in particular the strong subadditivity property (2.1), originally proven in [21], becomes a simple consequence of the obvious monotonicity (2.8) of the relative entropy [31].

3. In spite of this monotonicity (2.8) of the relative entropy terms $S_{\mathcal{A}}(\omega|\omega_i)$ in definition (2.7), however, the entropy $S_{\mathcal{A}}(\omega)$ is *not* monotonic w. r. t. the C^*-subalgebra $\mathcal{A} \subset \mathcal{M}$ for non-Abelian $\mathcal{M}$ (not even in the simplest situation $\mathcal{M} = \mathcal{B}(\mathcal{H})$ as above, cf. after (2.1)); the reason being that the supremum in (2.7) is taken over decompositions $\omega|_{\mathcal{A}} = \sum_i \lambda_i \omega_i|_{\mathcal{A}}$ of ω *as state on* $\mathcal{A} \subset \mathcal{M}$. Narnhofer and Thirring [31] thus stated in a "side-remark", that the entropy functional $H_\omega(\mathcal{A})$ (we use this notation *without* "abuse of language", see below), defined by the expression (2.7) with the supremum taken over $\omega|_{\mathcal{M}} = \sum_i \lambda_i \omega_i|_{\mathcal{M}}$ independent of $\mathcal{A} \subset \mathcal{M}$, would be *monotonic* because of (2.8):

$$H_\omega(\mathcal{A}) = \sup_{\{\sum_i \lambda_i \omega_i|_{\mathcal{M}} = \omega|_{\mathcal{M}}\}} \sum_i \lambda_i S_{\mathcal{A}}(\omega|\omega_i), \tag{2.10}$$

and that $H_\omega(\mathcal{A}) \leq S_{\mathcal{A}}(\omega)$ for $\mathcal{A} \subset \mathcal{M}$, because then $\omega|_{\mathcal{M}} = \sum_i \lambda_i \omega_i|_{\mathcal{M}}$ implies $\omega|_{\mathcal{A}} = \sum_i \lambda_i \omega_i|_{\mathcal{A}}$.

4. For an *Abelian* W^*-algebra $\mathcal{M} \ni 1$ with faithful normal state ω on $\mathcal{M}$ and a *finite-dimensional* $*$-subalgebra $\mathcal{A} \subset \mathcal{M}$ with minimal projectors $P_i \in \mathcal{A}$ as in (1.5), the restriction $\omega|_{\mathcal{A}}$ is determined by the probability distribution $\{\omega(P_i)\}$ on the finite partition (of the Gelfand space $X_{\mathcal{M}}$, cf. the remarks after (1.5)) corresponding to $\mathcal{A} \in \mathcal{F}$. Then in view of the Riesz' representation theorem for ω (cf. (1.4)), every finite decomposition $\omega|_{\mathcal{A}} = \sum_i \lambda_i \omega_i|_{\mathcal{A}}$ can be extended to a (not uniquely determined) decomposition $\omega|_{\mathcal{M}} = \sum_i \lambda_i \omega_i|_{\mathcal{M}}$, such that (2.7) and (2.10) even coincide: $S_{\mathcal{A}}(\omega) = H_\omega(\mathcal{A})$; and because of the special form of the relative entropy terms $S_{\mathcal{A}}(\omega|\omega_i)$ from (2.6) in this Abelian case (cf. [23, 31]), the supremum is attained for the decomposition $\lambda_i \omega_i = \omega(P_i \bullet)$ given by the partition corresponding to $\mathcal{A}$, and both (2.7),(2.10) in turn coincide with (1.5). Thus (together with 2. above) the formal identity of (1.3) and (1.5) can be viewed to stem from the common definition (2.7) of the entropy $S_{\mathcal{A}}(\omega)$; and our notation $H_\omega(\mathcal{A})$ in (2.10) is no "abuse of language" in comparison with (1.5).

Surprisingly, however, the notation $H_\omega(\mathcal{A})$ for the monotonic entropy functional (2.10) of Narnhofer and Thirring [31] is not even an "abuse of language" when compared with our notation H_ω after (2.4) above: For a (non-Abelian) von Neumann algebra $\mathcal{M}$ with (faithful, normal) *trace state* $\omega = \tau$ on $\mathcal{M}$ and with a *finite-dimensional* (von Neumann) subalgebra $\mathcal{A} \subset \mathcal{M}$, the functional $H_\omega(\mathcal{A})$ from (2.10) can be easily seen (cf. [12]) to coincide with the functional $H_\tau(\mathcal{A})$ of Connes and Størmer [19] (as used in (2.2), but here for only *one* argument). Thus the "by-product" (2.10) (of our detour (2.6) and (2.7)) turns out to be already the desired extension to non-tracial states ω of the H_ω-entropy functional [19] with only *one* argument; and this in turn also shows the way (cf. [12]) to the further extension for *more* than one argument, now even for a general C^*-algebra $\mathcal{M}$:

Definitions: Let $\mathcal{M} \ni 1$ be a C^*-algebra with state ω on $\mathcal{M}$. We denote by $\mathcal{F}$ the set of all finite-dimensional $*$-subalgebras of $\mathcal{M}$; and for a finite decomposition $\sum_{I_n} \omega_{I_n} = \omega$ of ω into positive linear functionals ω_{I_n} on $\mathcal{M}$ with multi-index $I_n \equiv (i_1, \dots, i_n) \in \mathbf{N}^n$, we denote the n single-index partial sums by $\omega^k_{i_k} = \sum_{I_n,\ i_k \text{ fixed}} \omega_{I_n}$ $(k = 1, \dots, n)$.

(i) The *entropy* of the set $\{\mathcal{A}_i \in \mathcal{F} | i = 1, \ldots, n\}$ is defined by

$$H_\omega(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \sup_{\{\sum_{I_n} \omega_{I_n} = \omega\}} \left[\sum_{I_n} \eta(\omega_{I_n}(1)) + \sum_{k=1}^{n} \sum_{i_k} S_{\mathcal{A}_k}(\omega | \omega_{i_k}^k) \right], \tag{2.11}$$

where the supremum is taken over all finite multi–index decompositions of ω as above.

(ii) The *conditional entropy* of $\mathcal{A} \in \mathcal{F}$ w. r. t. $\mathcal{B} \in \mathcal{F}$ is defined by

$$H_\omega(\mathcal{A}|\mathcal{B}) = \sup_{\{\sum_i \omega_i = \omega\}} \sum_i [S_{\mathcal{A}}(\omega|\omega_i) - S_{\mathcal{B}}(\omega|\omega_i)], \tag{2.12}$$

where the supremum is taken over all finite (single–index) decompositions of ω into positive linear functionals ω_i on $\mathcal{M}$.

Remarks:

1. The definition (2.11) was first given by A. Connes [32] for a *von Neumann* algebra $\mathcal{M}$ with normal state ω, in continuation of his joint work [19] with E. Størmer and *independently* from the definition (2.10) by H. Narnhofer and W. Thirring [31]; and (2.11) was then reformulated for a general C^*–algebra $\mathcal{M}$ (with state ω and finite–dimensional $*$–subalgebras $\mathcal{F}$) as above by Connes, Narnhofer and Thirring [20] *together*. As in the special case (2.10) of only *one* argument $\mathcal{A} \in \mathcal{F}$, the *monotonicity* of the entropy functional (2.11) w. r. t. algebraic inclusions of the $\mathcal{A}_i \in \mathcal{F}$ is "inherited" from the monotonicity (2.8) of the relative entropy terms; and as in the special case [19] of a *trace* state $\omega = \tau$, $H_\omega(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is again *subadditive* w. r. t. the set $\{\mathcal{A}_i \in \mathcal{F}\}$ on which it depends only (cf. [32, 20]). Thus the *characteristic invariant* $h_\omega(\theta)$ of a $*$–automorphism θ of the C^*–algebra $\mathcal{M} \ni 1$, with invariant state $\omega = \omega \circ \theta$, can be defined as in (2.2) and (2.3) (with τ replaced by ω); and $h_\omega(\theta)$ is again a conjugacy invariant of the ω–preserving Z–action θ on $\mathcal{M}$ as in (1.14) because of the θ–invariance of the functional (2.11).

2. Also the definition (2.12) was first given by Connes [32] as extension to non–tracial states ω of the conditional entropy defined in [19], and thus again as a tool to prove the non–Abelian "Kolmogorov–Sinai" theorem (2.4) (with τ replaced by ω) for $h_\omega(\theta)$ in the case of an AFD von Neumann algebra $\mathcal{M}$. This theorem was then proven more directly in [20], where also the analogous KS theorem for an AFD C^*–algebra $\mathcal{M} = \overline{\bigcup_{n \in \mathbf{N}} \mathcal{M}_n}$ (with norm closure) was derived (as a corollary of an even more general KS theorem for *nuclear* C^*–algebras, cf. also [23]). The conditional entropy (2.12) was reconsidered in [33] and [34], where it is explicitely shown to coincide with the classical conditional entropy (1.6) (as in (1.9)) for an *Abelian* W^*–algebra $\mathcal{M}$ with (faithful, normal) state ω; in which case of course also the functional (2.12) coincides with the classical expression (1.5): $H_\omega(\mathcal{A}_1, \ldots, \mathcal{A}_n) = H_\omega(\mathcal{A}_1 \vee \ldots \vee \mathcal{A}_n)$, as already for (2.10) with only one argument (cf. [20, 23]).

3. Already for a "generic" *Abelian C^*-algebra* $\mathcal{M} \ni 1$ with (faithful) state ω, however, which is $*$-isomorphic to $C(X_{\mathcal{M}})$ but *not* to $L^\infty(X_{\mathcal{M}}, \mu_\omega)$ on the Gelfand space $X_{\mathcal{M}}$ with Radon measure μ_ω as after (1.5), there will in general exist no *finite-dimensional* $*$-subalgebras of $\mathcal{M}$ at all, such that the set $\mathcal{F}$ in definition (2.11) and thus the definition itself will be empty. Connes, Narnhofer and Thirring [20] therefore extended definition (2.11) further to the entropy $H_\omega(\gamma_1, \ldots, \gamma_n)$ of sets of *completely positive maps* $\gamma_i : \mathcal{A}_i \to \mathcal{M}$ from ("abstract") finite-dimensional C^*-algebras $\mathcal{A}_i$ to $\mathcal{M}$, which always exist and reduce to the identical inclusions $\imath_{\mathcal{A}_i} = \gamma_i$ for *$*$-subalgebras* $\mathcal{A}_i \in \mathcal{F}$ as above. This further generalization in turn allows a more conceptual redefinition of (2.11) in terms of "Abelian models" (cf. [20, 23]) and also the general KS theorem for *nuclear C^*-algebras* $\mathcal{M}$ mentioned in 2.) above. – In the following physical applications of the quantum entropy invariant developed thus far, however, the (non-Abelian) C^*-algebra $\mathcal{M}$ will always be AFD, and we may be content with (2.11) in this form.

3 Symmetry group actions on quantum systems

Whereas the classical theory of section 1 has been successfully applied also to spatially *finite* classical systems with *finitely* many degrees of freedom, for a quantum system with *finitely* many particles (corresponding to a type I representation $\mathcal{M}$, cf. [8, 18]) the characteristic invariant $h_\omega(\theta)$ defined as in (2.3) for a symmetry $*$-automorphism θ of $\mathcal{M}$ always *vanishes* (cf. [35]), since then any physically reasonable (θ-invariant) state ω on $\mathcal{M}$ has *finite* quantum mechanical entropy $S_{\mathcal{M}}(\omega)$ as in (2.7) or (1.2). This would also be the case for a finite *classical* system with phase space X and "state" measure μ, however, if X had finite "grain-size" (i. e. for an *atomic* measure space X), since then the entropy $S_{\mathcal{M}_\mu}(\omega_\mu)$ from (2.7) of ω_μ on $\mathcal{M}_\mu$ as in (1.4) would coincide with (1.5) or (1.1) and hence be *finite*, too. Roughly speaking, the "quantum mechanical phase space" has grain-size h^N (with Planck's constant h and number of degrees of freedom N; cf. [36, 23]), and to get a non-trivial characteristic invariant (2.3), we have to pass to the thermodynamic limit of *infinite* volume (and number of degrees of freedom N, corresponding to type II or III representations $\mathcal{M}$; cf. also [37]). In the following, we shall describe simple applications to quantum *statistical* mechanics (cf. [38]); but in principle the theory developed in section 2 should be applicable also to (algebraic) quantum *field* theory (cf. also [18]).

The characteristic invariant $h_\omega(\theta)$, defined as in (2.3) with the H_ω-entropy functional (2.11), was first applied to quantum statistical mechanics by Connes [32], who showed that for a one-dimensional quantum spin system (cf. [38]), with translation invariant state ω on the AFD C^*-algebra $\mathcal{M}$ of quasilocal observables, the invariant $h_\omega(\sigma)$ of the unit lattice shift automorphism σ on $\mathcal{M}$ is always bounded above by the (spatial) *entropy density* $s(\omega)$ (cf. [38]) of the state ω: $h_\omega(\sigma) \leq s(\omega)$, cf. also [30]. This inequality was then sharpened to *equality* by Connes, Narnhofer and Thirring [20, 39] for a one-dimensional Fermi lattice gas with *quasifree* (cf. [38]) state $\omega = \omega \circ \sigma$ on the CAR algebra $\mathcal{M}$ over $\ell^2(\mathbf{Z})$ (equivalent to a spin system up to a Klein transformation, cf. [39]); and this equality was further extended to (one-dimensional) spin systems with translation invariant *short range* interactions (cf. [38]) and the corresponding KMS states ω by Narnhofer and Thirring [40], who showed that for these systems the spatial density of the entropy functional (on the local algebras $\mathcal{A}$) $H_\omega(\mathcal{A})$ from (2.10) coincides with the entropy density $s(\omega)$ (of the entropy $S_\mathcal{A}(\omega)$ from (2.7)), what they also used to prove the "Third Law of Thermodynamics" for (unique) temperature T-KMS states ω_T with sufficiently decaying correlation functions both in space and time (and uniformly in $T < T_0$): $\lim_{T\to 0} s(\omega_T) = 0$. And in [39] the continuum limit for the invariant $h_\omega(\sigma)$ of unit space translation σ as in [20] above was performed, leading to the one-dimensional Fermi gas with quasifree state $\omega = \omega\circ\sigma$ on the CAR algebra over $L^2(\mathbf{R})$, for which also the *dynamical entropy* $h_\omega(\tau)$ of the *quasifree* (i. e. one-particle implemented, cf. [38]) unit time evolution automorphism τ with KMS state $\omega = \omega\circ\tau$ was calculated from $h_\omega(\sigma)$ (by conjugacy as in (1.14)).

As already in the classical case [15] (resp. [17]), to be applicable as characteristic invariant of space translations in infinite quantum systems with spatial dimension $\nu > 1$ (resp. as "spacetime entropy"), the definition (2.3) (with state ω) has to be extended from $\mathbf{Z}$-actions to actions of $\mathbf{Z}^\nu$. This was done in [41] following Conze [16] on his "classical" route to define the invariant $h_\omega(G)$ of an Abelian automorphism group G generated by commuting $*$-automorphisms $\sigma_1, \ldots, \sigma_\nu$ of the C^*-algebra $\mathcal{M}$ with invariant state $\omega = \omega\circ\sigma_i$ $(i = 1, \ldots, \nu)$ on $\mathcal{M}$: For $\mathcal{A} \in \mathcal{F}$ as in (2.11) and an increasing sequence $\{\rho_n\}$ of parallelepipeds $\rho_n \subset \mathbf{Z}^\nu$ tending to infinity (in $\mathbf{Z}^\nu$), the following limit exists, again due to subadditivity and invariance of the entropy functional (2.11):

$$h_\omega(G, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{|\rho_n|} H_\omega\left(\bigcup_{k\in\rho_n} \{\sigma_1^{k_1} \circ \ldots \circ \sigma_\nu^{k_\nu} \mathcal{A}\}\right), \tag{3.1}$$

where $k \equiv (k_1, \ldots, k_\nu)$; and the limit is independent of the particular sequence $\{\rho_n\}$ and thus of the choice of generators $\sigma_1, \ldots, \sigma_\nu$ for G (and it had been used for a sequence of *cubes* $\rho_n \subset \mathbf{Z}^\nu$ already in [42]). The *characteristic invariant* of the ω-preserving $\mathbf{Z}^\nu$-action on $\mathcal{M}$ by G is then defined as for $\nu = 1$ in (2.3):

$$h_\omega(G) = \sup_{\mathcal{A}\in\mathcal{F}} h_\omega(G, \mathcal{A}). \tag{3.2}$$

The "scaling" property for $\nu = 1$ as in (1.12) was generalized in [41] (as in [16]) to a subgroup G_p of G with *finite* index $p \in \mathbf{N}$, acting on an AFD (or only nuclear) C^*-algebra $\mathcal{M}$ (such that again a KS theorem as for $\nu = 1$ in [20] holds):

$$h_\omega(G_p) = p \cdot h_\omega(G); \tag{3.3}$$

and as a corollary of this "scaling" property, if any subgroup $G(\mu)$ of G with *infinite* index (generated by $\sigma_{i_1}, \ldots, \sigma_{i_\mu}$ as above with $0 < \mu < \nu$) has *finite* invariant $h_\omega(G(\mu)) < \infty$, then $h_\omega(G) = 0$. This result was applied in [41] to the *spacetime group* $G \equiv G(\sigma, \tau)$, generated by the group $G(\sigma)$ of unit space translation automorphisms and by the unit time evolution automorphism τ, of translationally invariant (infinite) quantum systems. For quantum *spin systems* on a cubic lattice, with translation invariant state ω on $\mathcal{M}$, the inequality from

[32, 20] was generalized to more than one dimension: $h_\omega(G(\sigma)) \leq s(\omega) < \infty$ (cf. [38]), such that we can conclude $h_\omega(G(\sigma,\tau)) = 0$ (with any translation invariant interaction such that the time evolution τ exists, cf. [38]). Again (as for $\nu = 1$ in [20], cf. [12]), the above inequality can be sharpened to *equality* $h_\omega(G(\sigma)) = s(\omega)$ for a Fermi lattice gas with quasifree, translation invariant state ω on the CAR algebra $\mathcal{M}$ over $\ell^2(\mathbf{Z}^\nu)$; and after the continuum limit to $L^2(\mathbf{R}^\nu)$ (as for $\nu = 1$ in [39], cf. [12, 35]), $h_\omega(G(\sigma))$ can still be estimated from above in an explicit (momentum space) integral representation as for $\nu = 1$ in [41], where it was shown to be finite for KMS states ω of the *quasifree* time evolution τ with one–particle energy–momentum spectrum (Hamiltonian in momentum space) $\varepsilon(k) \geq c\|k\|^\delta$ as $\|k\| \to \infty$ for $c, \delta > 0$. But as remarked at the end of [41], in analogy to the *lattice* systems above it can be generally shown (cf. [12]) that $h_\omega(G(\sigma)) < \infty$ for *any* (not necessarily quasifree) translation invariant state ω with finite *particle density* (i. e. with integrable truncated two–point function density in momentum space, cf. [38]); and for *any* (unit) time evolution τ of the CAR algebra $\mathcal{M}$ over $L^2(\mathbf{R}^\nu)$ with $\omega \circ \tau = \omega$ we get again (by the corollary of (3.3), as for $\nu = 1$ in [41]) that the "spacetime entropy" *vanishes*: $h_\omega(G(\sigma,\tau)) = 0$. Whereas for $\nu = 1$ in [39, 41] also the *dynamical entropy* $h_\omega(\tau)$ of the quasifree time evolution τ with KMS state ω as above is *finite* for any physically reasonable $\varepsilon(k)$, however, it is shown in [12] (by a conjugacy argument as for $\nu = 1$ in [39], and again applying the corollary of (3.3) above; cf. also [33]) that $h_\omega(\tau) = \infty$ for spatial dimension $\nu > 1$, in analogy to the infinite classical systems considered by Goldstein [17]. In contradistinction to these *classical* gases, however, the *finite* entropy of space translations $h_\omega(G(\sigma)) < \infty$ for (even interacting) continous Fermi gases makes the quantum "spacetime entropy" $h_\omega(G(\sigma,\tau)) \equiv 0$ *too* "sharp" a tool of investigation (as a trivial invariant); the reason possibly being the "finite grain–size" of the "quantum mechanical phase space" mentioned above.

Finally, it should be remarked that the definition (3.1),(3.2) for the characteristic invariant of the symmetry automorphism group G could be further extended from $\mathbf{Z}^\nu$–actions G to actions of arbitrary (non–Abelian) *amenable* groups (such as the Euclidean group, cf. [18]) following the "classical" route of [13], if the functional (2.11) would be even *strongly* subadditive w. r. t. finite sets $X, Y, Z \subset \mathcal{F}$: $H_\omega(X \cup Y \cup Z) \leq H_\omega(X \cup Z) + H_\omega(Y \cup Z) - H_\omega(Z)$; generalizing the classical inequality (1.11), cf. [12, 41]. Furthermore, if such an inequality would be proven, we could define a "conditional entropy" $H_\omega(X|Y) = H_\omega(X \cup Y) - H_\omega(Y)$ as *information gain* generalizing the classical relation (1.9) (but *different* from the two–argument conditional entropy (2.12) for non–Abelian $\mathcal{M}$, cf. [33, 34]), such that also the classical relation (1.12) would generalize to $h_\omega(\theta, \mathcal{A}) = \lim_{n\to\infty} H_\omega(\tau^n\mathcal{A} | \bigcup_{i=0}^{n-1}\{\tau^i\mathcal{A}\})$ with $h_\omega(\theta, \mathcal{A})$ defined as in (2.2) for non–tracial state ω. – The above strong subadditivity of the entropy functional (2.11) has resisted proof (or disproof) up to now, however, and we leave this *intrinsically* nonlinear problem as a conjecture.

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