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FOREWORD

The summer workshop on Superstring theory was held at KEK on August 29-September 3, 1988. This workshop was devoted to various recent developments in conformal field theories and string theories. Special emphasis was laid on the algebraic approach to string compactification. More than 100 people participated in the workshop including young graduate students, and over 40 speakers announced their latest results. Several introductory lectures were also delivered for newcomers in the field. We hope this workshop serves as an oppotunity to stimulate future progress by these young physicists.

This workshop was supported by the Grant-in Aid for scientific research on Priority Arears from the Ministry of Education, Science and Culture. We wish to thank the speakers and participants for their successful efforts to provide a stimulating and friendly atmosphere.

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CONTENTS

Introduction to conformal field theories

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1 Motivation

Conformal field theories in two dimensions have been receiving much attention recently [1] (See [2] for excellent review). These theories are relevant to the two dimensional critical phenomena. In string theory, conformal field theories are building blocks of the perturbative vacuum.

Our motivation for considering conformal field theories here is as follows: How can we construct a "complete" field theory in which any N point correlation functions

$$\langle \phi_1(x_1)\phi_2(x_2)\dots\phi_N(x_N)\rangle \tag{1.1}$$

can be calculated exactly? It would be very nice if we had an explicit lagrangian for such theories, but let us content ourselves with the knowledge of all correlation functions.

To construct such field theories, we have to present a systematic way of calculating correlation functions anyhow. Let $\mathcal{A}$ be the set of local field operators. We postulate that $\mathcal{A}$ is closed under the multiplicative structure known as "operator product algebra (OPA)" or simply "operator algebra". That means the operator product expansion (OPE)

$$\phi_i(x)\phi_j(0) = \sum_k C_{ij}^k(x)\phi_k(0) \tag{1.2}$$

holds as an exact relation. Then any N point function can be expressed in terms of $N-1$ point functions:

$$\begin{aligned} &\langle \phi_i(x)\phi_j(0)\phi_{l_1}(x_1)\dots\phi_{l_N}(x_N)\rangle \\ =\ &\sum_k C_{ij}^k(x)\langle\phi_k(0)\phi_{l_1}(x_1)\dots\phi_{l_N}(x_N)\rangle \end{aligned} \tag{1.3}$$

Using this relation recursively, one can calculate, in principle, any N point functions if we know 1 point functions $\langle\phi_i(x)\rangle$ and structure functions $\{C_{ij}^k(x)\}$. But it must be guaranteed that the result does not depend on the way we applied the OPA eq. (1.2). This is nothing but the requirement of associativity of the OPA, which is indispensable to construct consistent field theories. *Bootstrap hypothesis* is the hypothesis which says that the structure of OPA is determined solely by this consistency of the theory.

This bootstrap condition puts infinite number of equations on the structure functions $\{C_{ij}^k(x)\}$, because one can think of correlation functions of arbitrary number of operators with arbitrary configurations. Is it possible to achieve this bootstrap program without fail? Remarkably, for conformal field theory $\stackrel{def}{=}$ two dimensional field theory with conformal invariance, the program is almost completed.

Conformal invariance implies that physical quantities are invariant under the local scale transformations i.e. the scaling factor being different from point to point. In two dimensions, this invariance turns out to be infinite dimensional symmetry, and this "gauge" symmetry is the master key to the miraculous solution of bootstrap problem.

2 Local conformal invariance

In this section, we discuss the local conformal invariance in two dimensions and its physical or mathematical implication. Here the term *local* means "infinitesimal", and we will consider a transformation which is conformal only within a small region.

Let us first consider d dimensional Euclidean space in general. A map $x_\mu \to x'_\mu \quad (\mu = 1, 2, \ldots, d)$ is called conformal if angles of any two vectors are preserved, or $dx'_\mu dx'^\mu = \rho(x)^{-2} dx_\mu dx^\mu$ holds for some function $\rho(x)$.

The following transformations are all conformal mappings.

translation	$x'_\mu = x_\mu + a_\mu$	
rotation	$x'_\mu = x_\mu + \epsilon_\mu{}^\nu x_\nu$	$\epsilon_\mu{}^\nu = -\epsilon_\nu{}^\mu$
dilatation	$x'_\mu = x_\mu + \lambda x_\mu$	
special conformal transformation	$x'_\mu = x_\mu + x^2\alpha_\mu - 2(\alpha\cdot x)x_\mu$	

If the dimension d is greater than 2, the table exhausts all the possibility. In two dimensions, however, it is well known that *a map* $x_\mu \to x'_\mu \quad (\mu = 1,2)$ *is conformal if and only if* $z' = f(z)$ *is holomorphic, where* $z = x_1 + ix_2, z' = x'_1 + ix'_2$.

Infinitesimal transformations of the form

$$z \to z' = z + \sum_{n\in\mathbf{Z}} \epsilon_n z^{n+1} \tag{2.1}$$

are all holomorphic, so there are infinitely many conformal transformations in two dimensions. In other words, conformal invariance is infinite dimensional symmetry.

As we will see later, in conformal field theories, this symmetry is expressed on the correlation functions in the form of Ward identities.

Let us introduce the generator $T(z)$ of conformal transformations. Suppose $x_\mu \to x'_\mu + \alpha_\mu(x)$ be an infinitesimal coordinate transformation, which is not necessarily conformal. Such a change in coordinates is described by the stress energy tensor $T_{\mu\nu}(x)$:

$$\delta\mathcal{H} = -\frac{1}{2\pi}\int \partial^\mu\alpha^\nu T_{\mu\nu} d^2x, \tag{2.2}$$

where the $\delta\mathcal{H}$ is the variation of the quantum action.

We can decompose the effect of the coordinate transformation into three parts:

$$\begin{aligned} \partial^\mu\alpha^\nu &= \tfrac{1}{2}(\partial\cdot\alpha)\delta^{\mu\nu} && \text{dilatation} \\ &\quad +\tfrac{1}{2}\{\partial^\mu\alpha^\nu - \partial^\nu\alpha^\mu\} && \text{rotation} \\ &\quad +\tfrac{1}{2}\{\partial^\mu\alpha^\nu + \partial^\nu\alpha^\mu - (\partial\cdot\alpha)\delta^{\mu\nu}\} && \text{shear} \end{aligned} \tag{2.3}$$

By definition of conformal invariance, dilatation and rotation induce no change to the system, so the stress energy tenser must be symmetric and traceless:

$$T_{\mu\nu} = T_{\nu\mu}, \quad T^\mu_\mu = 0. \tag{2.4}$$

If we switch to the complex notation and use the conservation law $\partial^\mu T_{\mu\nu} = 0$, we have

$$T_{z\bar{z}} = T_{\bar{z}z} = 0$$
$$\bar{\partial} T_{zz} = 0, \qquad T(z) \stackrel{def}{=} T_{zz}$$
$$\partial T_{\bar{z}\bar{z}} = 0, \qquad \bar{T}(\bar{z}) \stackrel{def}{=} T_{\bar{z}\bar{z}}$$

As we will see soon, this splitting of $T_{\mu\nu}$ into holomorphic and antiholomorphic part is crucial to derive the conformal Ward identities.

Let $\phi(z,\bar{z})$ be a primary field which transforms, by definition, as a tensor of type $(\Delta, \bar{\Delta})$:

$$\phi(z,\bar{z})(dz)^{\Delta}(d\bar{z})^{\bar{\Delta}} = \phi(w,\bar{w})(dw)^{\Delta}(d\bar{w})^{\bar{\Delta}} \tag{2.5}$$

Let us consider the correlation function of N primary fields, all of which are sitting inside some closed contour C. Suppose $z \to w(z) = z + \alpha(z)$ be the transformation which is conformal inside C, and $\alpha(z) \to 0$ $(z \to \infty)$. Then, from the definition of primary fields,

$$\langle \phi_1(z_1)\phi_2(z_2)\ldots\rangle = \prod_j \left(\frac{\partial w}{\partial z}\right)^{\Delta_j} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{\Delta}_j} \langle \phi_1(w_1)\phi_2(w_2)\ldots\rangle \tag{2.6}$$

holds. The effect of coordinate transformation can also be described by the energy momentum tensor. Using eq. (2.2) and eq. (2.6), we obtain the following equation from the terms of order $\mathcal{O}(\alpha)$:

$$\begin{aligned} &-\frac{1}{2\pi}\int_{R_2} \partial^\mu \alpha^\nu(x) \langle T_{\mu\nu}(x)\phi_1(z_1)\phi_2(z_2)\ldots\rangle d^2x \\ &= \sum_j \left\{ \alpha'(z_j)\Delta_j + \alpha(z_j)\frac{\partial}{\partial z_j} + \bar{\alpha}'(\bar{z}_j)\bar{\Delta}_j + \bar{\alpha}(\bar{z}_j)\frac{\partial}{\partial \bar{z}_j} \right\} \langle \phi_1(z_1)\phi_2(z_2)\ldots\rangle \end{aligned} \tag{2.7}$$

where R_2 is the region outside C. Using Stokes' theorem,

$$\begin{aligned} \text{lhs of eq. (2.7)} &= -\frac{1}{2\pi}\int_C n^\mu \alpha^\nu(x)\langle T_{\mu\nu}(x)\phi_1(x_1)\phi_2(x_2)\cdots\rangle \\ &\quad +\frac{1}{2\pi}\int_{R_2} \alpha^\nu(x)\langle \partial^\mu T_{\mu\nu}(x)\phi_1(x_1)\phi_2(x_2)\cdots\rangle \\ &= \frac{1}{2\pi i}\int_C dz\alpha(z)\langle T(z)\phi_1(z_1)\phi_2(z_2)\cdots\rangle \\ &\quad -\frac{1}{2\pi i}\int_C d\bar{z}\bar{\alpha}(\bar{z})\langle \bar{T}(\bar{z})\phi_1(\bar{z}_1)\phi_2(\bar{z}_2)\cdots\rangle, \end{aligned}$$

where we used conservation of energy-momentum $\partial^\mu T_{\mu\nu}(x) = 0$ and switched to complex notations. On the other hand, from the Cauchy's formula,

$$\begin{aligned}\text{rhs of eq. (2.7)} &= \frac{1}{2\pi i}\int_C dz\alpha(z)\sum_j\left\{\frac{\Delta_j}{(z-z_j)^2}+\frac{1}{z-z_j}\frac{\partial}{\partial z_j}\right\}\langle\phi_1\phi_2\cdots\rangle \\ &\quad -\frac{1}{2\pi i}\int_C d\bar{z}\bar{\alpha}(\bar{z})\sum_j\left\{\frac{\bar{\Delta}_j}{(\bar{z}-\bar{z}_j)^2}+\frac{1}{\bar{z}-\bar{z}_j}\frac{\partial}{\partial\bar{z}_j}\right\}\langle\phi_1\phi_2\cdots\rangle\end{aligned}$$

Comparing these expressions and using the freedom to deform the integration contour C, we obtain conformal Ward identity for single insertion of energy momentum tensor:

$$\langle T(z)\phi_1(z_1)\phi_2(z_2)\ldots\rangle = \sum_j\left\{\frac{\Delta_j}{(z-z_j)^2}+\frac{1}{z-z_j}\frac{\partial}{\partial z_j}\right\}\langle\phi_1(z_1)\phi_2(z_2)\ldots\rangle. \tag{2.8}$$

In a similar way, we can derive Ward identity for double insertion of the energy momentum tensor:

$$\begin{aligned}&\langle T(z)T(u)\phi_1(z_1)\cdots\rangle = \frac{c}{2}\frac{1}{(z-u)^4}\langle\phi_1(z_1)\cdots\rangle \\ &+\left\{\frac{2}{(z-u)^2}+\frac{1}{z-u}\frac{\partial}{\partial u}+\sum_j\left(\frac{\Delta_j}{(z-z_j)^2}+\frac{1}{z-z_j}\frac{\partial}{\partial z_j}\right)\right\}\langle T(u)\phi_1(z_1)\cdots\rangle.\end{aligned} \tag{2.9}$$

Here c is called conformal anomaly, trace anomaly or central charge. This comes from the Schwinger term of the commutation relation of energy momentum tensor, and it means that $T(z)$ does not transforms as a bona fide tensor in quantum level.

In the following we shall derive celebrated Virasoro algebra from the viewpoint of correlation functions.

In correlation functions, primary fields appear on a equal footing, but let us concentrate on a particular field, say $\phi_1(z_1)$. Let us put $z_1 = 0$ in eq. (2.8), and multiply by $\alpha(z) = z^{m+1}$, $m \geq -1$. Then integrating along a contour surrounding $z_1 = 0$, we have

$$\langle L_m\phi(0)\cdots\rangle \stackrel{def}{=} \frac{1}{2\pi i}\oint z^{m+1}\langle T(z)\phi(0)\cdots\rangle$$

$$= \frac{1}{2\pi i} \oint z^{m+1} \sum_j \left\{ \frac{\Delta_j}{(z-z_j)^2} + \frac{1}{z-z_j} \frac{\partial}{\partial z_j} \right\} \langle \phi(0) \cdots \rangle. \tag{2.10}$$

From (2.10), it is easy to see

$$\begin{aligned}
\langle L_m \phi(0) \cdots \rangle &= 0 \quad m = 1,2,3,\ldots, \\
\langle L_0 \phi(0) \cdots \rangle &= \Delta \langle \phi(0) \cdots \rangle, \\
\langle L_{-1} \phi(0) \cdots \rangle &= \langle \partial \phi(0) \cdots \rangle,
\end{aligned} \tag{2.11}$$

where $\Delta = \Delta_1$. Therefore, the Laurent expansion coefficients L_m act linearly on the vector space spanned by the correlation functions. In this sense, we can identify the correlation function of the form $\langle \phi(0) \cdots \rangle$ with a state vector $\cdots |\phi\rangle$ in some Hilbert space. From this point of view, eq. (2.11) can be interpreted as the condition put on the state vector $|\phi\rangle$:

$$\begin{aligned}
L_n |\phi\rangle &= 0 \quad n = 1,2,3,\ldots, \\
L_0 |\phi\rangle &= \Delta |\phi\rangle.
\end{aligned} \tag{2.12}$$

This is called highest weight condition.

In a similar way, from the conformal Ward identity eq. (2.9) for double insertion of energy momentum tensor, one can see that L_n's satisfy the Virasoro algebra commutation relations:

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m) \delta_{m+n,0}. \tag{2.13}$$

In quantum theories, every symmetry of the system is realized as a representation over the physical Hilbert space of the system. So representation theory has been quite helpful to investigate the quantum system.

We learned from above arguments that in conformal field theories the conformal symmetry (Virasoro algebra) is realized on the space of the correlation functions. So it is very natural to examine the representations of Virasoro algebra. For example, the following is very important question: "For which values of c and Δ positive definite representations of Virasoro algebra are possible ?"

We have come across a similar situation in quantum systems which are invariant under spatial rotations. In that case, the Hilbert space of the quantum system provides the representations of $SU(2)$. "For which values of j positive definite representation of $SU(2)$ are possible ?" We know very well the answer for that question. Let us review this case, for the method used there can be applied to the Virasoro algebra as well.

$SU(2)$ has three generators J_1, J_2 and J_3. Usually we choose the representation where the operator J_3 is diagonal. Then it is convenient to introduce the raising and lowering operator $J_+ = J_1 + iJ_2$ and $J_- = J_1 - iJ_2$, which changes the eigenvalue of J_3 by ± 1. In the representation space, there exist a state $|jj\rangle$ with the highest spin. That means

$$\begin{aligned} J_3|jj\rangle &= j|jj\rangle, \\ J_+|jj\rangle &= 0. \end{aligned} \tag{2.14}$$

We obtain new state vectors by applying lowering operator J_- one after another. Their norms can be calculated using commutation relations. The absence of negative norm state quantizes the value j, $j = 0, 1/2, 1, \ldots$. In this way, we can obtain all irreducible unitary representations.

In the case of Virasoro algebra, the representation can be constructed in quite similar fashion. The central charge c can be considered to be just a c-number because it commutes with all the other generators (Schur's lemma). We want to construct the representation in which L_0 is diagonal. The commutation relation

$$[L_n, L_0] = nL_n \tag{2.15}$$

means that L_n $(n < 0)$ are raising operators and L_n $(n > 0)$ are lowering operators. We assume that there exists a state vector $|c, \Delta\rangle$ which has the lowest eigenvalue of L_0 in the representation space:

$$\begin{aligned} L_n|c, \Delta\rangle &= 0 \quad n = 1, 2, 3, \ldots \\ L_0|c, \Delta\rangle &= \Delta|c, \Delta\rangle. \end{aligned} \tag{2.16}$$

This is the analogue of eq. (2.14) and nothing but the highest weight condition eq. (2.12).

Starting from the highest weight vector $|c, \Delta\rangle$, we repeatedly apply the raising operators on states to obtain new ones. In contrast to the $SU(2)$ case, we have infinitely many raising operators, so there are many linearly independent state vectors for each eigenvalue of L_0. We list the first several levels of representation space:

eigenvalue of L_0	eigenvectors
Δ	$\|c, \Delta\rangle$
$\Delta + 1$	$L_{-1}\|c, \Delta\rangle$
$\Delta + 2$	$L_{-2}\|c, \Delta\rangle, \quad L_{-1}^2\|c, \Delta\rangle$
$\Delta + 3$	$L_{-3}\|c, \Delta\rangle, \quad L_{-2}L_{-1}\|c, \Delta\rangle, \quad L_{-1}^2\|c, \Delta\rangle$
$\vdots$	$\vdots$

In general we have p_n linearly independent vectors corresponding to the eigenvalue $\Delta + n$, where p_n is the number of possible ways of partitioning n into the sum of positive integers.

In the second step, we have to calculate the norms of the vectors and see whether the positive definite representation space can be constructed. By definition, we have

$$\langle c, \Delta | c, \Delta \rangle = 1. \tag{2.17}$$

The next level,

$$\langle c, \Delta | L_1 L_{-1} | c, \Delta \rangle = \langle c, \Delta | 2L_0 | c, \Delta \rangle = 2\Delta \tag{2.18}$$

So we need

$$\Delta \geq 0 \tag{2.19}$$

to insure the positivity of the representation space. Going to the next level, we have two vectors $L_{-2}|c, \Delta\rangle$ and $L_{-1}^2|c, \Delta\rangle$. The positivity requires the norm of any linear combination of these two, $\lambda L_{-1}^2|c, \Delta\rangle + \mu L_{-2}|c, \Delta\rangle$ is non negative. Accordingly,

$$\begin{aligned} &\langle c, \Delta | (\lambda^* L_{-1}^2 + \mu^* L_{-2})(\lambda L_{-1}^2 + \mu L_{-2}) | c, \Delta \rangle \\ = \ & 4\left\{(\Delta + \frac{c}{8})|\lambda|^2 + 3\Delta \mathrm{Re}\lambda\dot{\mu} + \Delta(2\Delta + 1)|\mu|^2\right\} \geq 0 \end{aligned} \tag{2.20}$$

must hold for any μ, ν. Therefore, we have the following constraint on c and Δ:

$$8\Delta^2 - (5-c)\Delta + \frac{c}{2} \geq 0. \tag{2.21}$$

In this way, we have infinite number of inequalities from the requirement of positivity. It appears quite difficult to list up all the conditions as well as to find the possible values of c and Δ. But the answer is known [3].

The representation $V_{c,\Delta}$ of Virasoro algebra obtained in this way is unitary if and only if

$$c \geq 1 \quad \textit{and} \quad \Delta \geq 0 \tag{2.22}$$

or

$$\begin{aligned} c &= 1 - \frac{6}{m(m+1)}, \quad m = 2,3,4,\ldots \quad \textit{and} \\ \Delta &= \Delta_{p,q} = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}, \quad 1 \leq q \leq p \leq m-1. \end{aligned} \tag{2.23}$$

In the latter case (usually called minimal unitary series), the central charge c and conformal dimension Δ is quantized just as spin j is quantized in $SU(2)$. In fact, it is known that this series is pertinent to the multicritical phenomena of some two dimensional lattice systems [6].

The structure of the representation space has been well studied. For example, the explicit form of the character formulas are known:

$$\begin{aligned} \chi_{c,\Delta}(\tau) &\stackrel{def}{=} \mathrm{Tr}_{V_{c,\Delta}} q^{L_0 - \frac{c}{24}} \qquad (q = e^{2\pi i\tau}) \\ &= \frac{\Theta_{(m+1)p-mq,m(m+1)}(\tau) - \Theta_{(m+1)p+mq,m(m+1)}(\tau)}{\eta(\tau)} \end{aligned} \tag{2.24}$$

where

$$\Theta_{l,n}(\tau) = \sum_{j \in Z + \frac{l}{2n}} q^{nj^2}, \qquad \eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.25}$$

In the representation space corresponding to the minimal unitary series, there exist special kind of vectors $|\chi\rangle$ which satisfy the "pseudo" highest weight condition

$$\begin{aligned} L_n|\chi\rangle &= 0 \quad n = 1,2,3,\ldots \\ L_0|\chi\rangle &= (\Delta + K)|\chi\rangle. \end{aligned} \tag{2.26}$$

They are called null vectors because their norm is zero. So they must be put to be zero to obtain a positive definite irreducible representation space.

For example, if we choose c and $\Delta = \Delta_{2,1}$ from the list (2.23), then one can easily check that the vector

$$|\chi\rangle = \left\{ L_{-2} - \frac{3}{2(2\Delta+1)} L_{-1}^2 \right\} |c, \Delta_{2,1}\rangle \tag{2.27}$$

is a null vector. The physical implication of this fact is as follows. As stated before, the state $|c, \Delta\rangle$ corresponds to some primary field $\phi_{2,1}$. "Null" means "decouple from the theory", so the correlation function

$$\langle \phi_i(0) \left\{ L_{-2} - \frac{3}{2(2\Delta+1)} L_{-1}^2 \right\} \phi_{2,1}(z) \phi_j(1) \phi_k(\infty) \rangle \tag{2.28}$$

must identically vanish. Then using Ward identity, we can show that the correlation function

$$G(z) = \langle \phi_i(0) \phi_{2,1}(z) \phi_j(1) \phi_k(\infty) \rangle$$

is the solution to the following differential equation:

$$\left\{ \frac{3}{2(2\Delta+1)} \frac{d^2}{dz^2} - \left(\frac{1}{z} + \frac{1}{z-1} \right) \frac{d}{dz} - \frac{\Delta_i}{z^2} - \frac{\Delta_j}{(z-1)^2} - \frac{\Delta_i + \Delta_j - \Delta_k}{z(z-1)} \right\} G(z) = 0$$

(For details, please see the original paper [1].)

In conclusion, in minimal theories, correlation functions are the solution to some differential equations which originate from the existence of the null states. But in general, we have many linearly independent solutions although the physical correlation function is unique. So something is missing to complete our program. Let us discuss this point in the next section.

3 Global conformal invariance

In this section, we shall consider the global conformal invariance. Global means "cannot be achieved by the integration of infinitesimal transformations". Before going into details of global conformal invariance, let us stop for a while and see how far we have come to realize bootstrap program.

- In two dimensional conformal field theories, we have the conserved, symmetric traceless stress energy tensor, which generates infinitesimal conformal transformations. From this we can construct Virasoro generators $\{L_n\}$.

- Among the local fields, there exist a special set of operators called "primary fields" which are characterized by the transformation law eq. (2.28). Each primary field corresponds to an irreducible representation of Virasoro algebra.

- The space of local field operators $\mathcal{A}$ can be decomposed into the direct sum of conformal families, where each conformal family consists of a primary field and its descendants:

$$\mathcal{A} = \bigoplus_n [\phi_n], \qquad [\phi_n] = \{L_{-n_1} \cdots L_{-n_l}\phi \mid n_i > 0\}.$$

- We have Ward identities for insertion of the energy momentum tensors, so any correlation functions of the descendant fields can be obtained from those of primary fields. Therefore it is enough to calculate the correlators of primary fields.

- Using local conformal invariance, the OPA structure "functions" $\{C_{ij}^k(x)\}$ can be uniquely determined if we know the operator product expansion "coefficients" $\{C_{ij}^k\}$, which are defined as the coefficients of the leading terms of operator product expansion:

$$\phi_i(z)\phi_j(w) = \sum_k \frac{C_{ij}^k}{(z-w)^{\Delta_i+\Delta_j-\Delta_k}} \{\phi_k(z) + \mathcal{O}(z-w)\}.$$

- In the special class of conformal field theories called "minimal theories", the number of primary fields are finite. Moreover, the correlation functions are the solution to the differential equations which come from the null states.

Hence as far as minimal unitary conformal field theories are concerned, the bootstrap program will be completed if we can somehow determine the OPE coefficients $\{C_{ij}^k\}$. How shall we attack this problem?

Let us consider the N point functions of primary fields on the Riemann sphere. Invariance under the $SL(2,\mathbf{C})$ transformations (which is the integration of infinitesimal conformal transformations) completely fixes the form of correlation functions up to $N=3$:

$$\begin{array}{ll}
\text{1 point} & \langle \mathbf{1}(z)\rangle = 1, \qquad \langle\phi(z)\rangle = 0 \qquad (\phi\neq\mathbf{1}) \\
\text{2 point} & \langle \phi_i(z)\phi_j(w)\rangle = \delta_{ij}(z-w)^{-2\Delta_i} \\
\text{3 point} & \langle \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle = C_{ijk}(z_1-z_2)^{-\Delta_i-\Delta_j+\Delta_k}(\text{ cyclic perm. })
\end{array} \tag{3.1}$$

where $\mathbf{1}$ denotes the identity operator. One can also show that a four point functions has a following form:

$$\langle \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\phi_l(z_4)\rangle = \prod_{i<j}(z_i-z_j)^{\gamma_{ij}}U(z),$$

$$U(z) = \langle \phi_i(0)\phi_j(z)\phi_k(1)\phi_l(\infty)\rangle, \qquad z = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)},$$

where γ_{ij} are some constants determined by Δ's.

The form of $U(z)$ cannot be determined from $SL(2,\mathbf{C})$ invariance, but as we will see below, the associativity of OPA constrains the pole structure of $U(z)$. If we use the OPA for the field $\phi_i(z_1)$ and $\phi_j(z_2)$, then we have following expansion of $U(z)$:

$$\langle \phi_i(0)\phi_j(z)\phi_k(1)\phi_l(\infty)\rangle \sim \sum_p \frac{C_{ijp}C_{pkl}}{|z|^{2(\Delta_i+\Delta_j-\Delta_p)}}\{1+\mathcal{O}(z)\}. \tag{3.2}$$

On the other hand, if we use the OPA for the field $\phi_j(z_2)$ and $\phi_k(z_3)$, then we get another expansion

$$\langle \phi_i(0)\phi_j(z)\phi_k(1)\phi_l(\infty)\rangle \sim \sum_q \frac{C_{jkq}C_{qil}}{|1-z|^{2(\Delta_i+\Delta_k-\Delta_q)}}\{1+\mathcal{O}(1-z)\}. \tag{3.3}$$

We imposed the associativity of the operator product algebra, so the two expression must coincide. This is nothing but the requirement of crossing

symmetry. Schematically

$$\begin{array}{ccc}
(\phi_i \cdot \phi_j) \cdot \phi_k & = & \phi_i \cdot (\phi_j \cdot \phi_k) \\
\| & & \| \\
C^p_{ij} \phi_p \cdot \phi_k & & C^q_{jk} \phi_i \cdot \phi_q \\
\| & & \| \\
C^p_{ij} C^l_{pk} \phi_l & & C^q_{jk} C^l_{qi} \phi_l
\end{array}$$

$$\Longrightarrow C^p_{ij} C^l_{pk} = C^q_{jk} C^l_{qi} \quad \text{for any } i, j, k, l \tag{3.4}$$

Hence bootstrap conditions become important for four point functions, putting nontrivial constraints on the OPE coefficients. Note that the bootstrap condition comes from the two different configuration of primary fields, namely $z \sim 0$ and $z \sim 1$, which can never be continuously connected by conformal transformations. Therefore, associativity of OPA would be one of the important aspects of global conformal invariance.

Anyway, if we know the exact form of all four point functions, we can calculate C^k_{ij}'s through factorization like eq. (3.2) and check the associativity of OPA.

As explained in section 2, from the local conformal invariance of the theory, we learned that every correlation function must be among the solutions of some differential equations, but we do not know which solution is the correlation function of physical interest.

Likewise, we have enough information about the *allowed* values of c and Δ in conformal field theories, but we do not yet know which primary fields are *necessary* to have consistent conformal field theories.

These two questions are intimately related with each other and can be solved by quite similar methods. We have to obtain monodromy invariant correlation functions for the first problem [5], and modular invariant partitions for the second problem [4].

Let us first consider modular invariance of the partition function. Let $\{\phi_i\}_{i\in I}$ be the set of possible primary fields, and $\{\Delta_i\}_{i\in I}$ be that of possible conformal dimensions. The total Hilbert space can be decomposed with

respect to the two copies (left and right) of Virasoro algebra:

$$\mathcal{H} = \bigoplus_{i,j\in I} \mathcal{N}_{ij} \mathcal{V}_{c,\Delta_i} \bigotimes \overline{\mathcal{V}}_{c,\Delta_j} \tag{3.5}$$

The coefficients $\mathcal{N}_{ij}$ count the number of primary fields with dimension $(\Delta_i, \bar{\Delta}_j)$. By definition, $\mathcal{N}_{ij}$ are non-negative integers. The decomposition eq. (3.5) leads to the following expression of the partition function of the system:

$$Z(\tau) = \sum_{i,j\in I} \mathcal{N}_{ij} \chi_{c,\Delta_i}(\tau) \overline{\chi_{c,\Delta_j}(\tau)} \tag{3.6}$$

The partition function is nothing but the zero point function or free energy of the system on the torus whose moduli parameter is τ. Torus is usually defined by identifying the points on the flat plane which are different by a lattice vector. A change of fundamental region of the lattice leads to a change of parameter τ, which is known as modular transformation. But physical quantities should not depend on the choice of fundamental region. Accordingly, in particular, the partition function $Z(\tau)$ must be invariant under modular transformations. In other words, the substitutions

$$T \, : \, \tau \to \tau + 1 \qquad S \, : \, \tau \to -\frac{1}{\tau}$$

should leave $Z(\tau)$ invariant. In general, the characters reshuffle among themselves under the modular transformations:

$$\begin{aligned} \chi_i(\tau+1) &= \sum_j T_{ij} \chi_j(\tau), \\ \chi_i(-\frac{1}{\tau}) &= \sum_j S_{ij} \chi_j(\tau). \end{aligned} \tag{3.7}$$

From eq. (3.6), it is easy to see that the necessary and sufficient condition for $Z(\tau)$ to be modular invariant is that the coefficients $\mathcal{N}_{ij}$ satisfy the following equations:

$$\begin{cases} S\mathcal{N}S^\dagger = \mathcal{N}, \\ T\mathcal{N}T^\dagger = \mathcal{N}. \end{cases}$$

Thus modular invariance puts stringent constraints on the operator content $\mathcal{N}_{ij}$. In fact, the minimal unitary conformal field theories are completely classified along this line [7,8].

Now let us consider the monodromy invariance of the four point correlation functions. As explained in section 2, every correlation function is a solution to some differential equation. In general, the differential equation has several linearly independent solutions (conformal blocks). Let us denote by $\{I_p(z)\}_{p\in I}$ the set of solutions. Then the physical correlation function must be some sesquilinear combination of conformal blocks:

$$\begin{aligned} U(z) &= \langle \phi_i(0)\phi_j(z)\phi_k(1)\phi_l(\infty)\rangle \\ &= \sum_{p,q\in I} X_{pq} I_p(z)\overline{I_q(z)}, \end{aligned} \tag{3.8}$$

where X_{pq} are some numerical constants. The correlation function should be uniquely determined by the positions of the primary fields, so if we move z along some closed contour and return to the original position, the correlation function should be equal to the original one. But the functions $\{I_p(z)\}_{p\in I}$ have $z = 0, 1$ and ∞ as singular points and each of them are not invariant under such transformation. In general, if the conformal blocks are analytically continued along closed paths surrounding $z = 0$ or $z = 1$, they undergo the following monodromy transformations:

$$\begin{aligned} I_p(z) &\rightarrow \sum_{q\in I} \Omega^{(0)}_{pq} I_j(z) \\ I_p(z) &\rightarrow \sum_{q\in I} \Omega^{(1)}_{pq} I_j(z) \end{aligned} \tag{3.9}$$

So the necessary and sufficient condition for $U(z)$ to be modular invariant is

$$\begin{cases} \Omega^{(0)} X \Omega^{(0)\dagger} = X \\ \Omega^{(1)} X \Omega^{(1)\dagger} = X \end{cases} \tag{3.10}$$

The monodromy matrices $\Omega^{(0)}$ and $\Omega^{(1)}$ can be calculated by inspecting the differential equation for the correlation function. Solving eq. (3.10), one can completely determine the form of physical correlation function.

Once the monodromy invariant correlation function is obtained, we can compute the OPE coefficients C^k_{ij} by factorizing the correlation function and separating the pole residues (cf (3.2),(3.3)).

In this way, we have accomplished our original program of constructing "complete" field theory in the form of minimal unitary conformal field theories.

4 Concluding remarks

In this review, we have discussed the conformal fields theories from the viewpoint of bootstrap program, and we have been principally concerned with how correlation functions are calculated. In the course of investigating those theories, local and global conformal symmetry has played crucial role in every stages.

There still remain a variety of important aspects of conformal field theories, which we have not been able to discuss in detail. In order to apply conformal field theories to string theories, it is inevitable to study conformal field theory on general Riemann surfaces. In that case, the deformations of the complex structure of Riemann surfaces also come to play their role [10]. If we want to understand compactification of superstrings in terms of conformal field theories, we have to explore the $c > 1$ region, which is not minimal in the usual sense. Hence Virasoro algebra must be extended to some larger algebra, W algebra [9] for example, if we want to apply the same technique which has been developed in the minimal theories. Also, to study deformation of conformal field theories themselves is a very interesting problem. Such insight would shed light on our understanding of the configuration space of superstring vacua and the dynamics of string compactification. Much remains to be done in this area.

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Representation Theory of Current Algebra and Conformal Field Theory on Riemann Surfaces

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ABSTRACT

We study conformal field theories with current algebra (WZW-model) on general Riemann surfaces based on the integrable representation theory of current algebra. The space of chiral conformal blocks defined as solutions of current and conformal Ward identities is shown to be finite dimensional and satisfies the factorization properties.

1. Introduction

First let us recall general aspects of operator formulation of CFT's on Riemann surfaces. (We treat only the chiral half of the CFT.) Let $\mathcal{H}$ be a representation space of some chiral algebra $\mathcal{A}$ (e.g. Virasoro, current, W-algebras), and let $X = (R, Q_1, \cdots, Q_N, z_1, \cdots, z_N)$ be a N−punctured Riemann surface of genus g with local coordinates $z_i; z_i(Q_i) = 0$. For each set of N external states

$$|\phi_1> \otimes \cdots \otimes |\phi_N> \in \mathcal{H} \otimes \cdots \otimes \mathcal{H} = \mathcal{H}^{\otimes N}, \tag{1}$$

the correlation function on the surface R is given by

$$<\phi_1 \cdots \phi_N> = <\Psi||\phi_1> \otimes \cdots \otimes |\phi_N>, \tag{2}$$

where the linear functional $<\Psi| : \mathcal{H}^{\otimes N} \to \mathbf{C}$ is the vacuum state of type (g, N) depending on the geometrical data X.

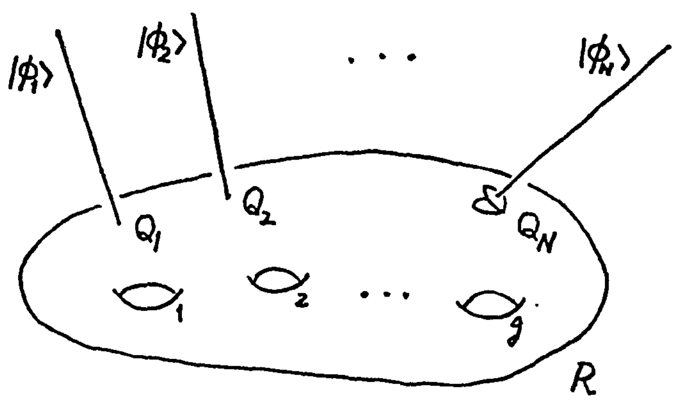


Fig.1 correlation function on Riemann surface

The conformal Ward identity describes the moduli dependence of the vacuum $<\Psi|$ in terms of energy-momentum tensor insertion[5]. The problem we discuss here is how can we characterize the vacuum state $<\Psi|$ in terms of representation of current algebra.

2. Representation Theory of Current Algebra.

We take the current ($\times$ Virasoro) algebra $\hat{\mathrm{g}}$ associated with some simple Lie algebra g as the chiral algebra $\mathcal{A}$. In terms of operator products expansion[1], the algebra takes the following form[2],

$$
\begin{aligned}
J^a(z)J^b(w) &= \frac{l\delta^{ab}}{(z-w)^2} + \frac{f^{abc}}{(z-w)}J^c(w) + \cdots, \\
T(z)J^a(w) &= \left\{\frac{1}{(z-w)^2} + \frac{1}{(z-w)}\frac{d}{dw}\right\}J^a(w) + \cdots, \\
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \left\{\frac{2}{(z-w)^2} + \frac{1}{(z-w)}\frac{d}{dw}\right\}T(w) + \cdots,
\end{aligned} \tag{3}
$$

where $T(z)$ is the energy-momentum tensor of the Sugawara form;

$$
T(z) = \frac{1}{2(g^* + l)} \lim_{w\to z} \left\{ \sum_{a=1}^{\mathrm{dimg}} J^a(z)J^a(w) - \frac{l\mathrm{dimg}}{(z-w)^2} \right\}, \tag{4}
$$

where $J^a, (a = 1, \cdots, \mathrm{dimg})$ are orthonormal basis of g, we also use the Cartan-Killing basis $H^i, E^\alpha, (i = 1, \cdots, \mathrm{rankg}, \quad \alpha \in$ roots of g). The Virasoro central charge c and the current central charge l (level) is related by

$$
c = \frac{l\mathrm{dimg}}{g^* + l}, \tag{5}
$$

where g^* is the dual Coxeter number of g (e.g. $g^* = n$ for $\mathrm{g} = \mathrm{su}(n)$).

The highest weight representation of $\hat{\mathrm{g}}$ is generated by the primary field $\phi(z)$, which takes values in representation V_λ of g with highest weight λ, satisfies the following highest weight condition;

$$
\begin{aligned}
J^a(z)\phi(w) &= \frac{\tau^a}{(z-w)}\phi(w) + \cdots, \\
T(z)\phi(w) &= \left\{\frac{\Delta}{(z-w)^2} + \frac{1}{(z-w)}\frac{d}{dw}\right\}\phi(w) + \cdots,
\end{aligned} \tag{6}
$$

where τ^a is the representation of J^a on V_λ. The conformal weight Δ of $\phi(z)$ is

given by

$$\Delta = \frac{(\lambda, \lambda + 2\rho)}{2(g^* + l)} \tag{7}$$

where ρ is the half sum of positive roots of g and (,) is the Cartan-Killing form normalized by $(\theta, \theta) = 2$ for highest root θ.

The integrable highest weight representations of $\hat{\mathbf{g}}$ (which correspond to the $c < 1$ minimal discrete series of Virasoro algebra) exist if the level l is non-negative integer, and they are characterized by the following null field constraint[12],

$$(E_{-1}^{\theta})^{l-(\lambda,\theta)+1}|\lambda> = 0. \tag{8}$$

Such a representation exists for only finite number of λ's such that $0 \leq (\lambda, \theta) \leq l$. The importance of this integrability condition in CFT was first noted by [8] and [13], and which also plays a crucial role in the followings.

3. Gauge Condition and Correlation Functions

Let us consider the correlation functions of N primary fields $\phi_i \in V_i$ with arbitrary numbers of current insertions,

$$< J^{a_1}(P_1) \cdots J^{a_M}(P_M) \phi_1(Q_1) \cdots \phi_N(Q_N) > . \tag{9}$$

In operator formulation, it is equivalent to consider the linear functional

$$< \Psi | : \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_N} \to \mathbf{C}, \tag{10}$$

because any correlations of descendants are given from (9) by contour integrations of currents.

For each variable P_i, we demand that the correlation function behaves as (single valued) meromorphic 1−form with poles at most only at $P_i = P_j (j \neq i)$ or $P_i = Q_k (k = 1, \cdots, N)$ as indicated by the OPE(3)(6). In operator languages, it is equivalent to the following condition;

$$\sum_{k=1}^{N} < \Psi|\mathrm{Res}_{Q_k}\Big(f(z)J^a(z)dz\Big) = 0, \tag{11}$$

for any meromorphic function with poles at most at Q_k's. This is nothing but the special case of the Ward identity for current insertion, and it can be interpreted as the residue theorem for operator valued 1-forms[9]. We call this condition as the gauge condition.

Note that the solution of the gauge condition is not unique, and ther may be infinite solutions. For example, in abelian current theory[10][11], the charge operators

$$J[C] = \int_C dz J(z), \qquad C \in H_1(R, Z), \tag{12}$$

preserve the gauge condition. These charges satisfy the following canonical commutation relations;

$$\Big[J[C], J[C']\Big] = 2\pi i < C, C' >, \tag{13}$$

with symplectic form of intersections $< C, C' >$. Thus the solution space of gauge condition is infinite dimensional.

In non-abelian case, we have an additional constraint from integrability(8), and we can prove that the solution space is of finite dimension. Here we give the sketch of the proof.

First let us discuss the case of $g = 0$. For given point Q_i and positive integer n, we have a meromorphic function f which has a pole of order n only at Q_i.

With this f, we get

$$\begin{aligned}
&\mathrm{Res}_{Q_i}\Big(f(z)J^a(z)dz\Big) = [J^a_{-n}]_i,\\
&\mathrm{Res}_{Q_k}\Big(f(z)J^a(z)dz\Big) = [\text{annihilators}]_k, (k \neq i).
\end{aligned} \tag{14}$$

Then, from the gauge condition, any creation operators are expressed as linear combinations of annihilation operators. Using this relation recursively the correlation of the descendants reduces to that of primaries, so the following linear function ψ (the initial term of $<\Psi|$)

$$\psi : V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N} \to \mathbf{C}, \tag{15}$$

determines $<\Psi|$, and the solution space is of finite dimension. Moreover from the gauge condition for the constant function f =const., the initial term must to be g-invariant, that is

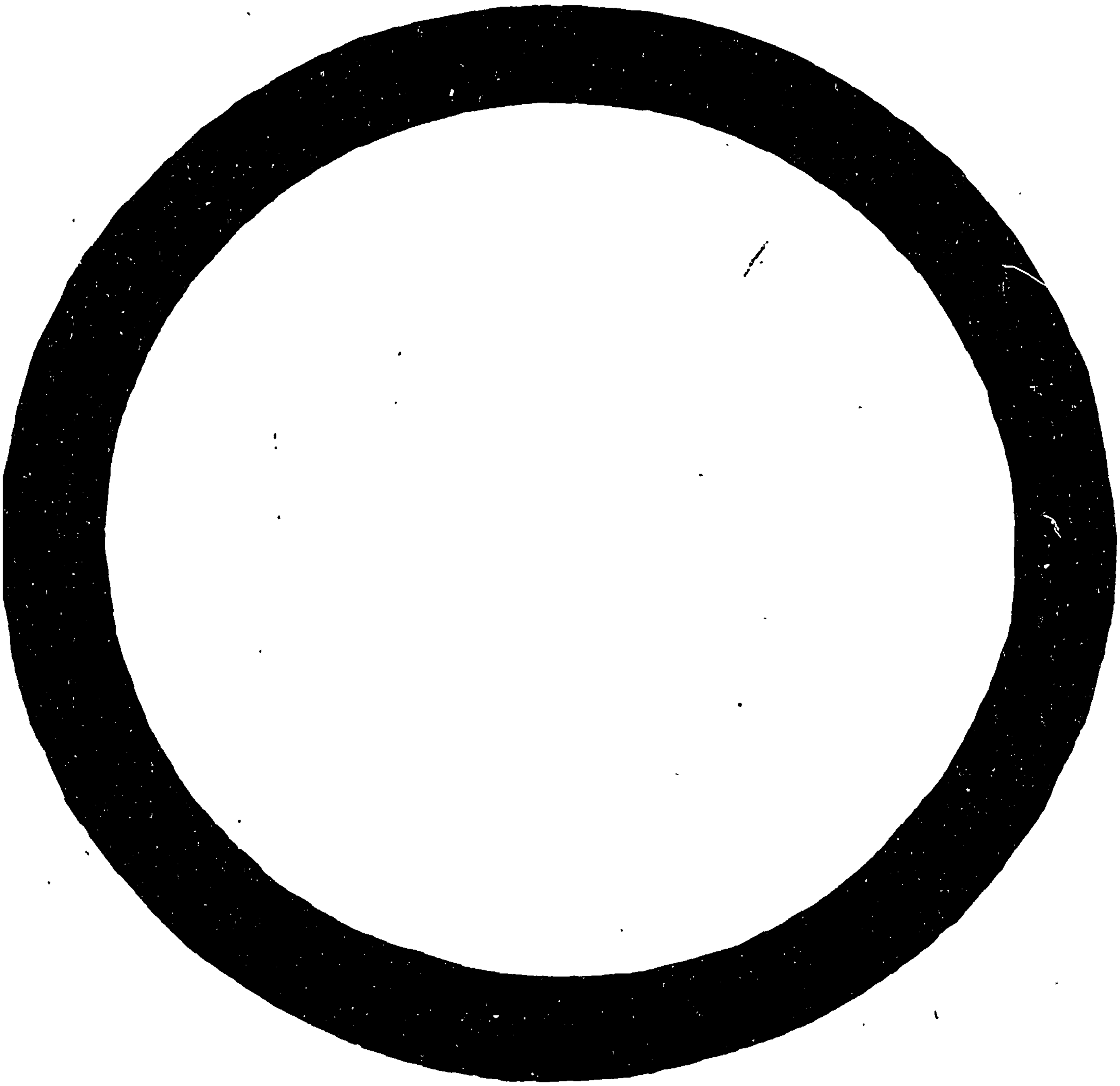
$$\sum_{k=1}^{N} \psi(\phi_1, \cdots, \tau^a \phi_k, \cdots, \phi_N) = 0. \tag{16}$$

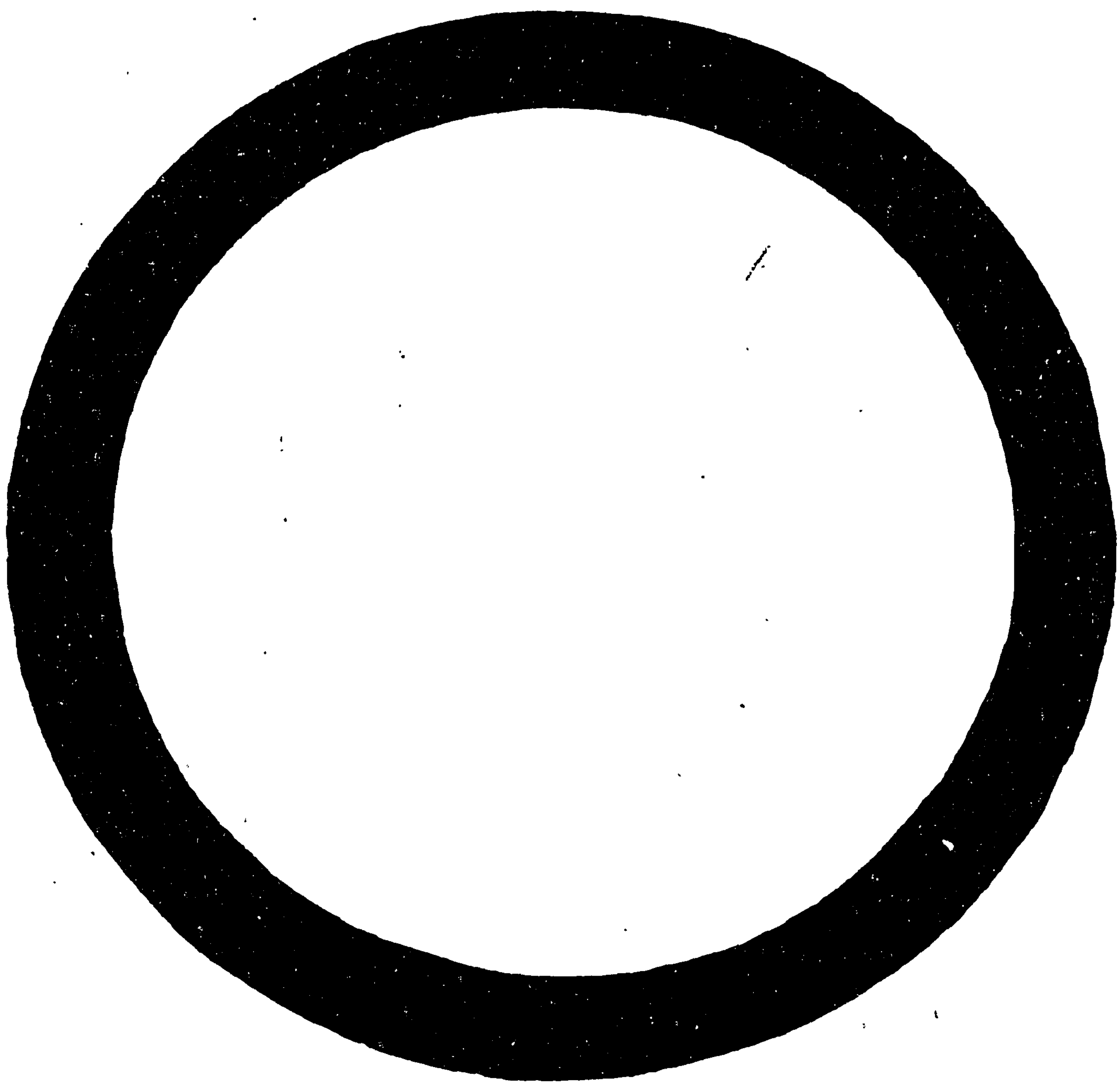
This condition gives us the usual Clebcsh-Gordan condition for $N = 3$ case. But, from the integrability condition(8), the real selection rule is more restricted than C-G condition. For example, for g = su(2), level = l case, the selection rule is expressed by the following fusion rule[8][13];

$$\phi_i \phi_j = \sum_{k=|i-j|}^{\min\{i+j, l-i-j\}} \phi_k, \tag{17}$$

where i, j and k run over the integrable spins $\{0, \frac{1}{2}, 1, \cdots, \frac{l}{2}\}$.

In $g \geq 1$ cases, we have finite numbers ($= g$) of gaps for meromorphic functions, and the creation operators corresponding to these gaps can not be expressed





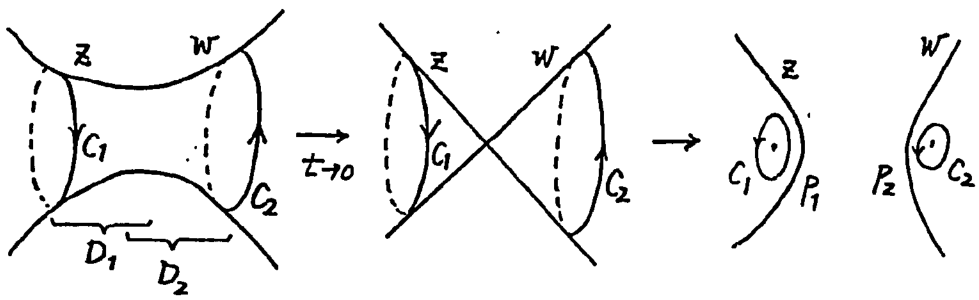


Fig.3 double point singularity and its normalization

Let us consider a double point singularity and its 1-parameter resolution $zw = t$.(Fig.3)

For any integer $n \in \mathbf{Z}$ define a holomorphic function f on $D = D_1 \cap D_2$ as follows;

$$\begin{aligned} f_1(z) &= z^n, \quad \text{on} \quad D_1, \\ f_2(w) &= (\frac{t}{w})^n, \quad \text{on} \quad D_2. \end{aligned} \tag{20}$$

From Cauchy's theorem we get;

$$\oint_{C_1} \frac{dz}{2\pi i} f_1(z) J^a(z) + \oint_{C_2} \frac{dw}{2\pi i} f_2(w) J^a(w) = 0. \tag{21}$$

In $t \to 0$ limit, cycles C_i shrink to small cycles around normalized points P_i and we get,

$$\begin{aligned} [J_n^a]_1 &= [J_n^a]_2 = 0, \\ [J_0^a]_1 &+ [J_0^a]_2 = 0. \end{aligned} \tag{22}$$

This means that the highest weight states ϕ_1, ϕ_2 with conjugate representation appear at the double points. The above fact shows that the currents $J^a(z)dz$ may have poles of order 1 at double points, where the sum of residues vanish. Thus the currents $J^a(z)dz$ are sections of the dualizing sheaf [7]. Furthermore, using the decoupling theorem of the non-integrable fields on sphere[8], we can conclude that *integrable primary fields are pair created at each double point.*

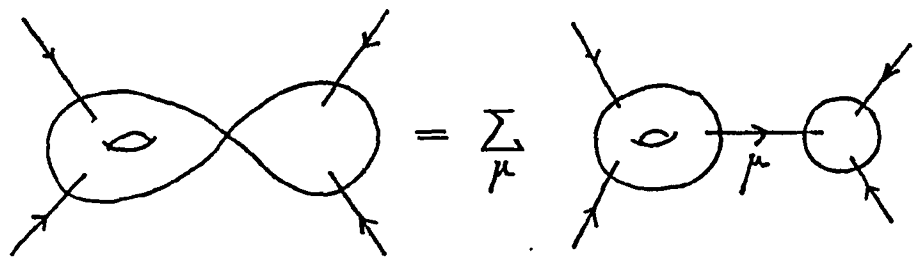


Fig.4 factorization property

From this factorization properties, we can take a canonical basis of conformal blocks in the limit of most degenerated surfaces. The most degenerated surfaces of type (g, N) consist of $2g-2+N$ vertices (sphere with 3-punctures) connected by $3g-3+N$ edges (double point nodes), and they correspond to some ϕ^3-diagrams as follows,

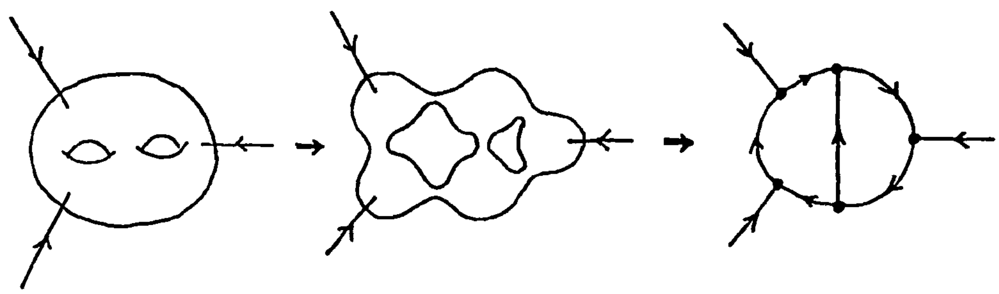


Fig.5 degenerate surface and corresponding ϕ^3–diagram

Bases corresponding to this diagram are labeled first assigning intermediate state to each edge and then assigning admissible 3-point function to each vertex. Then the dimension of conformal blocks are given by

$$\text{dim.} = \sum_{\mu:\text{edges}} \prod_{(\lambda,\mu,\nu):\text{vertices}} N_{\lambda,\mu,\nu}, \tag{23}$$

where $N_{\lambda,\mu,\nu}$ counts the number of independent 3-point functions at the vertex (λ, μ, ν).

Correlation functions corresponding to these bases can be constructed by the sewing procedure[19]. They are formal solution of the Ward identity (as formal power series in the blowing up parameters), but as the solution of differential equation of regular holonomic type, they converge.

5. Global Properties

First let us study rather trivial monodromy operations corresponding to the twisting around the vanishing cycles. In this case the ϕ^3-diagram is not changed and the monodromy matrices are simultaneously diagonalized. Fore example the monodromy for the twist in Fig.6 is diagonal matrix with diagonal phase $\exp\{2\pi i(\Delta_I - \sum \Delta_k)\}$, where Δ's are conformal weights of intermediate (I) and $n-$external ($k = 1, \cdots, n$) states.

Second monodromy matrices connect two basis corresponding to different ϕ^3-diagrams. They are generated by the following simple operations; 1) blowing up of one node, and 2) pinching another cycle.(Fig.7)

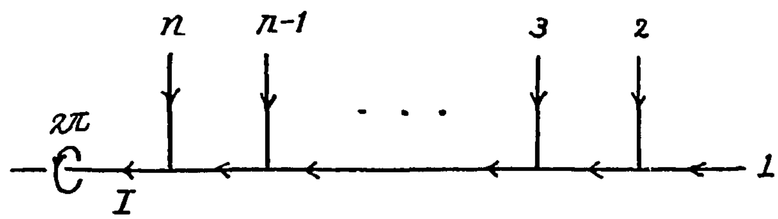


Fig.6 twisting around a vanishing cycle

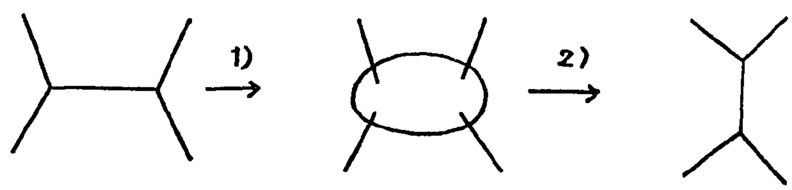


Fig.7 fundamental base change

Corresponding monodromy matrices satisfy some consistency condition of topological nature. Typical examples of these conditions are the fusion and the braid relation depicted as follows:

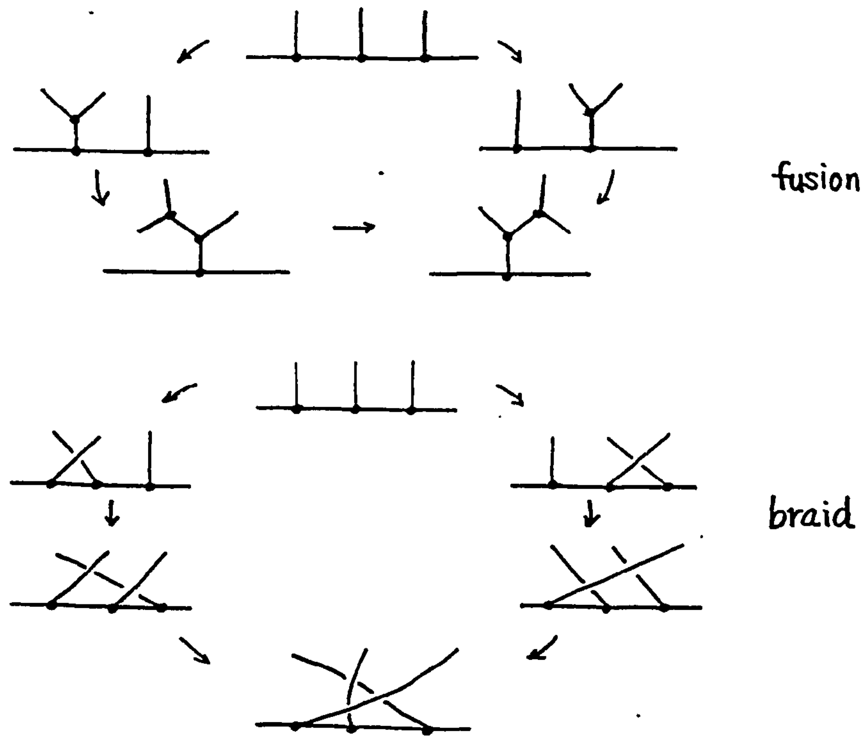


Fig.8 the pentagon (fusion) and the hexagon (braid) identities

Recently Moore and Seiberg have written down the generators and fundamental relations of these monodromy matrices [18]. The most fundamental relation is the fusion relation (or the Pentagon identity), which is used to prove the remarkable formulae conjectured by Verlinde[15][18][17]. These formulations will give us the starting point for the extensive study of the modular geometry and classification of CFT (or its good sub-category such as rational CFT)[16].

In [21], Witten proposed 3-dimensional Chern-Simons YM theory as exact solvable QFT whose correlations are the knot invariants. The essential point is that its Hilbert space in canonical quantization is nothing but the space of

conformal blocks described here. It will be interesting to see this relation more precisely.

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Application of Superconformal Symmetry to String Compactification

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Abstract

We discuss string compactifications on manifolds with $SU(n)$ holonomy by making use of representation theories of extended superconformal algebras. In particular string compactification on K_3 surfaces is discussed in detail. We calculate loop space indices and show that all $c = 6$ superconformal field theories describe string propagation on manifolds with $SU(2)$ holonomy. We also discuss $c = 9$ superconformal field theories and their relation to Calabi-Yau manifolds.

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1 Introduction

The study of low-energy supergravity theory has shown that the compactifying spaces of the string theory must be Ricci flat and Kähler manifolds [1,2]. In fact, n-dimensional complex manifold with $SU(n)$ holonomy possesses a unique covariantly constant spinor field which generates $N = 1$ space-time supersymmetry transformation. These desirable properties of string compactifications may be formulated as a constraints on superconformal field theories on the two-dimensional world-sheet. First important observation is that the $N = 2$ extended superconformal symmetry on the world-sheet is necessary and sufficient in order to realize $N = 1$ space-time supersymmetry [3,4,5,6,7,8,9]. In this context the covariantly constant spinors correspond to the Ramond ground state. The everywhere non-zero (anti-)holomorphic tensors, which are characteristic to the manifolds with $SU(N)$ holonomy, generate the chiral currents. These currents combined with the $U(1)$ current in the $N = 2$ superconformal algebra enhance the internal Kac-Moody symmetry.

The $N = 2$ superconformal algebra has the inner automorphism due to the $U(1)$ Kac-Moody subalgebra [9,10]. Consequently Neveu-Schwarz (NS) and Ramond (R) sectors are isomorphic and mapped onto each other by the spectral flow which is equivalent to the space-time supersymmetry transformation [9,10]. In order to have a well-defined space-time supersymmetric theory, the world-sheet superconformal field theory should be local. Traditionally this has been achieved by the celebrated GSO projection. In the formulation based on the $N = 2$ superconformal theory, this projection is generalized to the $U(1)$ charge integrality condition. This is seen by noticing that the spin fields of the R ground state play the role of the mapping operator, and hence enter into the construction of the space-time supercurrent. Remember again that geometrically this R ground states represent the covariantly constant spinor fields.

Recently Gepner [6,7] has constructed a new class of compactification models which are based on the tensoring of $N = 2$ minimal discrete theories, where he invented a nice way of imposing the both conditions of charge integrality and modular invariance. These models are not manifestly geometrical, however, he has reproduced some known topological numbers of $c_1 = 0$ (vanishing first Chern class) manifolds by calculating the massless spectra of his models.

In this report we make an extensive study of string compactification on $c_1 = 0$ manifolds by using the representation theory of $N = 2$ and $N = 4$ superconformal algebras. In particular we will concentrate on the case of the K_3 surface where we can use the results of $N = 4$ representation theory which has recently been worked out [13,14]. We shall calculate the loop space index [15,16,17] and show that all $c = 6$ superconformal field theories describe string propagation on K_3 manifolds.

In section 2 we make use of Gepner's models and describe how to construct modular invariant partition functions of non-linear σ-model on K_3 surface. We calculate loop space indices in section 3 and show how the theory reproduces known topological invariants of K_3 manifolds. In section 4 we discuss heterotic string compactification and its massless particle spectra. In section 5 we discuss $c = 9$ superconformal field theories and discuss their relation to Calabi-Yau manifolds.

For the details of the present report, see the original article [27]. Other approach to construct $N = 2$ superconformal field theories is found in ref. [28].

2 Non-linear σ-model on K_3 surface

Let us discuss string propagation on $c_1 = 0$ manifolds using the representation theory of $N = 2$ and $N = 4$ algebras. We will concentrate on the case of the K_3 surface. In this section we motivate our discussions making use of Gepner's models based on the tensoring of $N = 2$ minimal theories. Results described below, however, do not depend on the details of the $N = 2$ models but holds for generic K_3 surfaces. Specific examples of the tensoring of $N = 2$ minimal series will be discussed in detail in section 5.

In Gepner's method, one considers a tensor product $k_1^{m_1} k_2^{m_2} \cdots k_l^{m_l}$ ($k_i, m_i \in$ N) of $N = 2$ discrete series with levels $k_1, ..., k_l$ in such a way that the central charge adds up to $3n$ (n is the complex dimensionality of the $c_1 = 0$ manifold)

$$c = \sum m_i \frac{3k_i}{k_i + 2} = 3n \ . \tag{2.1}$$

$n = 2$ for K_3 surface and 3 for Calabi-Yau manifolds. In ref.[7] sixteen possibilities of (2.1) with $c = 6$ are listed which describes string propagation

on K_3 surfaces with a variety of complex structures. For a catalogue of Calabi-Yau cases, $c = 9$, see ref.[18].

A special feature of the $N = 2$ algebra is the isomorphism of the algebra under a continuous shift of the moding of the supercharge operators [10]. One can check that tha algebra remains invariant under the transformation,

$$\begin{aligned} L_n &\rightarrow L_n + \eta J_n + \frac{c}{6}\,\eta^2\,\delta_{n,0}\,, \\ J_n &\rightarrow J_n + \frac{c}{3}\,\eta\,\delta_{n,0}\,, \\ G_r &\rightarrow G_{r+\eta}\,, \\ \bar{G}_r &\rightarrow \bar{G}_{r-\eta}\,. \end{aligned} \tag{2.2}$$

Here L_n, J_n, G_r and $\bar{G}_r$ are the Virasoro, $U(1)$ current and supercharge generators, respectively, and η is an arbitrary real parameter. Thus, in particular, the R and NS sectors are isomorphic to each other ($\eta = 1/2$ and $r \in \mathbf{Z}$ or $\mathbf{Z} + 1/2$). The shift $\eta \rightarrow \eta \pm 1/2$ corresponds to the space-time supersymmetry transformation [9]. In fact, when $c = 3n$, the ground state $h = Q = 0$ (h and Q are the eigenvalues of L_0 and J_0) of the NS sector is mapped onto the states with $h = c/24 = n/8, Q = \pm c/6 = \pm n/2$ in the R sector which correspond to the covariantly constant spinor fields on the Calabi-Yau or K_3 manifold.

On the other hand, under a shift $\eta \rightarrow \eta \pm 1$, the theory comes back to its original sector. The highest weight states of the algebra are, however, transformed onto different highest weight states. The ground state of the NS sector is now mapped onto the states with $h = c/6 = n/2$, $Q = \pm c/3 = \pm n$ in the NS sector which correspond to the holomorphic or anti-holomorphic n-form of the Calabi-Yau or K_3 manifold. The generators of the transformation $\Delta\eta = \pm 1$ are conformal fields with $h = n/2$ and $Q = \pm n$, and in the case of $n = 2$, i.e. K_3 surface, they are nothing but the $SU(2)$ currents $J^{\pm}$ (note the factor 2 difference in the $U(1)$ charge and the 3rd component of the isospin $J_0 = 2J_0^3$). When the $N = 2$ algebra is extended by the addition of the flow generators $J^{\pm}$, one obtains the $N = 4$ superconformal algebra. Thus the string compactifications on the K_3 surface are described by the representation theory of $N = 4$ algebra. On the other hand, when $n = 3$, i.e. the case of Calabi-Yau manifolds, the flow generators are fermionic ($h = 3/2$ and $Q = \pm 3$) and their addition to $N = 2$ gives a new algebra which will be discussed in section 5.

The Witten index $Tr(-1)^F$ is defined in the R sector with the identification $F = Q + \bar{Q}$, where Q and $\bar{Q}$ are holomorphic and anti holomorphic $U(1)$ charges, respectively. This gives us the Euler characteristic of the σ-model target manifold.

Partition functions of Gepner's models are expressed in terms of the character functions of $N = 2$ algebra [24,6]. The $N = 2$ characters are defined by $tr\, q^{L_0 - c/24} e^{i\theta J_0}$ and the angle θ keeps track of the $U(1)$ charge of the representation contents. The isomorphism of the $N = 2$ algebra (2.2) manifests itself in the quasi-periodicity of θ in the character formulas. In fact the shift $\eta \longrightarrow \eta + 1/2$ corresponds to $\theta \longrightarrow \theta + \pi\tau$ and we have

$$ch^{NS,k}_{l,m}(\tau; \theta + \pi\tau) = q^{-c/24} e^{-ic\theta/6} ch^{R,k}_{l,m}(\tau; \theta) \ , \tag{2.3}$$

where l, m label the representations of the level-k minimal theories ($0 \leq l \leq k$, $-l \leq m \leq l$, $l - m \equiv 0 \pmod 2$). Under a "full" shift $\eta \longrightarrow \eta + 1$ or $\theta \longrightarrow \theta + 2\pi\tau$

$$ch^{k}_{l,m}(\tau; \theta + 2\pi\tau) = q^{-c/6} e^{-ic\theta/3} ch^{k}_{l,m-2}(\tau; \theta) \ , \tag{2.4}$$

which holds in both NS and R sectors.

Let us now concentrate on the case of K_3 surface and describe a method of constructing of modular invariant partition functions. For the sake of illustration we consider the 1^6 model defined by taking 6 copies of $k = 1$ minimal theories. There exist three representations $l = m = 0$, $l = m = 1$, $l = -m = 1$ in the $k = 1$ theory and we denote their characters (in the NS sector) as $A(h = Q = 0), B(h = 1/6, Q = 1/3)$ and $C(h = 1/6, Q = -1/3)$, respectively. Under the specral flow (2.4) A, B and C are cyclically permuted among each other.

Now we introduce a flow-invariant combination

$$NS_1 = A^6 + B^6 + C^6, \tag{2.5}$$

which we call as the "graviton" orbit. (2.5) contains the identity operator ($h = Q = 0$) which generates the graviton multiplet in the heterotic string compactification.

$SU(2)$ symmetry acts on the flow-invariant orbit (2.5) and hence the $N = 2$ symmetry is enhanced to $N = 4$. Therefore NS_1 can be decomposed into the representations of $N = 4$ algebra.

Highest weight states of the $N = 4$ algebra are parametrized by the conformal dimension h and isospin l. Unitarity puts a restriction $h \geq l$ in the NS sector and $h \geq 1/4$ in the R sector (when $c = 6$). There exist two distinct classes of representations of $N = 4$ algebra [13]; massless and massive representations. The massless representations exist at the unitarity bound

$$\begin{cases} h = l = 0, \\ h = l = 1/2, \quad \text{in NS sector,} \end{cases} \tag{2.6}$$

$$\begin{cases} h = 1/4, \ l = 0, \\ h = 1/4, \ l = 1/2, \quad \text{in R sector,} \end{cases} \tag{2.7}$$

Ground states of the R sector carry non-zero Witten index in these representations and they possess unbroken $N = 4$ world-sheet supersymmetry. The massless representations keep track of the non-trivial topology of the K_3 surface. On the other hand the massive representations exist in the range

$$\begin{aligned} &h > 0, \qquad l = 0, \qquad \text{in NS sector} \\ &h > 1/4, \quad l = 1/2, \quad \text{in R sector} \end{aligned} \tag{2.8}$$

and have ground states with the equal number of bosons and fermions, and thus have vanishing Witten index. They describe the degrees of freedom of deformation of the K_3 surface. Under the spectral flow, a NS representation with isospin l is mapped onto a R representation with isospin $\frac{1}{2} - l$.

The graviton orbit (2.5) contains the $l = 0$ massless character and an infinite sum of massive characters

$$\begin{aligned} NS_1(\tau; z) &= ch_0^{NS}(l = 0; \tau; z) + \sum_{n=1}^{\infty} f_n^{(1)} ch^{NS}(h = n; \tau; z) \\ &= ch_0^{NS}(l = 0; \tau; z) + F_1(\tau) ch^{NS}(h = 0; \tau; z) , \end{aligned} \tag{2.9}$$

$$F_1(\tau) = \sum_{n=1}^{\infty} f_n^{(1)} q^n. \tag{2.10}$$

(For the explicit form of $N = 4$ characters, see refs)

Under the modular transformation $S : \tau \rightarrow -1/\tau$, NS_1 transforms into a family of new orbits,

$$NS_2 = A^3 B^3 + B^3 C^3 + C^3 A^3,$$

$$
\begin{aligned}
NS_3 &= A^2B^2C^2, \\
NS_4 &= A^4BC + B^4CA + C^4AB.
\end{aligned} \tag{2.11}
$$

The matrix S_{ij} of the S-transformation,

$$
NS_i(\tau;\theta) = \sum S_{ij}\ NS_j(-\frac{1}{\tau};\frac{\theta}{\tau})\ e^{-i\frac{\theta^2}{2\pi\tau}} \tag{2.12}
$$

can be computed from the S-transformation of the $N = 2$ sub-theories (see Appendix A). In the case of 1^6 theory S_{ij} is given by

$$
S_{ij} = \frac{1}{27}\begin{pmatrix} 3 & 60 & 270 & 90 \\ 3 & -21 & 27 & 9 \\ 1 & 2 & 9 & -6 \\ 3 & 6 & -54 & 9 \end{pmatrix}. \tag{2.13}
$$

There are in general three types of NS orbits in K_3 compactification. They all possess integral values of the $U(1)$ charge.

(1) graviton orbit:
NS_1 is the only trajectory containing the ground state $h = Q = 0$.

(2) massless matter orbits:
NS_i $(i = 2, ..., d)$ contain states $h = 1/2,\ Q = \pm 1$ and are rewritten as

$$
\begin{aligned}
NS_i(\tau;z) &= ch_0^{NS}(l = 1/2;\tau;z) + F_i(\tau)ch^{NS}(h = 0;\tau;z), \\
F_i(\tau) &= \sum_{n=1}^{\infty} f_n^{(i)} q^n
\end{aligned} \tag{2.14}
$$

(In some cases, states $h = 1/2,\ Q = \pm 1$ appear more than once in one orbit. Then $ch_0^{NS}(l = 1/2)$ in (2.14) must be multiplied by the multiplicity. We ignore this complication in the following).

(3) massive orbits:
NS_j $(j = d+1, \ldots, d+d')$ contain massive characters only

$$
\begin{aligned}
NS_j(\tau;z) &= F_j(\tau)ch^{NS}(h = 0;\tau;z), \\
F_j(\tau) &= q^{r_j}\sum_{n=0}^{\infty} f_n^{(j)} q^n, \quad 0 < r_j \in \mathbf{Q}.
\end{aligned} \tag{2.15}
$$

In (2.10), (2.14), (2.15) the expansion coefficients $f_n^{(m)}$ are non-negative integers. Number of orbits, d and d', and the functions F_m depend on the tensoring of sub-theories.

In the case of 1^6, $d=2$, $d'=2$ and

$$\begin{aligned} F_1(\tau) &= 5q+29q^2+80q^3+\cdots, \\ F_2(\tau) &= 5q+26q^2+85q^3+\cdots, \\ F_3(\tau) &= q^{2/3}(1+5q+20q^2+59q^3+\cdots), \\ F_4(\tau) &= q^{1/3}(1+16q+38q^2+127q^3+\cdots). \end{aligned} \tag{2.16}$$

These $d+d'$ trajectories enter into modular invariant partition functions.

Using the symmetry property of the S-matrix of sub-theories, it is easy to show that the S-matrix of orbits (2.12) is symmetrizable by a diagonal matrix D with integral elements D_i

$$D_i S_{ij} = D_j S_{ji} \quad (\text{no sum on } i,j) \tag{2.17}$$

with

$$D_i = \frac{S_{1i}}{S_{i1}}, \quad i=1,\ldots,d+d' \tag{2.18}$$

(D is normalized as $D_1 = 1$). In the case of 1^6 theory $D_i = (1,20,270,30)$. D_i's are essentially the combinatorial factors in the tensoring of representations

$$D_i = \frac{(\text{combinatorial factor of orbit } i\ ) \times (\text{standard length of orbits})}{(\text{length of orbit } i)}. \tag{2.19}$$

The standard length of orbits of $k_1^{m_1}\cdots k_l^{m_l}$ is the least common multiple of $k_1+2,\ldots,k_l+2$.

The matrix D is the key ingredient in the construction of modular invariant partition functions. Indeed it is easy to check that, using (2.17) and $S^2=1$,

$$\sum_{i=1}^{d+d'} D_i (NS_i)^*(NS_i) \tag{2.20}$$

is S-invariant. The sum of D_i for massless matter orbits always adds up to 20,

$$\sum_{i=2}^{d} D_i = 20 \tag{2.21}$$

in K_3 compactification. This is the Hodge number $h^{1,1}$ and it gives the multiplicity of massless spinors in the $\underline{56}$ of E_7 in heterotic string compactification. We will derive (2.21) in the next section.

The structure of the trajectories in the other sectors is determined by the spectral flow. By shifting θ by $\pi\tau$ and $\pi\tau+\pi$ in (2.12), we find

$$R_i(\tau;\theta) = \sum S_{ij}\ \widetilde{NS}_j(-\frac{1}{\tau};\frac{\theta}{\tau})\ e^{-i\theta^2/2\pi\tau}, \tag{2.22}$$

$$\tilde{R}_i(\tau;\theta) = -\sum S_{ij}\ \tilde{R}_j(-\frac{1}{\tau};\frac{\theta}{\tau})\ e^{-i\theta^2/2\pi\tau}, \tag{2.23}$$

where $\widetilde{NS}$ and $\tilde{R}$ are NS and R sectors with $(-1)^F$ insertion and $\tilde{R}_i(\theta)$ gives the Witten index I_i at $\theta=0$. Since $I_1=-2,\ I_i=1\ (i=2,\ldots,d), I_j=0\ (j=d+1,\ldots,d+d')$, S-matrix has an eigenvector $(-2,1,\ldots,1,0,\ldots,0)$ with eigenvalue -1

$$\sum S_{ij}I_j=-I_i. \tag{2.24}$$

The modular invariant partition function of the non-linear σ- model on K_3 surface is then given by (in the case of A-type invariant)

$$Z_\sigma=\frac{1}{2}\sum_{i=1}^{d+d'} D_i\left\{|NS_i|^2+|\widetilde{NS}_i|^2+|R_i|^2+|\tilde{R}_i|^2\right\}. \tag{2.25}$$

Euler number is equal to

$$\chi=\sum_{i=1}^{d+d'} D_iI_i^2=4+\sum_{i=2}^{d} D_i=24. \tag{2.26}$$

3 Loop space index

Functions $F_i(\tau)$ $(i=1,\ldots,d+d')$ depend on the tensoring of $N=2$ sub-theories and thus are dependent on the complex structure of the K_3 surface. In order to characterize general aspects of K_3 compactifications, it is convenient to introduce topological invariants which are independent of the complex structure or the moduli of the K_3 surface. In this section we consider the loop space indices [15,16,17] (or elliptic genera) which are string theoretic generalizations of classical topological invariants.

We introduce

$$\Phi(\hat{A}) \equiv tr_{NS\times R}\, q^{L_0-1/4}(-1)^{F_L}\bar{q}^{L_0-1/4}\ , \tag{3.1}$$

where the trace is taken by imposing the NS and R (with $(-1)^F$ insertion) boundary conditions in the right- and left-moving sector of the theory, respectively. $\Phi(\hat{A})$ is the elliptic genus generalizing the Dirac index $\hat{A}$ [16].

(3.1) may be easily evaluated making use of the non-linear σ- model. We can explicitly compute (3.1) by using any of the tensorings of $N = 2$ models and find that they all give the same result. Thus $\Phi(\hat{A})$ is in fact common to all K_3 compactifications. Calculation is easy in the 2^4-theory and we obtain

$$\begin{aligned}
\Phi(\hat{A}) &= \sum_i D_i (NS_i)(\widetilde{R}_i), \\
&= 2\frac{(\vartheta_2^4 - \vartheta_4^4)}{\eta^4}\left(\frac{\vartheta_3}{\eta}\right)^2, \\
&= -q^{-1/4}(2 - 40q^{1/2} - 124q + \cdots),
\end{aligned} \tag{3.2}$$

(3.2) may also be derived directly from (3.1); we note that the boundary condition of $\Phi(\hat{A})$ is invariant under the transformations S and T^2 ($T : \tau \to \tau + 1$) and thus $\Phi(\hat{A})$ is a modular form invariant under Γ_2, the level-2 principal congruence subgroup . This fact uniquely determines $\Phi(\hat{A})$ up to an overall constant (this constant is fixed by comparing the first terms in the q-expansion). (3.2) agrees with the calculation of the elliptic genus for K_3 by using the theory of characteristic classes.

If we compare q-expansions in (3.2), we find that $\sum_i D_i I_i^2 = 24$ (eq.(2.26)) and the theory reproduces the Euler number of K_3 surface. Thus the $c = 6$ superconformal field theory describes the string propagation on K_3 manifolds. Our only assumption in section 2 is the absence of the mixture of $l = 0$ and $l = 1/2$ massless representations. If there is a contribution of $l = 1/2$ representation in the graviton orbit, then the interference term $ch_0(l = 0)^* ch_0(l = 1/2)$ gives primary fields with conformal dimension $\bar{h} = 0,\ h = 1/2$. These are nothing but free (complex) spinor fields and in this case the theory describes string propagation on the product of complex tori $T \times T$. This is actually what happens in Gepner's models $1^3 2^2, 1^2 2^1 (10')^1$ and $1^1(10')^2$ ($10'$ means the use of E_6 -type invariant for $k = 10$ sub-theory).

In this context we shall note that the function $F_1(\tau)$ in the graviton orbit generates (anti-) holomorphic fields of type $(h, \bar{h}) = (n, 0)$ or $(0, n)$, $n =$

$1, 2, \ldots$. In particular if $f_1^{(1)} \neq 0$, it generates extra $U(1)$ gauge fields in the heterotic string compactification (see section 4).

Instead of (3.1) we may impose the R boundary condition in the right-moving sector and define

$$\Phi(\sigma) = tr_{R\times R}\, q^{L_0-1/4}(-1)^{F_L}\bar{q}^{L_0-1/4}. \tag{3.3}$$

(3.3) can also be evaluated using the non-linear σ-model,

$$\begin{aligned}\Phi(\sigma) &= \sum_i D_i(R_i)(\tilde{R}_i) \\ &= \left\{\frac{q^{-1/8}}{\eta}(-2+\sum_{i=1}^{d} D_i I_i F_i(\tau)) + \chi\, h_3(\tau)\right\}(\frac{\vartheta_2(\tau)}{\eta(\tau)})^2 \\ &\quad +\chi\,(\frac{\vartheta_4(\tau)}{\vartheta_3(\tau)})^2.\end{aligned} \tag{3.4}$$

We obtain

$$\Phi(\sigma) = 2\frac{(\vartheta_3^4+\vartheta_4^4)}{\eta^4}(\frac{\vartheta_2}{\eta})^2 = 16(1+34q+\cdots). \tag{3.5}$$

$\Phi(\sigma)$ is the elliptic genus corresponding to the Hirzebruch signature σ.

Finally, if we insert $(-1)^{F_R}$ to (3.3), we obtain the Euler characteristic

$$\begin{aligned}\Phi(\chi) &= tr_{R\times R}\,(-1)^{F_R+F_L}q^{L_0-1/4}\bar{q}^{L_0-1/4} \\ &= \sum D_i I_i^2 = 24.\end{aligned} \tag{3.6}$$

Actually these three genera may be combined into a single function

$$\Phi(\theta) = tr_{NS\times R}\, q^{L_0-1/4}e^{i\theta J_0}(-1)^{F_L}\bar{q}^{L_0-1/4}. \tag{3.7}$$

$\Phi(\hat{A}), \Phi(\sigma)$ and $\Phi(\chi)$ are given by $\Phi(\theta)$ at $\theta = 0,\ \pi\tau$ and $\pi\tau+\pi$, respectively. Thus the classical topological invariants are nicely "unified" in the superconformal field theory.

The elliptic genus is quite useful to check if some $c = 6$ superconformal field theory describes compactification on K_3 surface. This characterization has enabled us to show in a proper way that the Z_l ($l = 2, 3, 4$ and 6) orbifold models in ref. [20] are in fact the orbifold limits of K_3 surface.

4 Heterotic string compactification

Let us now turn to the discussion on the heterotic string compactification on K_3 surface. In this case we must take into account the degrees of freedom of the uncompactified 6-dimensional Minkowski space and the internal space in the left-moving sector.

In the right sector of the theory, $N = 4$ characters are multiplied by the characters of the $SO(4)$ Kac-Moody algebra which is generated by the four (transverse) spinor fields of the uncompactified Minkowski space (four uncompactified bosons generate an additional $\eta(\tau)^{-4}$). The orbits are given by

$$\begin{aligned} X_{R,i}(z) &= \chi_v^{SO(4)}(z)\ NS_i^+(z) + \chi_b^{SO(4)}(z)\ NS_i^-(z) \\ &\quad -\chi_c^{SO(4)}(z)\ R_i^-(z) - \chi_s^{SO(4)}(z)\ R_i^+(z). \end{aligned} \tag{4.1}$$

Here $NS^{\pm} \equiv \frac{1}{2}(NS \pm \widetilde{NS})$, $R^{\pm} \equiv \frac{1}{2}(R \pm \tilde{R})$ and b, v, s, c are the conjugacy classes of the level-1 representations. (4.1) is written as

$$X_{R,i}(z) = \frac{1}{2}\left\{(\frac{\vartheta_3(z)}{\eta})^2 NS_i(z) - (\frac{\vartheta_4(z)}{\eta})^2 \widetilde{NS}_i(z) - (\frac{\vartheta_2(z)}{\eta})^2 R_i(z) + (\frac{\vartheta_1(z)}{\eta})^2 \tilde{R}_i(z)\right\}, \tag{4.2}$$

where we have used

$$\begin{aligned} \chi_b^{SO(2n)} + \chi_v^{SO(2n)} &= (\frac{\vartheta_3}{\eta})^n, \qquad \chi_b^{SO(2n)} - \chi_v^{SO(2n)} = (\frac{\vartheta_4}{\eta})^n, \\ \chi_s^{SO(2n)} + \chi_c^{SO(2n)} &= (\frac{\vartheta_2}{\eta})^n, \qquad \chi_s^{SO(2n)} - \chi_c^{SO(2n)} = (\frac{-i\ \vartheta_1}{\eta})^n. \end{aligned} \tag{4.3}$$

On the other hand, in the left sector the $N = 4$ characters are multiplied by those of E_8 and $SO(12)$ Kac-Moody algebra which describe the degrees of freedom of the internal space. Note that the standard $E_8 \times E_8$ gauge symmetry of the heterotic string is broken down to $E_8 \times E_7$ in K_3 compactification. $SU(2)'$ gauge symmetry in

$$E_8 \supset SU(2)' \times E_7 \supset SU(2)' \times SU(2) \times SO(12) \tag{4.4}$$

is lost due to the holonomy of the K_3 surface while the $SU(2)$ symmetry of $N = 4$ algebra is combined with $SO(12)$ and generates the E_7 gauge group.

The orbits in the left sector are given by

$$\begin{aligned} X_{L,i}(z) &= (\chi_b^{SO(12)}(z)\ NS_i^+(z) + \chi_v^{SO(12)}(z)\ NS_i^-(z) \\ &\quad + \chi_c^{SO(12)}(z)\ R_i^+(z) + \chi_s^{SO(12)}(z)\ R_i^-(z))\chi_1^{E_8}(z)\,, \qquad (4.5) \\ &= \frac{1}{2}((\frac{\vartheta_3(z)}{\eta})^6 NS_i(z) + (\frac{\vartheta_4(z)}{\eta})^6 \widetilde{NS}_i(z) \\ &\quad + (\frac{\vartheta_2(z)}{\eta})^6 R_i(z) + (\frac{\vartheta_1(z)}{\eta})^6 \tilde{R}_i(z))\ \frac{1}{2}\sum_i(\frac{\vartheta_i(z)}{\eta})^8. \qquad (4.6) \end{aligned}$$

$X_{R,i}$ and $X_{L,i}$ are constructed in such a way that they transform under S as in equation (2.12) with the same S-matrix. GSO projections in (4.1) and (4.5) ensure the correct spin-statistics connection. Modular invariants are formed as

$$Z = \frac{\text{const}}{(Im\tau)^2|\eta|^8}\sum_i D_i(X_{R,i})(X_{L,i}^*). \qquad (4.7)$$

It is easy to see that the right-moving orbits (4.2) actually vanish and hence the theory has zero cosmological constant. We first note that the $N = 4$ massive characters are proportional to the squares of elliptic theta functions ,

$$\begin{aligned} ch^{NS}(z) &\propto \frac{1}{\eta}(\frac{\vartheta_3(z)}{\eta})^2, \qquad ch^{\widetilde{NS}}(z) \propto \frac{1}{\eta}(\frac{\vartheta_4(z)}{\eta})^2, \\ ch^{R}(z) &\propto \frac{1}{\eta}(\frac{\vartheta_2(z)}{\eta})^2, \qquad ch^{\tilde{R}}(z) \propto \frac{1}{\eta}(\frac{\vartheta_1(z)}{\eta})^2. \qquad (4.8) \end{aligned}$$

Thus the contributions of the massive representations vanish in each orbit (4.2) due to the Jacobi identity. It is easy to see that also the contributions of the massless representations cancel in the right sector.

On the other hand, in the left-moving sector of the theory, the massive representations in $X_L(z)$ are combined into E_8 characters

$$\sum_i \vartheta_i(z)^8 \chi_1^{E_8}(z) \propto (\chi_1^{E_8}(z))^2. \qquad (4.9)$$

Thus the massive sector of the theory does not feel the holonomy of the K_3 surface and retains the original $E_8 \times E_8$ symmetry. On the other hand massless components of each orbit are expressed as a sum of E_7 characters,

and we have

$$X_{L,1} = \left\{ A_{1,1}(\tau)\chi_1^{E_7}(z) + A_{1,2}(\tau)\chi_{56}^{E_7}(z) + \frac{q^{-1/8}}{\eta} F_1(\tau)\chi_1^{E_8}(z) \right\} \chi_1^{E_8}(z), \tag{4.10}$$

$$X_{L,i} = \left\{ A_{2,2}(\tau)\chi_{56}^{E_7}(z) + A_{2,1}(\tau)\chi_1^{E_7}(z) + \frac{q^{-1/8}}{\eta} F_i(\tau)\chi_1^{E_8}(z) \right\} \chi_1^{E_8}(z). \tag{4.11}$$

Here $A_{2l+1,2l'+1}(\tau)$ are the branching functions of $N=4$ massless characters into those of $SU(2)$ and we have used

$$\chi_1^{E_7}(z) = \chi_b^{SO(12)}(z)\chi_1^{SU(2)}(z) + \chi_s^{SO(12)}(z)\chi_2^{SU(2)}(z), \tag{4.12}$$

$$\chi_{56}^{E_7}(z) = \chi_v^{SO(12)}(z)\chi_2^{SU(2)}(z) + \chi_c^{SO(12)}(z)\chi_1^{SU(2)}(z). \tag{4.13}$$

(Indices of the characters represent the multiplicity of the highest weight state.)

The massless spectra of the theory are easily read off from (4.10) and (4.11). Besides the standard gravity, E_7 gauge multiplets and 20 spinors of $\underline{56}$ of E_7, there exist $\sum_{i=2}^{d} D_i(2+f_1^{(i)})$ gauge singlets coming from the massless matter orbits. If $f_1^{(1)} \neq 0$, there also appear additional $f_1^{(1)}$ $U(1)$ gauge fields from the graviton orbit. At generic points in the moduli space of K_3 surface, $\sum D_i(2+f_1^{(i)}) = 130$, $f_1^{(1)} = 0$, while in $N=2$ and orbifold models there always exist extra $U(1)$ symmetry ($f_1^{(1)} > 0$) and an excess of gauge singlets $\sum D_i(2+f_1^{(i)}) > 130$. The elliptic genus $\Phi(\hat{A})$ predicts, however, that their difference $\sum D_i(2+f_1^{(i)}) - 2f_1^{(1)}$ must be always equal to 130.

5 $c=9$ superconformal field theories

In this section we discuss $c=9$ superconformal field theories and their relation to Calabi-Yau manifolds. As we have mentioned in section 2, for a systematic treatment of $c=9$ theories we need to study an enlarged version of $N=2$ algebra extended by the addition of the flow generators. The flow generators, denoted as X, $\bar{X}$, have $h=3/2$ and $Q=\pm 3$ and their commutators with G, $\bar{G}$ generate additional operators Y, $\bar{Y}$ with $h=2$, $Q=\pm 2$.

These generators, together with L, J, generate an algebra which contains bilinear terms in the right-hand side of commutation relations. This is a non-Lie algebra of the type introduced by Zamolodchikov [19] and we call it as the $c = 9$ algebra.

Commutation relations of the $c = 9$ algebra have recently been worked out [21]. Among its commutators, important ones are given by

$$\begin{aligned}
\{X_r, \bar{X}_s\} &= (r^2 - \frac{1}{4})\delta_{r+s,0} + (r-s)J_{r+s} + (J^2)_{r+s}, && (5.1)\\
[X_r, \bar{Y}_m] &= (r + \frac{1}{2})G_{r+m} + (JG)_{r+m}, && (5.2)\\
[\bar{X}_r, Y_m] &= (r + \frac{1}{2})\bar{G}_{r+m} - (J\bar{G})_{r+m}, && (5.3)\\
[Y_n, \bar{Y}_m] &= \frac{n}{2}(n^2 - 1)\delta_{n+m,0} + \frac{1}{2}(n(n+1) + m(m+1))J_{n+m} \\
&\quad + \frac{1}{4}(n-m)(J^2)_{n+m} - (m+1)L_{n+m} + (JL)_{n+m} \\
&\quad - \frac{1}{2}(G\bar{G})_{n+m}\,. && (5.4)
\end{aligned}$$

We also record

$$[(J)^2_n, X_r] = 3(n - 2r)X_{r+n}. \qquad (5.5)$$

Here the bilinear forms of operators are defined with the normal ordering; $(AB)_n \equiv \sum_{p\le -h_A} A_p B_{n-p} + \sum_{p>-h_A} (-1)^{AB} B_{n-p} A_p$ (h_A is the conformal dimension of A). Inside the $c = 9$ algebra, L, J, G, $\bar{G}$ form the standard $N = 2$ algebra with $c = 9$ and $\frac{1}{6}J^2$, $\frac{1}{3}J$, $\frac{1}{\sqrt{3}}X$ and $\frac{1}{\sqrt{3}}\bar{X}$ form an additional $N = 2$ algebra with $c = 1$. We note that the latter is isomorphic to the algebra of Waterson [25].

$c = 9$ algebra is again invariant under a transfomation which shifts the moding of the operators G, $\bar{G}$, X and $\bar{X}$, and thus the NS and R sectors of the algebra are isomorphic to each other.

It follows from (5.1),(5.5) that the allowed values of h and Q of the highest-weight states are given by (in the NS sector),

massless representations;

$$\begin{cases} h = 0, & Q = 0, \\ h = 1/2, & Q = 1, \\ h = 1/2, & Q = -1. \end{cases} \qquad (5.6)$$

massive representations;

$$\begin{cases} h > 0, & Q = 0, \\ h > 1/2, & Q = \pm 1. \end{cases} \tag{5.7}$$

Under the spectral flow, representations (5.6),(5.7) are mapped onto the R representations;

massless representations;

$$\begin{cases} h = 3/8, & Q = \pm 3/2, \\ h = 3/8, & Q = 1/2, \\ h = 3/8, & Q = -1/2. \end{cases} \tag{5.8}$$

massive representations;

$$\begin{cases} h > 3/8, & Q = \pm 3/2, \\ h > 3/8, & Q = \pm 1/2. \end{cases} \tag{5.9}$$

Although precise character fomulas of the $c = 9$ algebra have not yet been worked out, their dependence on the $U(1)$ angle θ can be described by the functions

$$f_Q(\theta) \equiv \frac{1}{\eta} \sum_n q^{\frac{3}{2}(n+\frac{Q}{3})^2} \; e^{3i(n+\frac{Q}{3})\theta}. \tag{5.10}$$

In fact these are the only functions whose θ-dependence is consistent with the spectral flow,

$$f_Q(\theta \pm \pi\tau) = q^{-3/8} e^{\mp i3\theta/2} f_{Q\pm3/2}(\theta). \tag{5.11}$$

f_Q with $Q = 0,\ \pm 1$ ($\pm 3/2,\ \pm 1/2$) give characters of the NS(R) representations (5.6), (5.7) ((5.8), (5.9)) up to factors depending only on τ. We note that

$$f_{Q=0}(\theta = \pi\tau + \pi) \;=\; 0, \tag{5.12}$$

$$\begin{aligned} f_{Q=1}(\theta = \pi\tau + \pi) &= -f_{Q=-1}(\theta = \pi\tau + \pi) \\ &= q^{-\frac{3}{8}} . \end{aligned} \tag{5.13}$$

Let us now repeat our analysis of section 2 and construct modular invariant partition functions of $c = 9$ superconformal theories. If we consider, for instance, 1^9 theory, the graviton orbit is given by

$$NS_1(z) = A(z)^9 + B(z)^9 + C(z)^9. \tag{5.14}$$

An essential difference from the case of K_3 surface is that the Witten index of (5.14) now vanishes,

$$\tilde{R}_i(\theta=0) \propto NS_1(\theta=\pi\tau+\pi)=0. \tag{5.15}$$

This is due to the cancellation of contributions from $h=3/8$, $Q=\pm 3/2$ states in the R sector. (This is related to the fact that the charge-conjugation operator (anti-) commutes with the helicity operator in $4m(+2)$ dimensions. Covariantly constant spinors $h=n/8$, $Q=\pm n/2$ on n-dimensional complex manifold with $SU(n)$ holonomy have the same (opposite) helicities when n = even (odd)).

NS_1 is then expanded as,

$$\begin{aligned} NS_1(\tau;z) &= G_1(\tau)(f_1(z)+f_{-1}(z))+H_1(\tau)f_0(z), && (5.16)\\ H_1(\tau) &= q^{-1/3}(1+\sum_{n=1} h_n^{(1)}q^n), \quad G_1(\tau)=\sum_{n=1} g_n^{(1)}q^n. && (5.17)\end{aligned}$$

Other orbits are again generated by the S transformation of the graviton trajectory. In the $c=9$ case, there are four types of NS orbits;

(1) graviton orbit;
NS_1 is the only trajectory containing the ground state $h=Q=0$.

(2) massless matter orbits;
NS_i $(i=2,\dots,d)$ contains a state $h=1/2$, $Q=1$ and is rewritten as

$$\begin{aligned} NS_i(\tau;z) &= f_1(z)+G_i(\tau)(f_1(z)+f_{-1}(z))+H_i(\tau)f_0(z), && (5.18)\\ G_i(\tau) &= \sum_{n=1} g_n^{(i)}q^n, \quad H_i(\tau)=\sum_{n=1} h_n^{(i)}q^{n-1/3}. && (5.19)\end{aligned}$$

Orbit i is paired with its conjugate orbit $i^* \equiv i+d-1$. $NS_{i^*}(\tau;\theta)$ is given by $NS_i(\tau;\theta)$ with θ replaced by $-\theta$,

$$NS_{i^*}(\tau;z) = f_{-1}(z)+G_i(\tau)(f_1(z)+f_{-1}(z))+H_i(\tau)f_0(z), \tag{5.20}$$

NS_{i^*} contains a state $h=1/2$, $Q=-1$.

(In some cases, a state $h=\frac{1}{2}, Q=1$ or -1 appears more than once in each orbit. Then, the first terms in the right-hand-side of (5.18),(5.20) must be multiplied by the multiplicity).

(3) self-conjugate massless orbits;

NS_j ($j = 2d, \ldots, 2d + d' - 1$) contains both states $h = \frac{1}{2}$ and $Q = \pm 1$ and is self-conjugate. It is rewritten as

$$\begin{aligned} NS_j(\tau; z) &= G_j(\tau)(f_1(z) + f_{-1}(z)) + H_j(\tau) f_0(z) , &(5.21)\\ G_j(\tau) &= 1 + \sum_{n=1} g_n^{(j)} q^n , \quad H_j(\tau) = \sum_{n=1} h_n^{(j)} q^{n-1/3}. &(5.22)\end{aligned}$$

(4) massive orbits;

NS_m ($m = 2d + d', \ldots, 2d + d' + d''$) does not contain any of the states $h = Q = 0, h = \frac{1}{2}, Q = \pm 1$. It is written as

$$\begin{aligned} NS_m(\tau; z) &= G_m(\tau)(f_1(z) + f_{-1}(z)) + H_m(\tau) f_0(z) , &(5.23)\\ G_m(\tau) &= q^{r_m} \sum_{n=0} g_n^{(m)} q^n, \quad H_m(\tau) = q^{r'_m} \sum_{n=0} h_n^{(m)} q^n , &(5.24)\end{aligned}$$

$$r_m, r'_m \in \mathbf{Q}, \quad 0 < r_m, \ -1/3 < r'_m .$$

These $2d + d' + d''$ trajectories enter into the modular invariant partition functions. In the 1^9 case, $d = 2, d' = 0, d'' = 3$.

As in the case of $c = 6$ theories, modular invariants are formed using the D-coefficients which symmetrize the matrix of S-transformation,

$$Z_\sigma = \frac{1}{2} \sum_{i=1}^{2d+d'+d''} D_i \{ | NS_i |^2 + | \widetilde{NS}_i |^2 + | R_i |^2 + | \tilde{R}_i |^2 \} \qquad (5.25)$$

Note that $D_{i^*} = D_i$. Euler number is given by

$$\chi = -2 \sum_{i=2}^{d} D_i \qquad (5.26)$$

(There exists a sign ambiguity in deriving Euler number using the Witten index. In (2.26), (5.26), we have defined the states $h = n/8$, $Q = \pm n/2$ in R sector to be bosonic (fermionic) when $n =$ even (odd)). In the case of $c = 9$ theories, however, a set of new modular invariants can be constructed from each "A-type" invariant (5.25). These new invariants have Euler numbers which differ from (5.26) by $4 \times$ integers as we shall see later.

Let us now consider partition functions of the heterotic string theory in the $c = 9$ case. They are constructed by multiplying $SO(2)$, and $SO(10)$ and E_8 characters to the orbits in the right and left-moving sectors, respectively.

Cosmological constant again vanishes in heterotic string compactification. Orbits in the right sector are proportional to

$$\vartheta_3 f_Q - \vartheta_4 \tilde{f}_Q - \vartheta_2 f_{Q+3/2} = 0, \qquad Q = 0, \pm 1 \tag{5.27}$$

where $\tilde{f}_Q \equiv f_Q(\theta = \pi)$. (5.27) can be shown by using the product formula of theta functions.

On the other hand, orbits in the left sector are reexpressed in terms of E_6 characters,

$$\begin{aligned} \chi_1^{E_6} &= \frac{1}{2}\{(\frac{\vartheta_3}{\eta})^5 f_0 + (\frac{\vartheta_4}{\eta})^5 \tilde{f}_0 + (\frac{\vartheta_2}{\eta})^5 f_{3/2}\}\,, & (5.28)\\ \chi_{27}^{E_6} &= \frac{1}{2}\{(\frac{\vartheta_3}{\eta})^5 f_{-1} + (\frac{\vartheta_4}{\eta})^5 \tilde{f}_{-1} + (\frac{\vartheta_2}{\eta})^5 f_{1/2}\}\,; & (5.29)\\ \chi_{27^*}^{E_6} &= \frac{1}{2}\{(\frac{\vartheta_3}{\eta})^5 f_1 + (\frac{\vartheta_4}{\eta})^5 \tilde{f}_1 + (\frac{\vartheta_2}{\eta})^5 f_{-1/2}\}\,. & (5.30) \end{aligned}$$

Partition functions are again formed as

$$Z = \frac{\text{const}}{(Im\tau)|\eta|^4} \sum D_i (X_{R,i})(X_{L,i}^*)\,. \tag{5.31}$$

(5.31) contains the graviton and E_6 gauge multiplets. There also exist massless scalar multiplets of $\underline{27}$ and $\underline{27}^*$ of E_6 which come from the combinations $|h = \frac{1}{2}, Q = +1 > \otimes |\bar{h} = \frac{1}{2}, \bar{Q} = +1 >$ and $|h = \frac{1}{2}, Q = +1 > \otimes |\bar{h} = \frac{1}{2}, \bar{Q} = -1 >$, respectively. Their multiplicities are

$$\begin{aligned} n_{27} &= \sum_{i=2}^{d} D_i + \sum_{j=2d}^{2d+d'-1} D_j, & (5.32)\\ n_{27^*} &= \sum_{j=2d}^{2d+d'-1} D_j & (5.33) \end{aligned}$$

In Calabi-Yau compactifications they are identified with the Hodge numbers, $n_{27} = h^{2,1}$ and $n_{27^*} = h^{1,1}$, and are related to the Euler characteristic as $\chi = 2(n_{27^*} - n_{27})$.

We now go back to the "A-type" modular invariant of the non-linear σ-model (5.25) and construct a new set of modular invariants which are

the analogue of the D-type invariants of the A-D-E classification [26]. We consider a contribution of a pair of massless matter orbits i and i^*,

$$Z_i^{NS}(A) = D_i\{| NS_i |^2 + | NS_{i^*} |^2\} . \tag{5.34}$$

We then introduce

$$\begin{aligned} Z_i^{NS}(D) &= m_i\{| NS_i |^2 + | NS_{i^*} |^2\} \\ &\quad + n_i\{(NS_i)(NS_{i^*})^* + (NS_{i^*})(NS_i)^*\} , \end{aligned} \tag{5.35}$$

where m_i, n_i are non-negative integers and $m_i + n_i = D_i$. Difference between (5.34) and (5.35) is given by

$$Z_i^{NS}(A) - Z_i^{NS}(D) = n_i \mid f_{+1}(\theta) - f_{-1}(\theta) \mid^2 . \tag{5.36}$$

Here the crucial fact is that $f_{+1}(\theta) - f_{-1}(\theta)$ is invariant under S-transformation. After summing over spin structures, the right-hand-side of (5.36) becomes a constant

$$Z_i(A) - Z_i(D) = 4n_i \tag{5.37}$$

at $\theta = 0$. Thus the D-type combination

$$\begin{aligned} Z(D) &= \frac{1}{2}[| NS_1 |^2 \\ &\quad + \sum_{i=2}^{d}\{m_i(| NS_i |^2 + | NS_{i^*} |^2) + n_i((NS_i)(NS_{i^*})^* + (NS_{i^*})(NS_i)^*)\} \\ &\quad + \sum_{j=2d}^{2d+d'+d''} | NS_j |^2 + \text{other spin structures}] \end{aligned} \tag{5.38}$$

yields a modular invariant partition function. Its Euler number is given by

$$\chi = -2\sum_{i=2}^{d}(m_i - n_i) . \tag{5.39}$$

Thus, given an A-type invariant (5.25) with Euler index χ, we can construct a new set of invariants with Euler numbers by varying m_i, n_i

$$|\chi|,\ |\chi| - 4, \cdots, -|\chi| + 4,\ -|\chi| . \tag{5.40}$$

In forming D-type invariants (5.35), the left-right pairing of states $h = \frac{1}{2}, Q = \pm 1$ is reshuffled and this corresponds to interchanging the roles of $h^{2,1}$ and $h^{1,1}$. Such a procedure does not have a well-defined geometrical significance and the modular invariants (5.38) could not all describe string propagation on Calabi-Yau manifolds. Thus $c = 9$ superconformal field theories seem to come in a larger varieties than those of Calabi-Yau manifolds. This is in contrast to the case of K_3 surfaces and may have some profound implications in the phenomenology of superstring compactification.

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Introduction to W-algebras

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ABSTRACT

We review W-algebras which are generated by stress tensor and primary fields. Associativity plays an important role in determining the extended algebra and further implies the algebras to exist for special values of central charges. Explicitly constructing the algebras including primary fields of spin less than 4, we investigate the closure structure of the Jacobi identity of the extended algebras.

1. Introduction

Conformal field theories (CFTs)[1] in two dimensions are indispensable for recent progress in string theories and two dimensional critical phenomena. Although we have many varieties of examples, we are far from possessing the complete classification of CFTs. It is one of such approaches to find all possible extension of the Virasoro algebra.

Recently Zamolodchikov[2] has constructed the extensions of the Virasoro algebra containing primary field $\phi^{(\sigma)}$ of higher conformal spin σ ($\sigma \leq 3$). We call the extended algebra formed by primary field of spin n W_n-algebra. This is a new trend of generalizing the Virasoro algebra, since the commutator includes products of the operators. So it is not a Lie algebra.

The extended algebras are determined by the associativity of operator product expansion (OPE) algebras. It is made clear by Bouwknegt[3] that the associativity not only fixes the algebras but also puts severe restrictions on the central charges. By investigating the crossing symmetry of the four point correlation functions of the primary fields, he has concludes that in general the extended algebras of higher spin could be associative for generic values of central charges.

Our strategy for W_n-algebras is to study the Jacobi identities of the algebra[4]. The Jacobi identity including a stress tensor $T(z)$ and two of primary field $\phi^{(\sigma)}(z)$, *i.e.*$(T, \phi^{(\sigma)}, \phi^{(\sigma)})$ fixes the commutator of the primary field. This implies that the commutation relation is compatible with the conformal invariance. The crucial Jacobi identity is the $(\phi^{(\sigma)}, \phi^{(\sigma)}, \phi^{(\sigma)})$ one[3]. The crossing symmetry of the four point correlation function of the primary fields is nothing but the Jacobi identity, so that the pioneers of W_n-algebra have checked the crossing symmetry instead of the identity. It is shown that the identity is violated by some descendant operator. For special values of central charge, we can show that the operator becomes primary and so a null operator, hence we can remove the operator from the theory. Here practically calculating the $(\phi^{(\sigma)}, \phi^{(\sigma)}, \phi^{(\sigma)})$ identity, we reveal the importance of the null states constructed of the primary field $\phi^{(\sigma)}(z)$.

2. Formulation of W_n-algebras

In this section, we briefly reviews the conformal field theory[1] to introduce necessary notations and tools for later arguments.

In scale invariant system, stress-energy tensor is traceless symmetric and so has the component T_{zz} and $T_{\bar{z}\bar{z}}$ in complex coordinate. Here complex variables z and $\bar{z}$ are two dimensional coordinates defined by $z = x^1 + ix^2$ and the conjugate respectively. Energy-momentum conservation law implies that T_{zz} ($T_{\bar{z}\bar{z}}$) is (anti-)holomolphic function. Hence we denote the stress tensors as $T(z)$ and $\bar{T}(\bar{z})$. Conformal transformation of the field $\phi(z,\bar{z})$ is generated by stress tensor $T(z)$:

$$\langle \delta_\epsilon \phi(z,\bar{z}) \rangle = \frac{1}{2\pi i} \oint_{C_z} dw \epsilon(w) \langle T(w) \phi(z,\bar{z}) \rangle \ , \tag{2.1}$$

where $\epsilon(w)$ is an infinitesimal parameter and C_z the path enclosing z. Field with particular transformation law under conformal transformation $z' = f(z)$

$$\phi'(z',\bar{z}') = \left(\frac{\partial f}{\partial z}\right)^{\Delta} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{\Delta}} \phi(z,\bar{z}) \tag{2.2}$$

is called primary field. The infinitesimal version of Eq.(2.2) becomes

$$\begin{aligned} \delta_\epsilon \phi(z,\bar{z}) &= \epsilon(z) \frac{\partial}{\partial z} \phi(z,\bar{z}) + \Delta \frac{\partial}{\partial z} \epsilon \phi(z,\bar{z}) \ , \\ \delta_{\bar{\epsilon}} \phi(z,\bar{z}) &= \bar{\epsilon}(\bar{z}) \frac{\partial}{\partial \bar{z}} \phi(z,\bar{z}) + \bar{\Delta} \frac{\partial}{\partial \bar{z}} \bar{\epsilon} \phi(z,\bar{z}) \ . \end{aligned} \tag{2.3}$$

Combining Eqs.(2.1) and (2.3), we find the following OPE of the stress tensor and the primary field

$$\begin{aligned} T(z)\phi(w,\bar{w}) &= \frac{\Delta}{(z-w)^2} \phi(w,\bar{w}) + \frac{1}{z-w} \partial \phi(w,\bar{w}) + \text{ regular terms,} \\ \bar{T}(\bar{z})\phi(w,\bar{w}) &= \frac{\bar{\Delta}}{(\bar{z}-\bar{w})^2} \phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}} \bar{\partial} \phi(w,\bar{w}) + \cdots \ . \end{aligned} \tag{2.4}$$

It is already known that the stress tensor has the following OPE

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \cdots . \tag{2.5}$$

Among primary fields the field $\phi^{(\sigma)}$ of dimension $(\Delta, \bar{\Delta}) = (\sigma, 0)$ plays an important role in CFTs. Here spin σ is integer or half-integer on account of the boundary condition. It can be easily shown that $\phi^{(\sigma)}$ satisfies the conservation law

$$\frac{\partial}{\partial \bar{z}}\phi^{(\sigma)}(z) = 0 , \tag{2.6}$$

so that it generates an additional symmetry in CFTs. Examples are Kac-Moody algebra[5] for $\sigma = 1$ and superconformal algebra[6] for $\sigma = 3/2$.

In the following, we investigate the possibilities of the higher spin extensions of Virasoro algebra. Zamolodchikov[2] has studied models for $\sigma \leq 3$. It is a remarkable fact that the extended algebra with the primary field of $\sigma = 5/2$ possesses the associativity only when it has a specific central charge $c = -13/14$. This is a general property of $\mathrm{W_n}$-algebra for large n. Bouwknegt[3] has derived the complete table of the allowed values of central charges for $\mathrm{W_n}$-algebra for n ≤ 5.

Table 1
The allowed values of central charges for associative $\mathrm{W_n}$-algebras.

Spin	Central charge c
5/2	$-13/14$
3	arbitrary
7/2	$21/22,\ -19/6,\ -161/8,$
4	$86 \pm 60\sqrt{2}$
9/2	$25/26,\ -7/20,\ -125/22,\ -279/10,\ -35$
5	$6/7,\ -350/11,\ -7,\ 134 \pm 60\sqrt{5}$

He has obtained the table by studying the crossing symmetry of the four point correlation function of the primary fields. One of our aim of the present paper is to make clear what happens in the algebra at the central charge.

Our strategy to $\mathrm{W_n}$-algebra is to investigate the closure of the Jacobi identities of the algebra. To this end, we introduce the Fourier expansions of the operators as

$$\begin{aligned} T(z) &= \sum_{n\in Z} \frac{1}{z^{n+2}} L_n , \\ \phi^{(\sigma)}(z) &= \sum_{r\in Z\ (+\frac{1}{2})} \frac{1}{z^{r+\sigma}} \phi_r^{(\sigma)} . \end{aligned} \tag{2.7}$$

Here if $\sigma =$ half-integer r is summed over half-integer for Neveu-Schwarz sector and over integer for Ramond sector. Of course for $\sigma =$ integer the summation is carried over integer. Inverse Fourier transformations are given by

$$\begin{aligned} L_n &= \oint dz z^{n+1} T(z) , \\ \phi_r^{(\sigma)} &= \oint dz z^{r+\sigma-1} \phi^{(\sigma)}(z) . \end{aligned} \tag{2.8}$$

OPE (2.5) leads L_n to satisfy the following commutation relation

$$[\ L_m,\ L_n\ ] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} , \tag{2.9}$$

*i.e.*Virasoro algebra. The defining relation (2.4) of the primary field $\phi^{(\sigma)}$ becomes

$$[\ L_m,\ \phi_r^{(\sigma)}\ ] = \{(\sigma-1)m - r\}\phi_{m+r}^{(\sigma)} . \tag{2.10}$$

For the purpose of determining the extended algebra, it is necessary to specify the singular terms of OPE of the primary fields

$$\phi^{(\sigma)}(z)\phi^{(\sigma)}(w) = \sum_{s=0}^{2\sigma-1} \frac{a_s}{(z-w)^{2\sigma-s}} R^{(s)}(w) + \cdots . \tag{2.11}$$

Here $R^{(s)}(w)$'s are local operators which consist of the identity operator I, the

primary field $\phi^{(\sigma)}$ and their descendant fields, *i.e.*their derivatives and products. Note that $R^{(0)}$ is the identity operator. Using the normalization ambiguity of the primary field, we can always choose $a_0 = c/\sigma$. Following Belavin, et al.[1], we write Eq.(2.11) symbolically as

$$\phi^{(\sigma)}\phi^{(\sigma)} = (\frac{c}{\sigma})[I] + b[\phi^{(\sigma)}] \, , \tag{2.12}$$

where the symbol $[I]$ ($[\phi^{(\sigma)}]$) stands for the conformal family of the operator I ($\phi^{(\sigma)}$). The symmetry requirement under exchanging the order of OPE leads b to be zero except for σ = even integer. So the spin 4 algebra[4] is the first non-trivial example including the conformal family of the primary field in OPE.

In the next section, we show how the Jacobi identities derive the $\mathrm{W_n}$-algebras. The (L, $\phi^{(\sigma)}$, $\phi^{(\sigma)}$) Jacobi identity represents the compatibility of the OPE algebra of the primary field with conformal invariance. From the identity, we can completely fix the commutator of the primary field. On the other hand, the ($\phi^{(\sigma)}$, $\phi^{(\sigma)}$, $\phi^{(\sigma)}$) Jacobi identity plays a crucial role in determining the central charge allowed by the associativity of the algebra.

3. Structures of $\mathrm{W_n}$-algebras

Since we have already known $\mathrm{W_n}$-algebras for $n \leq 2$, we begin with $n = 5/2$. Models of lower n are listed in the following table.

Table 2

The models including the primary field of spin lower than 2.

Spin	models
1/2	free fermion theory
1	Kac-Moody algebra
3/2	superconformal algebra
2	direct product of Virasoro algebras

3.1 SPIN 5/2

From the dimensional requirement and the symmetry under $z \leftrightarrow w$, we find that OPE (2.11) for $\sigma = 5/2$ should be

$$\begin{aligned}\phi^{(5/2)}(z)\phi^{(5/2)}(w) = \frac{2c/5}{(z-w)^5} + \alpha_1\Big\{\frac{2}{(z-w)^3}T(w) + \frac{1}{(z-w)^2}\partial T(w)\Big\} \\ + \frac{2}{z-w}\Big\{\alpha_2\partial^2 T(w) + \beta\Lambda(w)\Big\} + \cdots .\end{aligned} \tag{3.1}$$

Here the Greek letters are some numerical constant determined by the closure of the Jacobi identity. The operator $\Lambda(z)$ is regularized square of the stress tensor:

$$\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z) , \tag{3.2}$$

where the parentheses indicate the normal product of operators. We adopt the

following definition[7] for the normal product for operators $A(z)$ and $B(z)$

$$(AB)(z) = \oint_{C_z} \frac{dx}{x-z} A(x)B(z) \; . \tag{3.3}$$

If we write the OPE of $A(z)$ and $B(z)$ as

$$A(z)B(w) = \sum_{s=1} \frac{1}{(z-w)^s} (\!(AB)\!)^{(s)}(w) + (\!(AB)\!)^{(0)}(w) + O(z-w) \; , \tag{3.4}$$

we find

$$(AB)(z) = (\!(AB)\!)^{(0)}(z) \; . \tag{3.5}$$

The comparison of Eq.(3.4) with the one exchanging A and B leads

$$([\, A,\ B\,])(z) = \sum_{r=1} \frac{(-)^{r+1}}{r!} \partial^r (\!(AB)\!)^{(r)}(z) \; . \tag{3.6}$$

If both A and B have fermionic nature, the commutator should be replaced with the anti-commutator. Converting Eq.(3.3) into the mode expansion, we have the equivalent expression

$$(AB)_m = \sum_{\ell=\sigma_a}^{\infty} A_{-\ell} B_{m+\ell} + \sum_{\ell=-\infty}^{\sigma_a - 1} B_{m+\ell} A_{-\ell} \; . \tag{3.7}$$

The commutator of Λ_n with L_m can be easily calculated

$$[\, L_m,\ \Lambda_n\,] = (3m-n)\Lambda_{m+n} + \frac{5c+22}{30} L_{m+n} \; . \tag{3.8}$$

It is emphasized that $\Lambda(z)$ becomes a primary field at $c = -22/5$.

From the OPE (3.1), we find that the anti-commutator of $\phi^{(5/2)}$ is

$$\begin{aligned}\{\, \phi_r^{(5/2)},\ \phi_s^{(5/2)}\, \} =& \frac{2c}{5!}\Big(r^2-\frac{9}{4}\Big)\Big(r^2-\frac{1}{4}\Big)\delta_{r+s,0} \\ &+\Big\{-\alpha_1\Big(r+\frac{3}{2}\Big)\Big(s+\frac{3}{2}\Big)+\alpha_2(r+s+2)(r+s+3)\Big\}L_{r+s} \\ &+\beta\Lambda_{r+s}\,.\end{aligned} \tag{3.9}$$

The $(L,\phi^{(5/2)},\phi^{(5/2)})$ Jacobi identity fixes the unknown constants $\alpha_1, \alpha_2, \beta$:

$$\alpha_1 = 1,\ \alpha_2 = 3/20,\ \beta = \frac{27/2}{5c+22}\,. \tag{3.10}$$

These are of course agree with Zamolodchikov's result[2] derived by using the conformal Ward identity of the correlation functions.

Now we show that the $(\phi^{(5/2)},\ \phi^{(5/2)},\ \phi^{(5/2)})$ Jacobi identity is satisfied only if $c=-13/14$. Using Eq.(3.9) with Eq.(3.10), we can easily derive

$$\begin{aligned}&[\, \phi_r^{(5/2)},\ \{\, \phi_s^{(5/2)},\ \phi_t^{(5/2)}\,\}\,] + \text{ cyclic permutation} \\ &= \frac{27}{5c+22}\Phi_{r+s+t}+\frac{14c+13}{5c+22}\Big\{-\frac{9}{28}(r+s+t)^3+\frac{9}{28}(r+s+t) \\ &\qquad +\frac{9}{8}(r+s+t)(rs+st+tr)-\frac{15}{8}rst\Big\}\phi^{(5/2)}_{r+s+t}\,,\end{aligned} \tag{3.11}$$

where $\Phi(z)$ is a descendant field of $\phi^{(5/2)}(z)$:

$$\Phi(z) = 9(T\partial\phi^{(5/2)})(z) - 5\partial(T\phi^{(5/2)})(z) - \frac{4}{7}\partial^3\phi^{(5/2)}(z)\,. \tag{3.12}$$

It is a remarkable fact that if $c=-13/14$ $\Phi(z)$ becomes a primary field. This can be most easily confirmed by the OPE with the stress tensor:

$$\begin{aligned}T(z)\Phi(w) =& \frac{11/2}{(z-w)^2}\Phi(w)+\frac{1}{z-w}\partial\Phi(w) \\ &+\frac{14c+13}{7}\Big[-\frac{5}{(z-w)^5}\phi^{(5/2)}(w)+\frac{1}{(z-w)^4}\partial\phi^{(5/2)}(w)\Big]\,.\end{aligned} \tag{3.13}$$

The first two terms in r.h.s. of Eq.(3.13) is what would be required for $\Phi(z)$ to be

a primary field of dimension 11/2. The field $\Phi(z)$ is descendant and primary and hence a null state, so that it decouples from the theory. In this way the Jacobi identities of $W_{5/2}$-algebra closes.

3.2 SPIN 3

Now we can assume the OPE of $\phi^{(3)}$ to be

$$
\begin{aligned}
\phi^{(3)}(z)\phi^{(3)}(w) =& \frac{c/3}{(z-w)^6} \\
&+ \alpha_1\left\{\frac{2}{(z-w)^4}T(w) + \frac{1}{(z-w)^3}\partial T(w) - \frac{1/12}{z-w}\partial^3 T(w)\right\} \\
&+ \alpha_2\left\{\frac{2}{(z-w)^2}\partial^2 T(w) + \frac{1}{z-w}\partial^3 T(w)\right\} \\
&+ \beta\left\{\frac{2}{(z-w)^2}\Lambda(w) + \frac{1}{z-w}\partial\Lambda(w)\right\} + \cdots .
\end{aligned}
\tag{3.14}
$$

It is notable that each of the curly brackets in OPE (3.14) is symmetric under the substitution $z \leftrightarrow w$ and that it corresponds to linearly independent operators. Completely similar fashion to the case of spin 5/2, we obtain

$$
\alpha_1 = 1,\ \alpha_2 = 3/20,\ \beta = \frac{16}{5c+22} \ . \tag{3.15}
$$

Comparing Eqs.(3.10) and (3.15), one is aware of the universality of the coefficients of the OPEs of the primary fields. Zamolodchikov's method[2] is applicable to determine the first few terms of OPEs for arbitrary spin σ and gives the following result

$$
\begin{aligned}
\phi^{(\sigma)}(z)\phi^{(\sigma)}(w) =& \frac{c/\sigma}{(z-w)^{2\sigma}} + \frac{2}{(z-w)^{2\sigma-2}}T(w) + \frac{1}{(z-w)^{2\sigma-3}}\partial T(w) \\
&+ \frac{1}{(z-w)^{2\sigma-4}}\left[\frac{3}{10}\partial^2 T(w) + 2\beta^{(\sigma)}\Lambda(w)\right] \\
&+ \frac{1}{(z-w)^{2\sigma-5}}\left[\frac{1}{15}\partial^3 T(w) + \beta^{(\sigma)}\partial\Lambda(w)\right] + \cdots ,
\end{aligned}
\tag{3.16}
$$

where

$$\beta^{(\sigma)} = \frac{5\sigma + 1}{5c + 22} . \tag{3.17}$$

The universality of the coefficient of the OPE is the manifestation of the conformal invariance.

To study the $(\phi^{(3)}, \phi^{(3)}, \phi^{(3)})$ Jacobi identity, we write down the commutator of $\phi^{(3)}$

$$\begin{aligned}[\phi^{(3)}_m, \phi^{(3)}_n] =& \frac{3c}{5!} m(m^2-1)(m^2-4)\delta_{m+n,0} \\ &+ (m-n)\left\{ \frac{1}{30}(m^2 - mn + n^2 - 8)L_{m+n} + \frac{16}{5c+22}\Lambda_{m+n} \right\} .\end{aligned} \tag{3.18}$$

Then it can be easily made sure that the identity is satisfied for any value of central charge. We can say that this is rather exceptional example of $\mathrm{W_n}$-algebra for large n as shown in Table 1. This critical property is clearly explained in Ref.[3].

3.3 SPIN 7/2

In the OPE of $\phi^{(7/2)}$, we should introduce the descendant operators of dimension 6 in addition to the one of dimension 4 $\Lambda(z)$. There are two independent operators

$$\begin{aligned}\Xi(z) =& (\partial T \partial T)(z) - \frac{31/7}{5c+22}\partial^2\Lambda(z) - \frac{3}{70}\partial^4 T(z) , \\ \Delta(z) =& (T\Lambda)(z) - \frac{3}{10}\partial^2\Lambda(z) .\end{aligned} \tag{3.19}$$

We have two more operators of dimension 6, $(\partial^2 TT)(z)$ and $(T\partial^2 T)(z)$. The relation

$$(T\partial^2 T)(z) = (\partial^2 TT)(z) + \frac{1}{6}\partial^4 T(z) , \tag{3.20}$$

which is derived from Eq.(3.6), implies that these operators are not independent of $\Xi(z)$ and $\partial^2\Lambda(z)$. It is unnecessary to take care of $(\partial^2 TT)(z)$ and $(T\partial^2 T)(z)$

in constructing the extended algebras. In mode expansion, the commutation relations of the dimension 6 operators with the Virasoro operator L_m are

$$
\begin{aligned}
[L_m, \Xi_n] =& (5m - n)\Xi_{m+n} \\
& - \frac{1}{5c+22} m(m+1)\Big[\frac{124}{7}(m-1) + \Big(10c + \frac{29}{7}\Big)(m+n+4)\Big]\Lambda_{m+n} \\
& - \frac{1}{6}\Big(c + \frac{29}{70}\Big) m(m^2-1)(m-2)(m+n+2)L_{m+n} \ , \\
[L_m, \Delta_n] =& (5m - n)\Delta_{m+n} \\
& + m(m+1)\Big[\Big(\frac{c}{4} + \frac{11}{5}\Big)(m-1) + \frac{6}{5}(m+n+4)\Big]\Lambda_{m+n} \\
& + \frac{1}{10}\Big(c + \frac{22}{5}\Big) m(m^2-1)(m-2)(m+n+2)L_{m+n} \ .
\end{aligned}
\tag{3.21}
$$

Now we have the trial form of OPE of $\phi^{(7/2)}(z)$

$$
\begin{aligned}
&\phi^{(7/2)}(z)\phi^{(7/2)}(w) \\
&= \frac{c/4}{(z-w)^7} + \alpha_1\Big\{\frac{2}{(z-w)^5}T(w) + \frac{1}{(z-w)^4}\partial T(w) - \frac{1/12}{(z-w)^2}\partial^3 T(w)\Big\} \\
&\quad + \alpha_2\Big\{\frac{2}{(z-w)^3}\partial^2 T(w) + \frac{1}{(z-w)^2}\partial^3 T(w)\Big\} + \alpha_3\frac{2}{z-w}\partial^4 T(w) \\
&\quad + \beta_1\Big\{\frac{2}{(z-w)^3}\Lambda(w) + \frac{1}{(z-w)^2}\partial\Lambda(w)\Big\} + \beta_2\frac{2}{z-w}\partial^2\Lambda(w) \\
&\quad + \gamma\frac{2}{z-w}\Xi(w) + \delta\frac{2}{z-w}\Delta(w) \ .
\end{aligned}
\tag{3.22}
$$

As well as the previous examples, we transform the OPE (3.21) into commutator. Then we can determine the unknown constants in the OPE of $\phi^{(7/2)}$ from the $(L, \phi^{(7/2)}, \phi^{(7/2)})$ Jacobi identity with the aid of the commutation relations (3.21). To compare other $\mathrm{W_n}$-algebra, we explicitly write down the OPE of spin

7/2 field:

$$
\begin{aligned}
&\phi^{(7/2)}(z)\phi^{(7/2)}(w) \\
=&\frac{2c/7}{(z-w)^7}+\frac{2}{(z-w)^5}T(w)+\frac{1}{(z-w)^4}\partial T(w) \\
&+\frac{3/10}{(z-w)^3}\partial^2 T(w)+\frac{1/15}{(z-w)^2}\partial^3 T(w)+\frac{1/84}{z-w}\partial^4 T(w) \\
&+\frac{1}{5c+22}\left\{\frac{37}{(z-w)^3}\Lambda(w)+\frac{37/2}{(z-w)^2}\partial\Lambda(w)+\frac{155/28}{z-w}\partial^2\Lambda(w)\right\} \\
&+\frac{1}{z-w}\left\{-\frac{5c-118}{(2c-1)(7c+68)}\Xi(w)+\frac{5(270c+41)}{(2c-1)(7c+68)(5c+22)}\Delta(w)\right\}.
\end{aligned}
\tag{3.23}
$$

This is clearly agree with OPE (3.16) with Eq.(3.17).

The $(\phi^{(7/2)},\ \phi^{(7/2)},\ \phi^{(7/2)})$ Jacobi identity is also violated by the following descendant field

$$
\begin{aligned}
\Psi(z)=&15(\Lambda\partial\phi^{(7/2)})(z)-7\partial(\Lambda\phi^{(7/2)})(z) \\
&+\frac{10c+91}{10}\partial^3(T\phi^{(7/2)})(z)-\frac{390c+371}{70}\partial^2(T\partial\phi^{(7/2)})(z) \\
&+\frac{130c+357}{8}\left\{\frac{3}{5}\partial(T\partial^2\phi^{(7/2)})(z)-\frac{1}{3}(T\partial^3\phi^{(7/2)})(z)\right\} \\
&+\frac{2c+21}{154}\partial^5\phi^{(7/2)}(z)\,,
\end{aligned}
\tag{3.24}
$$

which is again primary if $c=21/22$ or $c=-19/6$. As for the case of $\sigma=5/2$, the primarity of $\Psi(w)$ can be shown by the OPE of $T(z)$ and $\Psi(w)$:

$$
\begin{aligned}
&T(z)\Psi(w)\\
&=\frac{17/2}{(z-w)^2}\Psi(w)+\frac{1}{z-w}\partial\Psi(w)\\
&\quad+(21c-22)\left[\frac{1}{(z-w)^4}\left\{7\partial(T\phi^{(7/2)})-11(T\partial\phi^{(7/2)})+\frac{4}{9}\partial^3\phi^{(7/2)}\right\}(w)\right.\\
&\qquad\qquad+5(6c+19)\left\{\frac{5/11}{(z-w)^6}\phi^{(7/2)}(w)+\frac{3/77}{(z-w)^5}\partial\phi^{(7/2)}(w)\right.\\
&\qquad\qquad\qquad\left.\left.+\frac{3/616}{(z-w)^4}\partial^2\phi^{(7/2)}(w)+\frac{1/5544}{(z-w)^3}\partial^3\phi^{(7/2)}(w)\right\}\right].
\end{aligned}
\tag{3.25}
$$

The descendant field $\Psi(z)$ is apparently primary for $c = 22/21$. For $c = -19/6$ the excess operator $\Psi'(z) = \{7\partial(T\phi^{(7/2)}) - 11(T\partial\phi^{(7/2)}) + \frac{4}{9}\partial^3\phi(7/2)\}(w)$ is primary at the central charge, so that it can be removed by the same reason of the decoupling of the null state in the spin $5/2$ algebra. In other words, $\Psi(z)$ has the double structure as primary field. This is the reason why $\Psi(z)$ becomes primary for two values of central charges.

3.4 SPIN 4

This is the first example of including the conformal family of the primary field in the OPE (2.12). Using this advantage, we can make W_4-algebra associative for any value of central charge.

The $(L, \phi^{(4)}, \phi^{(4)})$ Jacobi identity derives the commutator of spin 4 current algebra:

$$
\begin{aligned}
&[\ \phi^{(4)}_m,\ \phi^{(4)}_n\ ] \\
&= \frac{c}{4\cdot 7!} m(m^2-1)(m^2-4)(m^2-9)\delta_{m+n,0} \\
&+(m-n)\Big[\frac{1}{1680}(3m^4-2m^3n+4m^2n^2-2mn^3+3n^4 \\
&\qquad -39m^2+20mn-39n^2+108)L_{m+n} \\
&\qquad +\frac{1}{28(5c+22)}(39m^2-20mn+39n^2+57m+57n-102)\Lambda_{m+n} \\
&\qquad +\frac{1}{(2c-1)(7c+68)}\Big\{-\frac{3}{20}(19c-524)\Xi_{m+n} \\
&\qquad\qquad +\frac{12(72c+13)}{5c+22}\Delta_{m+n}\Big\}\Big] \\
&+b\frac{14}{3(c+24)}(m-n)\Big[\frac{1}{336}\Big\{-3(c+24)(m^2+4mn+n^2+15m+15n+38) \\
&\qquad +(5c+64)(m+n+4)(m+n+5)\Big\}\phi^{(4)}_{m+n} \\
&\qquad +(T\phi^{(4)})_{m+n}\Big]\ .
\end{aligned}
\tag{3.26}
$$

Here we emphasize that b is left as free parameter just after requiring the ($L, \phi^{(4)}$, $\phi^{(4)}$) Jacobi identity.

First let us consider the case of $b = 0$. The direct calculation leads to

$$
\begin{aligned}
&[\ \phi^{(4)}_m,\ [\ \phi^{(4)}_n,\ \phi^{(4)}_p\ ]\ ] + \text{ cyclic permutation} \\
&= \frac{c^2-172c+196}{(2c-1)(7c+68)(5c+22)}(m-n)(n-p)(p-m) \\
&\quad \Big[21\Big\{2\partial(T\phi^{(4)})-3(T\partial\phi^{(4)})+\frac{1}{10}\partial^3\phi^{(4)}\Big\}_{m+n+p} \\
&\quad + (\text{ some totally symmetric polynomial in } m,\ n,\ p\ )\ \phi^{(4)}_{m+n+p}\Big]\ .
\end{aligned}
\tag{3.27}
$$

Although the descendant field $\{2\partial(T\phi^{(4)}) - 3(T\partial\phi^{(4)}) + \frac{1}{10}\partial^3\phi^{(4)}\}(z)$ is primary

for $c = -22/5$, we can not take the value since the coefficient of the commutator (3.26) has the pole in c at the value. Hence we conclude that the $(\phi^{(4)}, \phi^{(4)}, \phi^{(4)})$ Jacobi identity is satisfied if $c = 86 \pm 60\sqrt{2}$. It is a surprising accident that the r.h.s. is exactly canceled out if we choose the undetermined parameter b as

$$b^2 = \frac{54(c+24)(c^2 - 172c + 196)}{(2c-1)(7c+68)(5c+22)}. \tag{3.28}$$

Thus spin 4 algebra is always associative on account of the freedom of the descendent field of $\phi^{(4)}$ in OPE aalgebra.

4. Concluding Remarks

We have discussed the general structure of $\mathrm{W_n}$-algebra, in particular the Jacobi identities, which completely fix the algebra as expected. It has been an interesting fact that the identities are not operator identities, *i.e.*they contain null fields.

So far we have considered the extended algebra generated by a single primary field. We think of algebras including more primary fields. $\mathrm{N} = 1$ supersymmetrization of $\mathrm{W_n}$-algebra[8] is an example of such algebras. In superstring theories, space-time supersymmetry demands $\mathrm{N} = 2$ superconformal symmetry on world sheet. Some models of superstring theories are constructed by assembling several superconformal minimal models in order to equate the central charges to that of super-CFT describing a compactifying internal manifold, *i.e.*9. In hopes of unifying the minimal models into a single representation of an extended algebra, it is intended to generalize $\mathrm{W_n}$-algebra to possess $\mathrm{N} = 2$ superconformal symmetry.

Although various kinds of work related to $\mathrm{W_n}$-algebra, *e.g.*Goddard-Kent-Olive coset construction,[7] Feigen-Fuch's construction,[9] *etc.*are done, a lot of task still reminds.

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Realistic Superstring Models

— An Introduction —

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Abstract

Realistic models from the heterotic superstring theory are reviewed briefly. Special attention is paid to the discrete symmetries linked to the topological and geometrical structure of the Calabi-Yau compacified manifold. The relation between the geometrical approach and the algebraic approach is also discussed.

§1. Introduction

It is expected that the superstring theory is the most promising unified theory. The superstring theory may provide a unified framework not only for all known interactions but also for matter and space-time.[1),2)] To ascertain this, it is an important problem for us to explore the low-energy effective theory which follows from the superstring theory and to clarify relations between the characteristic structure of the low-energy effective theory and the superstring theory. And also we would like to make a step towards the confrontation

of supersting theory with experimental physics. We have already known some properties of the standard model which are required for the low- energy effective theory as "a realistic model". Now we list such properties of the standard model:

- gauge group $G \supset G_{st} = SU(3)_C \times SU(2)_W \times U(1)_Y$
- chiral fermions
- N=1 supersymmetry in four-dimension
- standard matter contents (quarks, leptons, Higgs doublets)
- generations
- proton stability
- quark-lepton mass hierarchies
- no flavor-changing neutral currents
- weak CP violation
- no strong CP violation
- supersymmetry breaking
- $SU(2) \times U(1)$ breaking

...

A model with these characteristics could be called a "realistic superstring model". At present it is far from clear that one can find a superstring model fulfilling all the requirements mentioned above. The superstring theory and low-energy effective theory are connected with each other through compactification. The topological and geometrical properties of the compactified manifold have an essential influence upon the low-energy effective theory.

The first attempt to look for a realistic superstring model was made through field theory limit of the $E_8 \times E_8'$ heterotic superstring.[3)] Then we have ten-dimensional supergravity theory coupled with super Yang-Mills fields. This theory is expected to reduce to a four-dimensional theory near the Planck scale by virtue of some comactification mechanism. Although the dynamics of this compactification is not yet understood, Calabi-Yau

compactification is thought as most attractive candidate. In fact, Calabi-Yau compactification leads to $N = 1$ supersymmetric GUT with gauge symmetry of E_6 or its subgroup. Since the compactification scale has to be of the order of the string size, the field theory limit does not seem to be a good approximation. Rather the compactification should be considered on the string itself. The orbifold compactification is one of the approaches to stringy construction.[4] The orbifold is an essentially flat manifold except for finite number of singularities on which topological and geometrical structures are concentrated. In this talk we confine ourselves to the Calabi-Yau compactification and mainly discuss the connection between the topological and geometrical properties of the compactified manifold and the low- energy effective theory.

In section 2, we study the relation between realistic gauge hierarchies and Wilson loop breaking. The topological structure of the compactified manifold determines the gauge symmetry, the generation number and available matter fields at the compactification scale. In section 3, phenomenological constraints for models are discussed and phenomenology is presented briefly. In section 4 it is shown how discrete symmetries on the compactified manifold give strong constraints on Yukawa couplings by taking the four-generation model as an example. The final section is devoted to the discussion about the relation between the algebraic approach and the geometrical approach. To seek a fully consistent "realistic model", it is useful to combine two approaches.

§2. Wilson loop breaking and realistic gauge hierarchies

After the Calabi-Yau compactification near the Planck scale, the theory brings about the $N = 1$ supersymmetric GUTs at lower energy.[3] The available gauge group G in this theory is E_6 for simply connected $K = K_0$ but becomes a subgroup of E_6 for multiply connected $K = K_0/G_d$, where G_d is some discrete symmetry. The available fields are restricted to zero modes on the compactified manifold K. In addition to vector superfields we have $N_f\mathbf{27} + \delta(\mathbf{27} + \mathbf{27}^*)$ chiral superfields as the zero mode spectra, where $\mathbf{27}$ is the fundamental representation of E_6 and N_f is the generation number of quarks and leptons. $N_f + \delta = h^{21}$ and $\delta = h^{11}$ are the Hodge numbers of the compactified manifold K.[3),5),6] In the Calabi-Yau compactified superstring theory four-dimensional massless fields are given by the coefficient functions of the six-dimensional harmonic form expansion of the ten-

dimensional fields. Thus we must consider the gauge hierarchy and also the low energy effective theory only within these ingredients which are determined by the topology of the compactified manifold K. This is in sharp contrast to the ordinary GUTs, in which the generation numbers and Higgs fields are introduced arbitrarily.

In the Calabi-Yau compactification the effective Higgs mechanism can be described in terms of the Wilson loop on the manifold K

$$U(\Gamma) = \mathrm{P} \exp\left(i \int_{\Gamma} A^{am} T^{a} dy_{m}\right), \tag{2.1}$$

where T^a are generators of E_6 and A^{am} are extra-dimensional components of the ten-dimensional vector field.[7),8),9)] If the manifold K is multiply connected, there exists the non-trivial $U(\Gamma) \neq 1$. Only in this case realistic gauge hierarchies are possibly realized.[5),6),8),9)] The non-trivial U composes of the discrete symmetry $\tilde{G}_d$, which is a homomorphic embedding of G_d into E_6. Then the available gauge group G is determined as $G = \{g \mid U \in \tilde{G}_d,\quad [g, U] = 0\}$.[10)] U can be rewritten as[11)]

$$U = \exp\ i\left(\sum_{i} z_i H_i + \sum_{\alpha} x_\alpha E_\alpha\right), \tag{2.2}$$

where H_i and E_α are generators of E_6 and

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \qquad [H_i, H_j] = 0. \tag{2.3}$$

The coefficients z_i and x_α represent real parameters corresponding to the zero root breaking and the non-zero root breaking, respectively. The case $Z \equiv \{z_i\} \neq 0$ and $N \equiv \{x_\alpha\} = 0$ means the multiply connected manifold $K = K_0/G_d$ with the abelian $\tilde{G}_d$, while $\{x_\alpha\} \neq 0$ implies the non-abelian $\tilde{G}_d$.

In order to get phenomenologically acceptable models, it is required that the group G contains the standard gauge group $G_{st} = SU(3)_C \times SU(2)_W \times U(1)_Y$. In the case of the zero root breaking in which the rank of G is six, we have

$$< \rho \mid U \mid \rho > = \exp\ i(Z, \rho) \tag{2.4}$$

with $Z = \alpha\Theta_1 + \beta\Theta_2 + \gamma\Theta_3$, where $\mid \rho >$ stands for the state with the weight ρ in E_6 and Z represents the breaking direction in the six-dimensional root space. Θ_i's (i=1,2,3) are

Table 1. The **78** vector multiplet decomposition under $G_{min} = SU(3)_C \times SU(2)_W \times U(1)^3$. The roots are represented in terms of the Dynkin label. Three types of the hypercharge assignment are taken as $U(1)^3$. The last column stands for the representation in the case Y_I with respect to $SU(2)'$ which breaks down in the rank five G.

	Roots ξ	G_{min}	Number of Fields	(Z,ξ)	$SU(2)'$
a	(000001)	$(8,1)_{0,0,0}$	8	0	(1)
b	(10001-1)	$(1,3)_{0,0,0}$	3	0	(1)
c	(000000)	$(1,1)_{0,0,0}$	3	0	(1), (3)
d	(01-1100)	$(3,2)_{-5,1,1}$	12	α	(1)
e	(00100-1)	$(3,2)_{1,-5,1}$	12	β	(2)
f	(010-110)	$(3,2)_{1,1,-5}$	12	γ	(2)
g	(10-11-11)	$(3,1)_{-2,4,4}$	6	$-(\beta+\gamma)$	(1)
h	(1-110-10)	$(3,1)_{4,-2,4}$	6	$-(\gamma+\alpha)$	(2)
i	(100-101)	$(3,1)_{4,4,-2}$	6	$-(\alpha+\beta)$	(2)
j	(2-10000)	$(1,2)_{3,3,3}$	4	$-(\alpha+\beta+\gamma)$	(1)
k	(01-1-111)	$(1,1)_{0,6,-6}$	2	$-(\beta-\gamma)$	(3)
l	(00-12-10)	$(1,1)_{-6,0,6}$	2	$-(\gamma-\alpha)$	(2)
m	(0-12-10-1)	$(1,1)_{6,-6,0}$	2	$-(\alpha-\beta)$	(2)

perpendicular to the non-zero roots of G_{st} and coincide with three independent weight in $\mathbf{27}^*$ which are neutral under $SU(3)_C \times SU(2)_W$. Parameters (α, β, γ) are treated up to modulus of the discrete symmetry $\bar{G}_d$.

In order to see the relation between the gauge group G and the breaking direction Z, in Table 1 we show the decomposition of **78** vector superfields of E_6 under $G_{min} \equiv SU(3)_C \times SU(2)_W \times U(1)^3$. For a given Z, the mass of **78** vector bosons is proportional

Table 2. The **27** chiral multiplet decomposition under G_{min} and quark/lepton assignments for the case Y_I. Non-trivial (N,ρ) in the non-zero root breaking is also given for the case Y_I.

	Weights ρ	G_{min}	(Z,ρ)	(N,ρ)	Field assignment (Y_{Ia})	(Y_{Ib})	(Y_{Ic})	(Y_{Id})
A	(100000)	$(3,2)_{1,1,1}$	$-\varepsilon$	0	Q_L	Q_L	Q_L	Q_L
B	(00-1101)	$(3*,1)_{-4,2,2}$	$\alpha-\varepsilon$	0	$\bar{u}_R$	$\bar{u}_R$	$\bar{u}_R$	$\bar{u}_R$
C	(0-11000)	$(3*,1)_{2,-4,2}$	$\beta-\varepsilon$	$-x$	$\bar{d}_R$	$\bar{d}_R$	$\bar{g}$	$\bar{g}$
D	(000-111)	$(3*,1)_{2,2,-4}$	$\gamma-\varepsilon$	$+x$	$\bar{g}$	$\bar{g}$	$\bar{g}$	$\bar{g}$
E	(-110000)	$(3,1)_{-2,-2,-2}$	2ε	0	g	g	g	g
F	(001-11-1)	$(1,2)_{3,-3,-3}$	$2\varepsilon-\alpha$	0	h	h	h	h
G	(01-1010)	$(1,2)_{-3,3,-3}$	$2\varepsilon-\beta$	$+x$	l_L	h'	l_L	h'
H	(00010-1)	$(1,2)_{-3,-3,3}$	$2\varepsilon-\gamma$	$-x$	h'	l_L	h'	l_L
I	(1-11-100)	$(1,1)_{6,0,0}$	$-\alpha-\varepsilon$	0	$\bar{e}_R$	$\bar{e}_R$	$\bar{e}_R$	$\bar{e}_R$
J	(10-1001)	$(1,1)_{0,6,0}$	$-\beta-\varepsilon$	$+x$	S_1	S_1	S_1	S_1
K	(1-101-10)	$(1,1)_{0,0,6}$	$-\gamma-\varepsilon$	$-x$	S_2	S_2	S_2	S_2

to (Z,ξ), where ξ is the root of the corresponding vector boson. The explicit form of (Z,ξ) is also shown in Table 1 for each root.

Furthermore, in Table 2 we show the decomposition of the **27** chiral multiplet under G_{min}. Here it is noted that we have three hypercharge assignments; $3Y_I \equiv -5\Theta_1 + \Theta_2 + \Theta_3$, $3Y_{II} \equiv \Theta_1 - 5\Theta_2 + \Theta_3$, $3Y_{III} \equiv \Theta_1 + \Theta_2 - 5\Theta_3$. Θ_i's (i=1,2,3) coincide with the weights $-I$, $-J$, $-K$ in Table 2, respectively and $(\Theta_i, \Theta_j) = \frac{1}{3} + \delta_{ij}$. In Table 1 three types of the hypercharge assignment are taken as $U(1)^3$. As seen from Table 1, if $\alpha \equiv \beta \equiv \gamma \equiv 0$ (modG_d) G becomes E_6. While, in the case that all of α, β, γ, $\alpha \pm \beta$, $\beta \pm \gamma$, $\gamma \pm \alpha$ and $\alpha + \beta + \gamma$ are non-zero under mod $\bar{G}_d$ and that $\bar{G}_d$ is an abelian discrete group,

G amounts to G_{min}. Thus the group G can be easily determined depending on α, β and γ under mod $\bar{G}_d$. The assignments of ordinary quarks and leptons are also presented in Table 2 for the case Y_I. The other cases are obtained by the cyclic permutation.

In the case of the nonabelian G_d, G_{st}-preserving non-zero roots are restricted to $\pm k$, $\pm l$ and $\pm m$ in Table 1 for Y_I, Y_{II} and Y_{III}, respectively. Since they compose non-zero roots of an $SU(2)'$ for each case, $SU(2)'$ in E_6 breaks down. Therefore, in the non-zero root breaking the rank of G is five and we get at most $G = SU(6)$. Among the **78** vector multiplet remnant massless fields are confined only to $SU(2)'$ singlet components, which are shown in the last column in Table 1 for the case Y_I. In the rank five case Z should be normal to the roots of $SU(2)'$ and we have $\beta \equiv \gamma$ (mod G_d) for Y_I and so on.

Now let us consider important constraints arising from the longevity of the proton. The **78** vector multiplet contains $(3,2)_{-5}$and $(3,2)_1$ leptoquarks which cause the proton to disintegrate, where $(3,2)_{-5,1}$ means $SU(3)_C$ triplet, $SU(2)_W$ doublet and $3Y = -5, 1$. Since there are no $(3,2)_{-5}$ leptoquarks in the **27** chiral superfield, $(3,2)_{-5}$ leptoquark vector bosons can not become massive by the Higgs mechanism with the chiral superfield. Therefore, unless $(3,2)_{-5}$ leptoquark vector bosons get masses at the compacification scale M_c, the proton life-time becomes too short.[8] As for $(3,2)_1$ leptoquarks in the **78** vector multiplet, they may become massive by the Higgs mechanism at the intermediate scale M_I. However, even if it were the case, the proton will decay too fast because the scale M_I is rather small compared with the unification scale of the standard GUTs. Then, $(3,2)_1$ leptoquarks in the **78** vector multiplet also should be massive at the scale M_c. Consequently we have the constraints $\alpha, \beta, \gamma \not\equiv 0$ (mod $\bar{G}_d$) for the rank six groups and $\alpha \not\equiv 0$ and $\beta \equiv \gamma$, for Y_I in the rank five groups. All the groups with $SO(10)$, $SU(6)$ and $SU(5)$ in which $SU(3)_C$ and $SU(2)_W$ are embedded together are ruled out. That is to say, the unification of $SU(3)_C$ and $SU(2)_W$ cannot occur at the energy scale below M_c.

In order to find the low energy effective theory, it is necessary to study zero modes in the $\delta(\mathbf{27} + \mathbf{27}^*)$ chiral multiplets. Generally, $\delta(\mathbf{27} + \mathbf{27}^*)$ chiral multiplets can get the masses of order M_c through the Yukawa coupling $\mathbf{27} \cdot \mathbf{27}^* \cdot \mathbf{78}$. If and only if $(Z, \rho) \equiv 0$ and $(N, \rho) \equiv 0$ (mod$\bar{G}_d$) for the components of **27** with the weight ρ, the corresponding chiral superfields remain massless. On the other hand, $N_f\mathbf{27}$ chiral superfields are all massless. We can easily find the relations between the discrete symmetry G_d and massless chiral superfields at M_c. If the symmetry breaking $G \to G'$ occurs at the intermediate scale M_I,

more stringent constraints should be satisfied, that is, the standard gauge group G_{st} and the supersymmetry should be preserved at M_I. If the supersymmetry breaks down at M_I, the naturalness problem cannot be solved. These contraints are satisfied if and only if G_{st}-neutral fields in $\delta(\mathbf{27}+\mathbf{27}^*)$ (S_1 in Table 2 and $\bar{S}_1$ in $\mathbf{27}^*$) appear as zero modes and if $< S_{16} >=< \bar{S}_{16} >= O(M_I)$. Otherwise, we have only the grand desert scenario. Only the grand desert scenario is allowed for the rank five G. In the rank six case, considering the zero mode conditions we can find the allowed symmetry breaking $G \to G'$ at $M_I$and the discrete symmetry G_d. In the superpotential there is no trilinear couplings of S_1 and S_1 in $(\mathbf{27}+\mathbf{27}^*)$ because of $U(1)$ charge. The quartic terms of S_1 and $\bar{S}_1$ appear effetively by the exchange of the E_6 superheavy gauge bosons. The superpotential leads to the intermediate scale $M_I \sim O(M_s M_c)^{1/2} \sim 10^{10}$GeV, where M_s stands for the supersymmetry breaking scale.[10),11)]

§3. Phenomenology

In §2 we discussed the phenomenological constraints on the gauge group G coming from the proton stability. However, the proton decay potentially occurs also through the Yukawa interactions. Here, let us consider the rank six case. In this case the most general Yukawa couplings which are invariant under G_{min} are

$$\begin{aligned} W =&\lambda_1 AAE + \lambda_2 ABF + \lambda_3 ACG + \lambda_4 ADH + \lambda_5 BCD + \lambda_6 EBI \\ &+\lambda_7 ECJ + \lambda_8 EDK + \lambda_9 FGK + \lambda_{10} HFJ + \lambda_{11} GHI, \end{aligned} \tag{3.1}$$

where superfields A to K are given in Table 2 for the case Y_I. For each assignment of quarks and leptons, we write down each term of Eq.(3.1) in Table 3.

In the group G larger than G_{min} we have some relations among λ_1 to λ_{11}. From Table 3 we find that there are ordinary quark/lepton mass terms and at the same time some phenomenologically dangerous terms. The coexistence of QQg and $Ql\bar{g}$ or $\bar{g}\bar{u}\bar{d}$ and $g\bar{u}\bar{e}$ leads to the g-quark exchange proton decay processes $O(\lambda\lambda/M_g^2)(QQQl)$ and $O(\lambda\lambda/M_g^2)(\bar{u}\bar{u}\bar{d}\bar{e})$ with $M_g < \lambda M_I$. These processes make th e proton unstable because of $M_I \sim 10^{10}$ GeV. Thus we must prohibit at least the coexistence of these terms in the superpotential W.[11)] Furthermore, as seen in Table 3 in the cases Y_{Ib} and Y_{Ic} the quark/lepton mass terms exist, provided that $< h >$ and $< h' >$ have non-zero vacuum expectation values as in

Table 3. The contents of the Yukawa couplings for the case Y_I.

	Term	(Y_{Ia})	(Y_{Ib})	(Y_{Ic})	(Y_{Id})
λ_1	AAE	QQg	QQg	QQg	QQg
λ_2	ABF	$Q\bar{u}h$	Quh	$Q\bar{u}h$	$Q\bar{u}h$
λ_3	ACG	Qdl	$Q\bar{d}h'$	$Q\bar{g}l$	$Q\bar{g}h'$
λ_4	ADH	$Q\bar{g}h'$	$Q\bar{d}l$	$Q\bar{d}h'$	$Q\bar{d}l$
λ_5	BCD	$\bar{u}d\bar{g}$	$\bar{u}\bar{d}\bar{g}$	$\bar{u}\bar{d}\bar{g}$	$\bar{u}\bar{d}\bar{d}$
λ_6	BEI	$\bar{u}g\bar{e}$	$\bar{u}g\bar{e}$	$\bar{u}g\bar{e}$	$\bar{u}g\bar{e}$
λ_7	CEJ	$\bar{d}gS_1$	$\bar{d}gS_1$	$\bar{g}gS_1$	$\bar{g}gS_1$
λ_8	DEK	$\bar{g}gS_2$	$\bar{g}gS_2$	$\bar{d}gS_2$	$\bar{d}gS_2$
λ_9	FGK	lhS_2	$hh'S_2$	lhS_2	$hh'S_2$
λ_{10}	FHJ	$hh'S_1$	lhS_1	$hh'S_1$	lhS_1
λ_{11}	GHI	$lh'\bar{e}$	$lh'\bar{e}$	$lh'\bar{e}$	$lh'\bar{e}$

the standard model. While, in the cases Y_{Ia} and Y_{Id} there is no direct d-quark mass term although the g-quark can be massive. Although the d-quark can be massive through the $g-d$ mixing, it is impossible to give the appropriate mass by the diagonalization without unnatural assumption for the Yukawa coupling. Consequently, the Yukawa couplings in the case Y_I must satisfy the following conditions;

$$\begin{aligned} Y_{Ib} : \lambda_1 = \lambda_5 = 0 \quad \text{or} \quad \lambda_4 = \lambda_6 = 0, \\ \lambda_2, \lambda_3, \lambda_{11} \neq 0 \end{aligned} \tag{3.2}$$

$$\begin{aligned} Y_{Ic} : \lambda_1 = \lambda_5 = 0 \quad \text{or} \quad \lambda_3 = \lambda_6 = 0, \\ \lambda_4, \lambda_2, \lambda_{11} \neq 0, \end{aligned} \tag{3.3}$$

The conditions in the cases Y_{II} and Y_{III} are also obtained by the cyclic permutation of $(\lambda_2,\lambda_3,\lambda_4)$, $(\lambda_6,\lambda_7,\lambda_8)$ and $(\lambda_{11},\lambda_{10},\lambda_9)$.

Next we discuss the neutrino mass. For the cases $Y_{Ib}$and Y_{Ic} we have neutrino Dirac-mass terms $\lambda_{10} lhS_1$ and $\lambda_9 lhS_2$, respectively. To keep neutrinos massless or light, it is

necessary for us to have $\lambda_{10} = 0$ for the case Y_{Ib} and $\lambda_9 = 0$ for the case Y_{Ic}. The above conditions severely constrain the group G. To explain the constraints it is necessary to introduce some discrete symmetries which might be related to the topological and geometrical structure of the compactified manifold.

The realistic models are classified into four types of G as follows,[10),11)]

$SU(4)_c \times SU(2)_W \times U(1)^2$,

$SU(3)_C \times SU(2)_W \times SU(2)' \times U(1)^2$,

$SU(3)_C \times SU(2)_W \times U(1)^3$,

$SU(3)_C \times SU(2)_W \times U(1)^2$.

In the intermediate scale scenario G' is $G'_{min} = SU(3)_C \times SU(2)_W \times U(1)^2$ or $SU(3)_C \times SU(2)_W \times SU(2)' \times U(1)$, provided that we take the conditions coming from the proton stability and the quark/lepton mass generation. The discrete symmetry $\bar{G}_d$ is also found for each case. Therefore, we can constrain the topological structure of the compactified manifold phenomenologically.

Although superstring theory is the theory at Planck scale, in the effective theory at TeV region there are remnants of the compactification at Planck scale. If these effects are found in the experimental discrepancy from the standard model, it may be considered as an indirect evidence that nature selects the higher dimensional superstring scenario.

In all the realistic cases we have at least an extra $U(1)$ gauge symmetry above the weak scale. But it should be noted that the extra $U(1)_{Y'}$ in G'_{min} for the intermediate scale scenario differs from the extra $U(1)_{Y''}$ in $G = SU(3)_C \times SU(2)_W \times U(1)^2$ for the grand desert scenario with rank five. An extra $U(1)$ phenomenology at low energies is important to check the reality of the $E_8 \times E'_8$ superstring theory. The low energy effects of an extra $U(1)$ vector boson have been investigated.[12)] The renormalization group analysis was made for U(1) gauge coupling constants, taking account of the abelian kinetic term mixing.[13),14)] In the $E_8 \times E'_8$ superstring theory there emerges the mixing in the abelian kinetic terms which arise from the anomaly cancellation term $H_{PMN}H^{PMN}$ in the ten-dimensional Lagrangian and from possible existence of the extra heavy fields g, $\bar{g}$, h and h'. Phenomenologically acceptable solutions have been found.

§ 4. Discrete symmetries and Yukawa couplings[15]

In the Calabi-Yau compactified $E_8 \times E_8'$ superstring theory, the compactified six-dimensional manifold K is considered to be multiply connected i.e. $\Pi_1(K) = G_d(K = K_0/G_d$ and K_0 is a simply connected manifold) so that we obtain a realistic gauge group and a relatively small generation number. If this is the case, due to the Wilson-loop breaking mechanism the unbroken gauge group $G(\subset E_6)$ can be determined such that $G = \{\ g \mid {}^\forall U \in \hat{G}_d,\quad [g,U] = 0\}$, where $\hat{G}_d$ is the embedding of G_d into E_6 by a homomorphic mapping. Phenomenologically viable gauge groups G were systematically studied together with the $\hat{G}_d$ for each G and with the $\hat{G}_d$ charge assignments for the fields[9].

In general on the manifold K there can exist some discrete symmetries other than G_d. Such a symmetry H, which is also the symmetry of K_0, is given by $H = \{\ h \mid {}^\forall g \in G_d,\quad {}^\exists g' \in G_d \quad s.t.\quad hgh^{-1} = g'\}$ [5]. If such an H exists, it may naturally forbid the unfavorable Yukawa couplings $\mathbf{27} \otimes \mathbf{27} \otimes \mathbf{27}$. To see this it is necessary to remind how Yukawa couplings are introduced in the theory. In the Calabi- Yau compactification 10-dimensional fields $\psi(\omega)$ and $A_M(\omega)$ (**248** in E_8) can be expanded in terms of the 6-dimensional harmonic forms $\psi_i(y)$, $\phi_i(y)$ and $A_{mi}(y)$ as

$$\begin{aligned}\psi(\omega) &= \sum_i \psi^i(x)\psi_i(y),\\ A_M(\omega) &= \sum_i A^i_\mu(x)\phi_i(y)\delta_{M\mu} + \sum_j \phi^j(x)A_{mj}(y)\delta_{Mm},\end{aligned} \tag{4.1}$$

where x and y are 4-dimensional and 6-dimensional coordinates, respectively.[6),16] From 10-dimensional interaction terms

$$L_I = \tilde{g}\int d\omega\sqrt{-g^{10}}\bar{\psi}(\omega)\gamma^M A_M(\omega)\psi(\omega), \tag{4.2}$$

we derive the Yukawa interaction

$$L_Y = \lambda_{ijk}\int d^4x\sqrt{-g^4}\bar{\psi}^i(x)\phi^j(x)\psi^k(x). \tag{4.3}$$

Using the property of Calabi-Yau manifold, Yukawa coupling constant is expressed as[10,11]

$$\begin{aligned}\lambda_{ijk} &= \tilde{g}\int_K \sqrt{g^6}\psi^\dagger(z)^a_i\gamma^m A(z)^b_{mj}\psi(z)^c_k\epsilon_{abc},\\ &= \tilde{g}\int_K \omega\wedge A^a_i\wedge A^b_j\wedge A^c_k\epsilon_{abc},\end{aligned} \tag{4.4}$$

where we use complex coordiante z instead of real coordinate y, and $A_i^a = A_{\bar{m}i}^a d\bar{z}^{\bar{m}}$ is the i-th one form with its value on the anti-holomorphic tangent bundle T. An index a labels the **3** of $SU(3)$ so that A_i corresponds to **27** of E_6. ω is the holomorphic three form constructed by using the covariantly constant spinor such that $\omega_{mnp} = \eta^T \gamma_m \gamma_n \gamma_p \eta$. The knowledge how A_i transforms under H can tell us how the Yukawa couplings are restricted due to the requirement of H invariance. As a mathematical theorem it is known that the elements A_i of $H^1(T)$ are represented by the linearly independent deformation of the manifold[17,18]. From this fact (2,1)-forms $\phi_i(z)$ have one-to-one correspondence to the polynomials $q_i(z)$; $\phi_i(z) \simeq A_i(z) \simeq q_i(z)$. This shows that we can readily read off the transformation property of $\phi_i(z)$ under H. Now we consider the manifold K which is defined as the hypersurface in the complex 4-dimensional space constrained by the defining polynomial $P(z)$. Using the defining polynomial $P(z)$ of the manifold K, we can deform the polynomials $q_i(z)$ without changing the Yukawa couplings

$$\tilde{q}_i(z) \simeq q_i(z) + C_b^a z_a \frac{\partial P(z)}{\partial z_b}, \tag{4.5}$$

where C_b^a is the element of $GL(5, C)$. Therefore, we can calculate the Yukawa coupling constants (4.4) as the H invariant polynomial integrals and even see relations between them. Of course in this calculation the kinetic term normalization must be taken into account. This calculation can be also made in the same manner. The above arguments are appropriate only for (2,1)-forms or **27** of E_6 but not for (1,1)-forms or **27***. In general for (1,1)-form we need more complicated analysis[17].

Now we proceed to the analysis of the realistic models according to the above scenario. As mentioned above, the low energy group structure can be determined only through $\bar{G}_d$ but not through the manifold. Whereas, to get the generation number we must fix the manifold. Up to now some three and four generation models are known[3,19]. In this section we analyze a four generation model with $h^{1,1} = 1$ listed in ref.3. Comparison between G_d of these manifolds and $\bar{G}_d$ suggests that $K_0 = Y(4;5)$ with $G_d = Z_5 \times Z_5'$ is the most interesting one. Therefore, in the following analysis we confine ourselves to the manifold $K = K_0/G_d = Y(4;5)/Z_5 \times Z_5'$[3]. $Y(4;5)$ is the manifold defined as the zeros of the fifth-order polynomials in the CP^4. The discrete symmetries of the manifold K depend

on the selection of the defining polynomials $P(z)$. Here we take

$$P(z) = \frac{1}{5}\sum_{i=1}^{5} z_i^5 - cz_1z_2z_3z_4z_5 = 0, \tag{4.6}$$

where z_i ($i = 1 \sim 5$) are the local coordinates of CP^4 and c is a complex parameter. Then $P(z)$ has the freely acting discrete symmetries,

$$S; \quad z_i \to z_{i+1}, \tag{4.7}$$

$$T; \quad z_i \to \alpha^i z_i \qquad (\alpha^5 = 1) \tag{4.8}$$

for $c \neq 1$ and obviously $S^5 = T^5 = 1$ and $TS = \alpha ST$. We use S and T to construct K. The remaining symmetries of K are generated by

$$B; \quad z_i \to \alpha^{2i^2} z_i, \tag{4.9}$$

$$Y; \quad z_i \to z_{2i}. \tag{4.10}$$

These satisfy B^5=1 and Y^4=1.

In the manifold K_0 cohomology groups $H^{2,1}$ and $H^{1,1}$ compose of 101 elements and one, respectively. On the other hand, most generally there are 126 fifth-order monomials

$$z_1z_2z_3z_4z_5, \quad z_i^5, \quad z_i^3z_j^2(i \neq j), \quad z_i^3z_jz_k(i \neq j \neq k),$$
$$z_i^4z_j(i \neq j), \quad z_i^2z_j^2z_k(i \neq j \neq k), \quad z_i^2z_jz_kz_l(i \neq j \neq k \neq l),$$

but these are not independent. Using eq's(4.5) and (4.6) $z_i^4z_j$ and z_i^5 are related to $z_i^2z_jz_kz_l$ and $z_1z_2z_3z_4z_5$, respectively. Therefore we have 101 independent monomials which are just corresponding to $h^{2,1}(K_0) = 101$. On the manifold K which is constructed as the quotient space by $Z_5 \times Z_5'$, we have to construct the fifth-order polynomials as linear combinations of the above 101 monomials which are eigenstates with respect to S and T. We denote the polynomials as $T_{nm}^{(i)}$ which have the (S,T)- eigenvalues $(\alpha^n, \alpha^m)$and their definitions are given in Table 4. Five polynomials $T_{00}^{(i)}(i = 0 \sim 4)$ correspond to the five independent (2,1)-forms on the manifold $K = K_0/G_d$.

Transformation properties of the polynomials $T_{nm}^{(i)}$ under B and Y can be readily found also in Table 4. It is worthy to note that the polynomials $T_{nm}^{(i)}$ are classified into

Table 4. Definitions and transformation properties of the polynomials

	polynomials	B	Y
$T^{(0)}_{00}$	$z_1z_2z_3z_4z_5$	1	1
$T^{(1)}_{n0}$	$\sum_{i=1}^5 \alpha^{-ni} z_i^3 z_{i+2} z_{i-2}$	α	$\rightarrow T^{(4)}_{3n,0}$
$T^{(2)}_{n0}$	$\sum_{i=1}^5 \alpha^{-ni} z_{i+2}^2 z_{i-2}^2 z_i$	α^2	$\rightarrow T^{(3)}_{3n,0}$
$T^{(3)}_{n0}$	$\sum_{i=1}^5 \alpha^{-ni} z_{i+1}^2 z_{i-1}^2 z_i$	α^3	$\rightarrow T^{(2)}_{3n,0}$
$T^{(4)}_{n0}$	$\sum_{i=1}^5 \alpha^{-ni} z_i^3 z_{i+1} z_{i-1}$	α^4	$\rightarrow T^{(1)}_{3n,0}$
$T^{(1)}_{nm}$	$\sum_{i=1}^5 \alpha^{-ni} z_i^3 z_{i+3m}^2$	$\rightarrow \alpha^{\hat{m}_1} T^{(1)}_{n+m,m}$	$\rightarrow T^{(1)}_{3n,2m}$
$T^{(2)}_{nm}$	$\sum_{i=1}^5 \alpha^{-ni} z_i^3 z_{i+2m} z_{i-m}$	$\rightarrow \alpha^{\hat{m}_2} T^{(2)}_{n+m,m}$	$\rightarrow T^{(2)}_{3n,2m}$
$T^{(3)}_{nm}$	$\sum_{i=1}^5 \alpha^{-ni} z_i^2 z_{i+m}^2 z_{i-m}$	$\rightarrow \alpha^{\hat{m}_3} T^{(3)}_{n+m,m}$	$\rightarrow T^{(3)}_{3n,2m}$
$T^{(4)}_{nm}$	$\sum_{i=1}^5 \alpha^{-ni} z_i^2 z_{i+m} z_{i+2m} z_{i+3m}$	$\rightarrow \alpha^{\hat{m}_4} T^{(4)}_{n+m,m}$	$\rightarrow T^{(4)}_{3n,2m}$

$\hat{m}_1 = \hat{m}_3 = m(m-1)$
$\hat{m}_2 = -m$
$\hat{m}_4 = (3m-1)m$

four categories according to B and Y transformation property as

$$
\begin{array}{lcl}
T^{(0)}_{00} & ; & \text{(B,Y)- singlet,} \\
T^{(i)}_{00}\ (i = 1 \sim 4) & ; & \text{(B,Y)- doublet,} \\
T^{(i)}_{n0}\ (i, n = 1 \sim 4) & ; & \text{(B,Y)- quartet,} \\
T^{(i)}_{nm}\ \left({}^{i,m=1\sim 4}_{n=0\sim 4}\right) & ; & \text{(B,Y)- 20-plet.}
\end{array}
$$

It will be found that these properties severely restrict the phenomenologically consistent embedding of G_d into E_6 as seen later. The Yukawa coupling constants are given by the integrals over K of the products of three polynomials $T^{(i)}_{ll'} T^{(j)}_{mm'} T^{(k)}_{nn'}$. After integrating over

K, only (S,T) invariant products give non-vanishing value. (S,T) invariance requires the conditions

$$l+m+n \equiv 0 \quad (mod\, 5), \tag{4.11}$$

$$l'+m'+n' \equiv 0 \quad (mod\, 5). \tag{4.12}$$

Next we study the relation between the polynomials $T^{(i)}_{nm}$ and the **27** fields of E_6. To do this, we must remind the Wilson-loop breaking mechanism, that is, the homomorphic embedding of $G_d = Z_5(S) \times Z_5'(T)$ into E_6,

$$S \rightarrow U_S \in \bar{G}_d \subset E_6,$$

$$T \rightarrow U_T \in \bar{G}_d \subset E_6.$$

Here homomorphism requires $U_g U_{g'} = U_{gg'}$ for any $g, g' \in G_d$. In Ref. 11, for all the possible $\bar{G}_d$ we exhausted gauge group structures coming from the Wilson-loop mechanism and the $\bar{G}_d$ charge of the **27** fields. Then it is sufficient for us only to select the models which allow the homomorphic embedding $G_d \rightarrow \bar{G}_d$ and we can relate the polynomials to the **27** fields automatically. Naive consideration leads to the three independent embedding schemes (i) $Z_5 \times Z_5' \rightarrow Z_5 \times Z_5'$ (ii) $Z_5 \times Z_5' \rightarrow Z_5$ (iii) $Z_5 \times Z_5' \rightarrow Z_5'$. Phenomenologically, however, the cases (i) and (iii) are forbidden At the compactification scale the standard gauge group $SU(3)_C \times SU(2)_W \times U(1)_Y$ is unbroken and under this group **27** is decomposed into eleven multiplets. On the other hand, $T^{(i)}_{nm} (m \neq 0)$ forms (B,Y)-20-plet and then only the eleven physical multiplets can not compose of (B,Y) invariant Yukawa couplings. For this reason only the embedding (ii) is allowed. In this embedding scheme the allowed gauge structures potentially with the intermediate mass scale are

$$G = SU(3)_C \times SU(2)_W \times SU(2)' \times U(1)^2,$$

$$G = SU(4)_C \times SU(2)_W \times U(1)^2$$

at the compactification scale. The relations between the polynomials $T^{(i)}_{n0}$ and **27** fields are given in Table 5 for each case. It is evident from the above construction that in the expression of $T^{(i)}_{n0}$ the index i plays a role of the generation index so that we can calculate the generation dependent Yukawa coupling constants. In fact, (B,Y) invariance requires

$$i+j+k \equiv 0 \quad (mod\, 5). \tag{4.13}$$

Table 5. Relations of Polynomials and fields

discrete charge Z_5	polynomials $T^{(i)}_{n0}$	field assignments		
$G = SU(3)_C \times SU(2)_W \times SU(2)' \times U(1)^2$		III(b)	III(c)	
0	$T^{(i)}_{00}$	$(l,h)(S_1)$	$(h',h)(S_1)$	
1	$T^{(i)}_{10}$	$(\bar{g},\bar{u})$	$(\bar{d},\bar{u})$	
2	$T^{(i)}_{20}$	$(g),(S_2,\bar{e})$	$(g),(S_2,\bar{e})$	
3	$T^{(i)}_{30}$	$(\bar{d})(h')$	$(\bar{g})(l)$	
4	$T^{(i)}_{40}$	(Q)	(Q)	
$G = SU(4)_C \times SU(2)_W \times U(1)^2$		I(b)	I(c)	II(b)
0	$T^{(i)}_{00}$	$(\bar{g},S_1)$	$(\bar{d},S_1)$	$(\bar{u},S_1)$
1	$T^{(i)}_{10}$	$(l)(\bar{e})$	$(h')(\bar{e})$	$(h)(S_2)$
2	$T^{(i)}_{20}$	$(\bar{u},g)$	$(\bar{u},g)$	$(\bar{g},g)$
3	$T^{(i)}_{30}$	$(\bar{d},S_2)(h')$	$(\bar{g},S_2)(l)$	$(\bar{d},\bar{e})(h')$
4	$T^{(i)}_{40}$	(Q,h)	(Q,h)	(Q,l)

We are now in a position to evaluate Yukawa couplings, which are given by integrals of the products of three polynomials over the compactified manifold K

$$\int_K T^{(i)}_{n0}\, T^{(j)}_{m0}\, T^{(k)}_{l0}. \tag{4.14}$$

As mentioned above, the global symmetry on the manifold K gives strong constraints on the Yukawa coupling (4.14). In fact, we get a non-zero Yukawa coupling only when $i+j+k \equiv 0$ and $n+m+l \equiv 0 \pmod 5$ which are consequences of B and $\tilde{G}_d(S)$ invariance. Here we can derive further restrictions on the Yukawa couplings (4.14) which are given by an integral of a certain fifteenth-order polynomials over the manifold. A valuable

proposition at this stage is that at tree amplitude in the string theory the integrals have the same value whether the theory is formulated on K_0 or $K = K_0/G_d$[5]. By virtue of this fact, we can replace the integration over K by the one over K_0, and then we can make full use of symmetries on K_0, which do not necessarily exist on K. One of such symmetries is

$$z_i \to \alpha^{n_i} z_i, \qquad \text{with} \quad \Sigma n_i \equiv 0 \pmod 5, \tag{4.15}$$

which retains the defining polynomial invariant. All of the monomials in expansion of the fifteenth-order polynomials $T^{(i)}_{n0} T^{(j)}_{m0} T^{(k)}_{l0}$ are not invariant under (4.15) except for four types of monomials. Only fifteenth-order monomials which are invariant under the above transformation contribute to the integral. Non-vanishing terms are restricted to the following ones,

$$\begin{aligned} &(z_1 z_2 z_3 z_4 z_5)^3, \\ &(z_1 z_2 z_3 z_4 z_5)^2 z_i^5, \\ &(z_1 z_2 z_3 z_4 z_5) z_i^5 z_j^5, \\ &z_i^5 z_j^5 z_k^5. \end{aligned} \tag{4.16}$$

Since we can deform fifth-order polynomials by eq.(4.5), we may replace z_i^5 by $c(z_1 z_2 z_3 z_4 z_5)$. Thus each integral is expressed only by two parameters c and μ as listed in the Table 6[18].

Table 6. Contributions of fifteenth-order monomials

monomials	integrals over K
$(z_1 z_2 z_3 z_4 z_5)^3$	μ
$(z_1 z_2 z_3 z_4 z_5)^2 z_i^5$	$c\mu$
$(z_1 z_2 z_3 z_4 z_5) z_i^5 z_j^5$	$c^2\mu$
$z_i^5 z_j^5 z_k^5$	$c^3\mu$

Table 7. Values of Yukawa couplings

Types of coupling		value $(c \neq 1)$	notation
$T_0^{(0)}T_0^{(0)}T_0^{(0)}$		$\mu/5$	o
$T_0^{(0)}T_n^{(2)}T_m^{(3)}$		$\mu\xi^2$	p
$T_0^{(0)}T_n^{(1)}T_m^{(4)}$		$c\mu\zeta^2$	q
$T_n^{(3)}T_n^{(3)}T_l^{(4)}$ $T_n^{(2)}T_n^{(2)}T_l^{(1)}$		$(2+2c+c^3)\mu\xi^2\zeta$	r_0
$T_n^{(3)}T_m^{(3)}T_l^{(4)}$ $T_n^{(2)}T_m^{(2)}T_l^{(1)}$	$\|n-m\|=1$ $\|n-m\|=2$	$\{(\alpha^{-1}+\alpha)+(\alpha^{-2}+\alpha^2)c+c^3\}\mu\xi^2\zeta$	r_1
$T_n^{(3)}T_m^{(3)}T_l^{(4)}$ $T_n^{(2)}T_m^{(2)}T_l^{(1)}$	$\|n-m\|=2$ $\|n-m\|=1$	$\{(\alpha^{-2}+\alpha^2)+(\alpha^{-1}+\alpha)c+c^3\}\mu\xi^2\zeta$	r_2
$T_n^{(1)}T_n^{(1)}T_l^{(3)}$ $T_n^{(4)}T_n^{(4)}T_l^{(2)}$		$(2+c+2c^2)\mu\xi\zeta^2$	s_0
$T_n^{(1)}T_m^{(1)}T_l^{(3)}$ $T_n^{(4)}T_m^{(4)}T_l^{(2)}$	$\|n-m\|=1$ $\|n-m\|=2$	$\{(\alpha^{-1}+\alpha)+c+(\alpha^{-2}+\alpha^2)c^2\}\mu\xi\zeta^2$	s_1
$T_n^{(1)}T_m^{(1)}T_l^{(3)}$ $T_n^{(4)}T_m^{(4)}T_l^{(2)}$	$\|n-m\|=2$ $\|n-m\|=1$	$\{(\alpha^{-2}+\alpha^2)+c+(\alpha^{-1}+\alpha)c^2\}\mu\xi\zeta^2$	s_2

In order to connect the integrals (4.14) with the physical Yukawa couplings, we must use orthonormal bases such that

$$\int_K \phi_i^*(z)\,\phi_j(z) = \delta_{ij} \tag{4.17}$$

for any two harmonic forms $\phi_i(z)$ and $\phi_j(z)$ on the manifold K. Fortunately the orthogonality of the bases $T_n^{(i)}$'s (hereafter $T_{n0}^{(i)}$ is abbreviated to $T_n^{(i)}$) with respect to the index n

and i is guaranteed by $\bar{G}_d$ and B symmetry, respectively. Furthermore it is worthy to note that the normalization constant of $T_n^{(i)\prime}$s is independent of n by virtue of the symmetry (4.15) and the normalizations for the index $i = 1,4(2,3)$ are the same by Y symmetry. Taking account of these facts only three independent normalization constants are left and we introduce the parameters

$$
\begin{aligned}
\|T_0^{(0)}\| \;/\; \|T_n^{(1,4)}\| &\equiv \zeta, \\
\|T_0^{(0)}\| \;/\; \|T_n^{(2,3)}\| &\equiv \xi.
\end{aligned}
\tag{4.18}
$$

Using Table 6 and eq.(4.18), properly normalized Yukawa couplings are expressed in terms of only three real parameters μ, ζ, ξ and one complex parameter c as in Table 7.

§5. Algebraic approach and geometrical approach

In the preceding sections, we discuss the Calabi-Yau compactification based on the geometrical approach. There are a vast number of Calabi-Yau manifolds and most of them are very complicated manifolds with unknown metrics. If we take one of them, the manifold provides a solution of the classical string equation of motion. However, in order to keep the theory to be consistent, we have to redefine the manifld metric order by order in the world-sheet σ-model.[20] This means that the compactification to an arbitrary Calabi-Yau manifold does not always lead to a fully consistent string theory. We need some further constraints on the manifold to get a consistent string theory. Then, the question arises as to what constraint is required to obtain a consistent string theory.

Recently Gepner proposed the new space-time supersymmetric compactification which is fully consistent as a string theory.[21] This new compactification is represented as a tensor product of $N = 2$ minimal superconformal models with a trace anomaly $c = 3k/(k+2)$ ($k = 1, 2, 3, \cdots$). Since the critical dimension of superstring theories is $D = 10$, we have the trace anomaly $c = 12$. The internal (non-space-time) degrees of freedom are described by combining minimal superconformal models so as to make the correct trace anomaly

$$
c = \sum_{i=1}^{r} \frac{3k_i}{k_i + 2} = 9. \tag{5.1}
$$

In this tensoring of minimal models we have to eliminate all states with $U(1)$ charges which

are not odd integers (the generalized GSO projection). There are 168 different models which satisfy the condition (5.1). For two of them Gepner and others gave the geometrical interpretation of non-geometrical compactifications based on the algebraic approach. The one is the 3^5-model which corresponds to the $Y(4;5)$ manifold with the defining polynomial (4.6) and $c = 0$.[22] The 3^5-model and this Calabi-Yau manifold have the same discrete symmetry $S_5 \times Z_5^5/Z_5$. As shown in §4, on the quotient manifold $Y(4;5)/Z_5 \times Z_5'$ we have a four-generation model.[15] The other is $1^1 \cdot 16^3$-model which corresponds to the hypersurface in the $CP^3 \times CP^2$ constrained by the defining polynomials

$$
\begin{aligned}
P_1 &= z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0, \\
P_2 &= z_1 x_1^3 + z_2 x_2^3 + z_3 x_3^3 = 0,
\end{aligned}
\tag{5.2}
$$

where $\{z_0, z_1, z_2, z_3\} \in CP^3$ and $\{x_1, x_2, x_3\} \in CP^2$. In this case $1^1 \cdot 16^3$- model and the Calabi-Yau manifold both have the discrete symmetry $S_3 \times (Z_3 \times Z_9^3)/Z_9$. On the Calabi-Yau manifold devided by $Z_3 \times Z_3'$, we obtain a three-generation model.[23]

On the Calabi-Yau manifold which has a correspondence with a fully consistent algebraic theory, it is not necessary to redefine the manifold metric order by order in the world-sheet σ- model. In other words, a fully consistent theory sits on a fixed point on the renormalization scheme in the parameter space, which represents degrees of freedom of deforming the manifold. When we choose a Calabi-Yau manifold with certain Hodge number h^{21} and h^{11}, we can continuously deform this manifold without changing topological structure (for instance, generation number $N_f = h^{21} - h^{11}$ remain unchanged). These manifolds obtained thus are topologically equivalent and diffeomorphic to each other but have different complex structures. The diffeomorphic parameter space contains the h^{21}-dimensional parameter subspace, in which each point represents the hypersurface given by the defining polynomial with each set of parameter values. If we require the larger discrete symmetry for the manifold, the smaller subspace in the h^{21}-dimensional parameter space is selected out. Gepner's analysis suggests that the Calabi-Yau compactifications are fully consistent only in a limited subspace which posses a specific kind of discrete symmetries. It is important to study some kind of phase diagram in the parameter space and to explore fixed points in conjunction with discrete symmetries. In order to seek a "realistic" compactification, it seems to be very efficient to combine the geometrical approach and the algebraic approach.

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Quantum Gravity and Cosmological Constant†)

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ABSTRACT

Recent development of wormhole instanton physics is reviewed with an emphasis on its analogy to string theories.

§1. Introduction

Quantum gravity becomes relevant only in the extremely small scales of the order of the Planck length $\ell_p \sim 10^{-33}$ cm, or the Planck time $t_p \sim 10^{-43}$ sec. So it has long been thought that the very early universe may be the only (if at all) application of it. Some people talk about the wave-function of unvierse, which may give initial conditions to the classical cosmology and hopefully some clues to the large scale structure of the universe.[1]

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Very recently, it seems, the landscape has drastically changed since Colemsn's breakthrough paper on the vanishing cosmological constant.[2] The observed cosmological constant Λ (I define it as the vacuum energy rather than the traditional one) is vanishingly small,

$$\ell_p^4 \Lambda < (\frac{\text{Planck radius}}{\text{Hubble radius}})^2 \approx 10^{-120}. \tag{1.1}$$

Evidently, the right-hand side is the square of the ratio of the smallest scale to the largest scale in physics. In order to explain this number, we obviously need a mechanism which intimately relates the ultraviolet and infrared phenomena.[3] On the basis of Hawking's earlier works on the Euclidean quantum gravity,[4] wormholes[5] and de Sitter instanton,[6] Coleman argued that wormhole instantons do the job.

As briefly reviewed in §2, the radius of the wormhole instanton is typically $r_b \sim \sqrt{L\ell_p}$ with L being the characteristic length scale of the theory (e.g. $L \sim 10^{-13}$ cm for QCD). So its size is microscopic. However, its effect is non-local. The wormhole can go out of any point in space-time and enter any place in the universe and even in other universes.

We are going to explain how the wormhole instanton contributes to the Euclidean path-integral with a particular emphasis on the analogy to the Polyakov integral in the string theories (§3). §4 is a review of Coleman's explanation of the vanishing cosmological constant. §5 is devoted to discussions and summary.

§2. Wormhole Instantons

Giddings and Strominger[7] presented a simple (superstring motivated) model which has wormhole instanton solutions. The Lagrangian is given by

$$L = -\frac{1}{16\pi G}R + f^2 H^2_{\mu\nu\lambda}. \tag{2.1}$$

The first term is the standard Einstein-Hilbert Lagrangian. $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \cdots$ is the totally antisymmetric field strength with f being its coupling constant of the dimension of mass.

The field equations derived from the Lagrangian (2.1) are the Einstein equation:

$$G_{\mu\nu} = 16\pi G f^2 (3H_{\mu\alpha\beta} H_\nu^{\alpha\beta} - \frac{1}{2} g_{\mu\nu} H^2), \tag{2.2}$$

and the "Maxwell equation":

$$\partial^\mu H_{\mu\nu\lambda} = 0. \tag{2.3}$$

The "Maxwell equation" and the Bianchi identity are trivially satisfied by the ansatz:

$$\begin{aligned} H_{ijk} &= \frac{n}{f^2}\epsilon_{ijk} \qquad (i,j,k = 1,2,3) \\ H_{oij} &= 0. \end{aligned} \tag{2.4}$$

The magnetic field H_{ijk} is quantized in the sense that n in Eq. (2.4) takes integer values only. This quantization is achieved via coupling to matter fields, e.g. strings in the background of H and guarantees the stability of the solution. As for the Euclidean space-time, we make an ansatz:

$$ds^2 = d\tau^2 + a^2(\tau) d^2\Omega_3 \tag{2.5}$$

with $a(\tau)$ being the scale factor and $d^2\Omega_3$ the metric of S_3. The Einstein equation (2.2) reduces to

$$(\frac{da}{d\tau})^2 = 1 - \frac{r_b^4}{a^4}, \tag{2.6}$$

with $r_b^4 = 8\pi G n^2/f^2$. We can easily obtain a solution to Eq. (2.6) by using the eliptic integral but we do not need its explicit form. For $|\tau| \to \infty, a \to \tau$ as can be seen from (2.6). We choose the time $\tau = 0$ when the radius of the universe attains its minimum value, $a = r_b$. We have obtained an instanton solution which describes a tunneling from a flat 4 dimensional Euclidean space time E^4 to a (possibly different) E^4 via tiny baby universe of the radius r_b.

Fig.1

The tunneling amplitude is semiclassically calculated and is given for the half of the process (from baby to mother unviverses) by

$$
\begin{aligned}
&K e^{-S_b/\hbar}, \\
&S_b = \frac{3\pi^2}{4}\left(\frac{r_b}{\ell_p}\right)^2,
\end{aligned}
\tag{2.7}
$$

where K is an unimportant WKB prefactor.

Fig.2

As we shall see in the next section, Coleman's arguments do not depend on the details of the model (2.1), which is supposed to describe the sub-Planck physics. The mere existence of wormhole instantons suffices in the semi-classical approximation such that $r_b \gg \ell_p$. Even we do not need any knowledge about sub-Planck physics.

There are variety of models which admit wormhole solutions. First we can add scalar fields to (2.1) which turn out to have non-trivial configurations.

The Einstein-Yang-Mills system has a simple wormhole solution in the presence of a cosmological term.[8] (This may have a relevance at the time of QCD phase transition in the early universe.). Also we remark that if the space-time dimension were d (≥ 3) and the rank of antisymmetric field strength H were $d-1$, we would also get wormhole solutions.[9] The simplest case $d=3$ is the Einstein-Maxwell theory in the three dimensional space-time. The magnetic vortex $H_{ij} = \frac{n}{e}\epsilon_{ij}$ induces the wormhole instanton $a(\tau) = \sqrt{\tau^2 + r_b^2}$.

§3. Wormhole Dominance Approximation to the Euclidean Path-integral[10]

Let us start with the Euclidean path integral over 4-surfaces following Hawking:[4]

$$\Psi[g(\cdot),\phi(\cdot)] = \sum_{\text{topologies}} \int_{\substack{g(\cdot) \\ \phi(\cdot)\, at\, B}} [dg][d\phi]e^{-S[g,\phi]}, \tag{3.1}$$

where the 3-geometry $g(\cdot)$ and 3-dimensional configuration of matter fields collectively denoted by $\phi(\cdot)$ are specified at the boundary B. Since it is almost impossible to exactly carry out such a path integration, we appeal to "wormhole dominance approximation" which enables us to take account of contributions from nontrivial topologies (partially at least). For example, a manifold with a handle is replaced by the one with a wormhole bridge (Fig. 3).

Fig.3

In this approximation, Eq. (3.1) becomes an expansion depicted in Fig. 4.

Fig.4

In Fig. 4, many closed connected 4-manifolds are bridged by wormholes in all possible ways. As a warming up, let us consider the first three diagrams in Fig. 4. The first term may be written in path integral form as

$$\int_{\substack{g(\cdot)\, at\, B \\ smooth}} [dg]e^{-S} \equiv <1>^{B} \tag{3.2}$$

The path integral has to be carried out over smooth manifold with no further wormholes. (We have omitted matter fields ϕ for notational simplicity.) As for the second term, for each wormhole (cross mark) we associate a factor $\int L\sqrt{g}d^4x$ with L being some local operator consisting of metric and matter fields. L is calculable if underlying model of sub-Planck physics is given and a wormhole solution is found.

For our present problem of cosmological constant, it is sufficient to consider a single specie of wormhole which carries no particles. In this case L is simply a constant typically given by an instanton amplitude $\propto e^{-S_b}$ with S_b being a classical action of the wormhole solution. However, we keep the generic symbol L for generality of our diagrammatic method. The integration over space-time is needed, since the location of wormhole can be everywhere on the manifold. In other words, the space time position of wormhole is a collective coordinate.[11] We obtain

$$\int_{\substack{g(\cdot)\, at\, B \\ smooth}} [dg] e^{-S} \frac{(\int L\sqrt{g} d^4x)^2}{2!} \equiv < \frac{V_B^2}{2!} >^B . \tag{3.3}$$

Here we have written V_B for $\int L\sqrt{g}d^4x$ as a short hand. The factor 1/2! is necessary to avoid double counting of the two wormholes. It is easy to see that for the third term in Fig. 4 we have

$$< V_B >^B < V >^C \tag{3.4}$$

where

$$< V >^C = \int_{\substack{closed \\ connected\ \ smooth}} [dg] e^{-S} (\int L\sqrt{g} d^4x) . \tag{3.5}$$

We can proceed further to sum up handles (see Fig. 5)

$$< \sum_{s=0}^{\infty} (\frac{V_B^2}{2})^s \frac{1}{s!} >^B = < e^{\frac{V_B^2}{2}} >^B . \tag{3.6}$$

Fig.5

Now that we have gotten enough experience, consider the most general case that the i-th closed connected manifold is bridged to the manifold with a boundary B (mother manifold) through n_i wormhole corridors and also to the other j-th closed connected manifold through $n_{ij} (= n_{ji})$ wormhole corridors. See Fig. 6.

$\boxed{\text{Fig.6}}$

Let the number of closed connected manifolds be N.

$$\begin{aligned}
&\frac{1}{N!} < e^{\frac{1}{2}V_B^2}(V_B)^{n_1+\ldots+n_N} >^B \frac{1}{n_1!\,n_2!\ldots n_N!} \\
&\quad < (V_1)^{n_1+n_{12}+\ldots n_{1N}} e^{\frac{1}{2}V_1^2} >^{C_1} \frac{1}{n_{12}!n_{13}!\ldots n_{1N}!} \\
&\quad \cdots \\
&\quad < (V_N)^{n_N+n_{N1}+\ldots+n_{N-1,N}} e^{\frac{1}{2}V_N^2} >^{C_N} .
\end{aligned} \tag{3.7}$$

The sums over $n_1, \ldots n_N, n_{12} \ldots n_{1N} \ldots n_{N-1,N}$ are easily carried out. By summing over N we obtain for Ψ in Eq. (1) in the approximation explained in Fig. 4,

$$\Psi = \sum_{N=0}^{\infty} \frac{1}{N!} < \exp[\frac{1}{2}(V_1 + \ldots + V_N + V_B)^2] > , \tag{3.8}$$

where $<>$ denotes averages over B, $C_1 \ldots$ manifolds. We are very near the goal. We rewrite Eq. (3.8) as

$$\begin{aligned}
&\sum_{N=0}^{\infty} \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} \frac{1}{N!} < e^{(V_1+\ldots+V_N+V_B)\alpha} > e^{-\frac{1}{2}\alpha^2} \\
&= \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} < e^{V_B\alpha} >^B \sum_{N=0}^{\infty} \frac{1}{N!}(< e^{V\alpha} >^C)^N \cdot e^{-\frac{1}{2}\alpha^2}.
\end{aligned} \tag{3.9}$$

Therefore we obtain

$$\Psi = \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} < e^{V_B\alpha} >^B Z(\alpha) , \tag{3.10}$$

where

$$Z(\alpha) = \exp(< \exp V\alpha >^C) e^{-\frac{1}{2}\alpha^2}. \tag{3.11}$$

The equation (3.10) is nothing but Coleman's expression for wave function of universe.

To some extent, our present approach corresponds to Polyakov's[12] in string theories, while Coleman's to string field theories.[13] They are actually equivalent in string theories and that is the case also in our present wormhole problem. Perhaps most ambitious way will be the third quantization of universe.[14] Although there already appeared many interesting approaches, there remain a bulk of unsolved problems, e.g., physical interpretations, interaction of universe fields $\cdots$. Nevertheless, the third quantization of universe will be the only consistent way of formulation when we talk about baby universes which can carry off some particles.[12]

§4. Vanishing of Cosmological Constant

Let us evaluate the weight $Z(\alpha)$ given by Eq. (3.11). In the path-integral form we can write

$$< \exp V\alpha >^{C} = \int_{\text{closed}} [dg] \exp\left(-S + \int d^4x\sqrt{g}Ke^{-S_b}\alpha\right) \tag{4.1}$$

where the path-integration is performed over arbitrary closed compacat manifold. S is the action for the sub-Planck physics, which we do not need to know. However, we *know* the low energy effective action relevant at the cosmological scale: *the Einstein-Hilbert action plus possibly the cosmological constant.* This is firmly established by experiments! Therefore the functional integral (4.1) reduces to

$$< \exp V\alpha >^{C} = \int_{\text{closed}} [dg]^{\text{low}} \exp\left(\frac{1}{16\pi G}\int d^4x\sqrt{g}R - \Lambda(\alpha)\int d^4x\sqrt{g}\right). \tag{4.2}$$

after integrating out the "high frequency parts". Here $\Lambda(\alpha)$ is the effective cosmological constant which includes the effect of wormholes,

$$\Lambda(\alpha) = \Lambda_o - \alpha K e^{-S_b} + \cdots, \tag{4.3}$$

where $\ldots$ denotes the renormalization effects which occur when we integrate out the high frequency modes in the path-integral (4.1).

We assume that the de Sitter instanton ($\approx S^4$) dominates the path-integral (4.2). Namely, the scale factor $a(\tau)$ is determined by

$$(\frac{da}{d\tau})^2 = 1 - \frac{8\pi G}{3}\Lambda(\alpha)a^2. \tag{4.4}$$

The equation (4.4) tells us that the radius of the de Sitter S^4 is $a_{\max} = \sqrt{3/8\pi G\Lambda(\alpha)}$ and therefore its four volume is given by

$$\int \sqrt{g}d^4x = \frac{8\pi^2}{3}a_{\max}^4 = \frac{3}{8G^2\Lambda^2(\alpha)}. \tag{4.5}$$

On the other hand, the Einstein equation implies $R/16\pi G = 2\Lambda(\alpha)$. Collecting them all we obtain the semiclassical evaluation of (4.2) as

$$< \exp V\alpha >^C \approx K' \exp(\frac{+3}{8G^2\Lambda(\alpha)}), \tag{4.6}$$

with K' being a WKB prefactor which is in principle calculable but is not important for our purpose.

We arrive at Coleman's expression for the wavefunction of universe,

$$\Psi[g(\cdot)] = \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} Z(\alpha) \int_{\substack{g(\cdot)\,at\,B \\ smooth}} [dg]^{\text{low}} \exp[\frac{1}{16\pi G}\int R\sqrt{g}d^4x - \Lambda(\alpha)\int \sqrt{g}d^4x], \tag{4.7}$$

where the weight $Z(\alpha)$ is given by

$$Z(\alpha) = e^{-\frac{\alpha^2}{2}} \exp(K' \exp\frac{+3}{8G^2\Lambda(\alpha)}). \tag{4.8}$$

To get Eq. (4.7) we have also carried out the integration over the high frequency part of fields as we did before to evaluate $< \exp V\alpha >^C$. This should be the same calculation and therefore so should be $\Lambda(\alpha)$, except the boundary effect, which is presumably of order of $(\text{4-volume})^{-1}$, a very small effect.

It is now clear that the weight $Z(\alpha)$ has an extremely sharp peak at at $\Lambda(\alpha) = +0$. This implies that the effective cosmological constant $\Lambda(\alpha)$ at low energy i.e. the real one vanishes.

§5. Discussions and Summary

Several remarks are in order. First, Coleman heavily relies on the Euclidean path-integral for quantum gravity. There is the notorious problem on the path-integration over the conformal factor, which makes the path-integral badly divergent in the case of the Einstein-Hilbert action. However, the terms like R^2 will be present to ensure the convergence of the path-integral but they are not important in the far-infrared regime. Second, he also used the dilute gas approximation in the instanton summation. In order to check the validity of this appr̈oximaiton, we have to consider the interaction of wormhole instantons. But at the moment no one has done it yet. Third, how can we know about the wavefunction of the universe in the Lorentzian region? The wavefunction of the universe in that regime is presumably obtained from the WKB continuation formula from the wavefunction at the maximal expansion in the Euclidean regime. As we already know from Hawking's work, this would give a standing wave solution. So we end up with the notorious interpretation problem of the wavefunction of universe. Finally, as the audience already noticed, we deliberately ignored the diagrams which contain baby universes as external lines in the wormhole summations. We have implicitly assumed Hawking's boundary condition: No boundary except our present universe B. In order to see to what extent the conclusion of vanishing cosmological constant depends on Hawking's boundary condition, we consider the case that n baby universes initially exist. (Previously we had $n = 0$.) See Fig. 7.

Fig.7

The n tubes may hang down either from the mother manifold or from the vacuum blobs. It is easy to see that the wave function of the universe is then given by

$$\Psi_n = \frac{d^n}{dV^n}(\sum_{N=0}^{\infty} \frac{1}{N!} \exp[\frac{1}{2}(V_1 + \ldots + V_N + V_B)^2 + (V_1 + \ldots V_N + V_B)V])|_{V=0}/\sqrt{n!}\,. \tag{5.1}$$

The factor $1/\sqrt{n!}$ accounts for the Bose statistics of baby universes. In Eq. (5.1) we have omitted the average $<>$ for notational simplicity. In the same way as the previous n = 0 case, we introduce the α integration to obtain

$$\frac{d^n}{dV^n}(\int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2/2} \sum_{N=0}^{\infty} \frac{1}{N!} \exp[(V_1 + \ldots + V_N + V_B)(V + \alpha)]])|_{V=0}/\sqrt{n!}$$

$$= \int_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2/2} \frac{d^n}{d\alpha^n}(\sum_{N=0}^{\infty} \frac{1}{N!} \exp[(V_1 + \ldots + V_N + V_B)\alpha]/\sqrt{n!}.$$

Integrating by parts we obtain

$$\int_{-\infty}^{\infty} d\alpha\, u_n(\alpha)\, u_o(\alpha) \sum_{N=0}^{\infty} \frac{1}{N!} \exp[(V_1 + \ldots + V_N)\alpha]$$

where $u_n(\alpha) = 1/4\sqrt{\pi} H_n(\alpha) e^{-\alpha^2/4}, H_n(\alpha) = (-)^n \frac{d^n}{d\alpha^n}(e^{-\alpha^2/2}) e^{\alpha^2/2}$ is the n-th excited state of harmonic oscillator. The rest is the same as before (Eq. (3.10)). We finally arrive at

$$\Psi_n = \int_{-\infty}^{\infty} d\alpha\, u_o(\alpha) u_n(\alpha) < e^{V_B \alpha} >^B (< e^{V\alpha} >^c)\,. \tag{5.2}$$

The meaning of each factor of the integrand will be self-explanatory. If we took an exact coherent state instead of n baby universes state as an initial state, we would get

$$\Psi(\alpha) = < e^{V_B \alpha} >^B \exp(< e^{V\alpha} >^c)\,.$$

Of course we do not have a vanishing cosmological constant for this choice of initial state. However as Coleman explained, this is an extraordinary fine tuning.[2]

Now let us consider the case of two disconnected boundaries, say, B and B'. Without further ado, we obtain

$$\Psi[g \text{ at } B, g' \text{ at } B'] = \int d\alpha \, Z(\alpha) < e^{V_B \alpha} >^B < e^{V_{B'} \alpha} >^{B'}, \tag{5.3}$$

where $Z(\alpha)$ is given in Eq. (4.8). In general the three geometries g and g' at B and B' respectively are correlated, because of the α integration in Eq. (5.3). Physically speaking, wormholes can communicate informations between disconnected spaces. However, thanks to Coleman's mechanism, the distribution $Z(\alpha)$ is sharply peaked at some α_o such that $\Lambda(\alpha_o) = 0$. Then Eq. (5.3) becomes a factorized form. This makes the correlation vanish with an extreme accuracy.

It is almost trivial to generalize our whole argument to many species of wormhole instantons. We will just have many α's. Presumably, the minimum of the effective potential fixes all the α's.

To summarize, it seems that Coleman's theory of vanishing cosmological constant is very attractive and worth further investigations.[15]

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Figure captions

Fig. 1 A wormhole instanton: from E^4 to E^4 via baby universe of radius r_b.

Fig. 2 A half of Fig. 1: a baby universe branches off from the mother universe.

Fig. 3 A handle is replaced by a wormhole corrider. The cross mark denotes a wormhole attached to a manifold with a boundary B.

Fig. 4 The wormhole dominance expansion of Hawking's sum over topologies.

Fig. 5 Sum over s-pair of wormholes attached to the manifold.

Fig. 6 A generic configuration of the mother manifold and N closed connected manifolds. They are bridged by wormhole corridors.

Fig. 7 n baby universes initially exist.

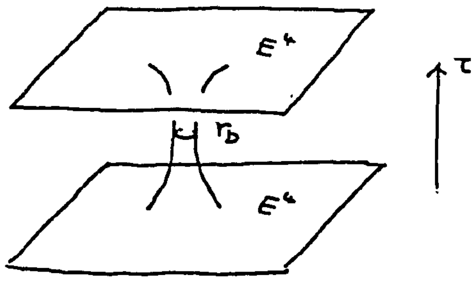

Fig. 1

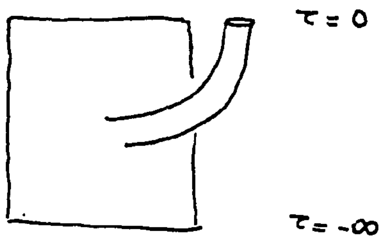

Fig. 2

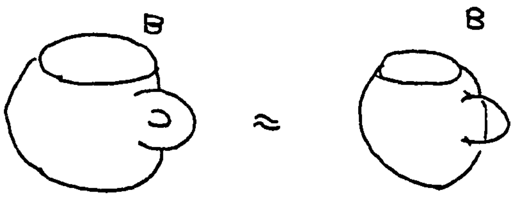

Fig. 3

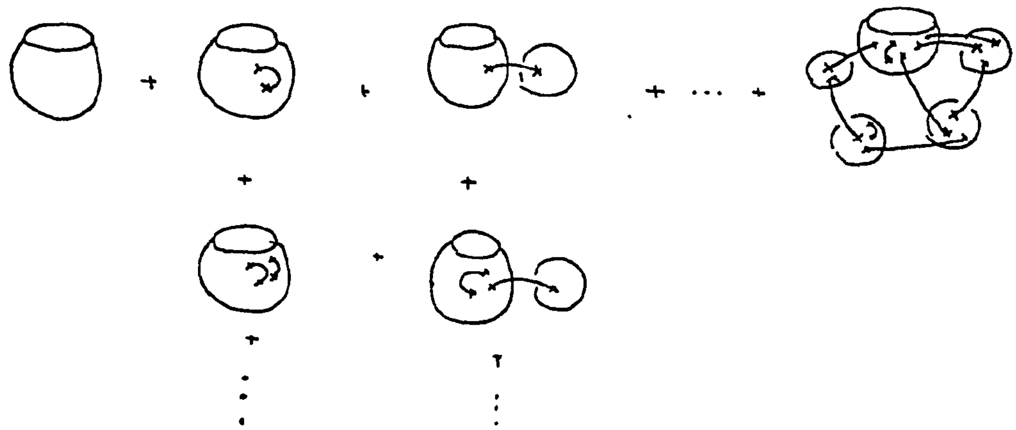

Fig. 4

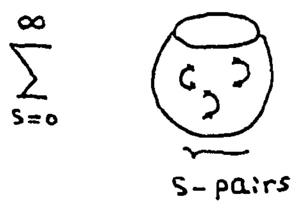

Fig. 5

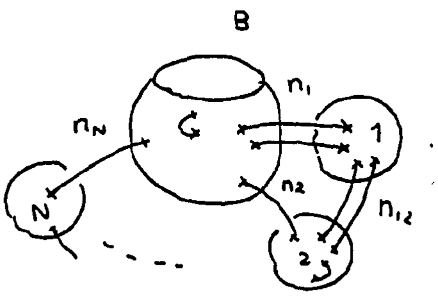

Fig. 6

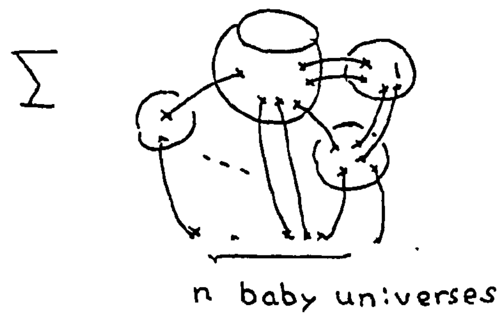

Fig. 7

A Four Dimensional Open Superstring Model
四次元超開弦理論とその性質

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The active study of superstring theory in recent years began with the discovery of the anomaly and infinity cancellation of $SO(32)$ type I superstring theory [1] [2]. Since then, numerous superstring models are constructed and the natures of multi-loop amplitudes are investigated. However, all these developments have been made mainly for the heterotic type string theories which consist of closed string only. Type I superstring theories, which consists of open and closed strings, have not been studied so actively. The biggest reason for this is that it is very difficult to construct four dimensional models with chiral fermions by compactifying the type I theory. In this note, we will consider four dimensional superstring models compactified on Z_n orbifolds in general. We will construct such a model explicitly in Z_3 case and show that it is a model which contains three generations of chiral fermions.

The open string models we consider here are unoriented ones. The nonorientability of the models makes it possible for the divergences of the annulus and the Möbius strip amplitudes to cancel [2]. The closed string sectors of such unoriented models are also unoriented. Let us begin our model building by considering unoriented closed strings on Z_n orbifolds.

Z_n orbifolds which are used in string compactifications are the quotients of a d dimensional torus T^d by its symmetries S which have fixed points in T^d, i.e.

$$\begin{aligned}&T^d/S,\\ &T^d \equiv \mathrm{R}^d/\Lambda(\Lambda;\mathrm{lattice}),\end{aligned} \tag{1}$$

where S is isomorphic to $\mathbf{Z}/n\mathbf{Z}$ and generated by α which satisfies $\alpha^n = 1$. We will take the complex coordinates $z^I, z^{\bar{I}}$ on T^d so that the actions of α on $z^I, z^{\bar{I}}$ are diagonalized as

$$\begin{aligned} \alpha \quad ; \quad & z^I \to e^{\frac{2\pi i l}{n}} z^I, \\ ; \quad & z^{\bar{I}} \to e^{-\frac{2\pi i l}{n}} z^{\bar{I}}, \\ & l; \text{ integer.} \end{aligned} \tag{2}$$

The first quantized Hilbert space $\mathcal{H}$ of the oriented closed string propagating on a Z_n orbifold can be constructed easily. $\mathcal{H}$ consists of the untwisted sector ($\mathcal{H}_0$) and the twisted sectors ($\mathcal{H}_k(k = 1, \cdots, n-1)$). Each sector is characterized by the boundary conditions of the variables $z^I(\tau, \sigma)$. Namely, the bosonic variables z^I, for example, satisfy

$$z^I(\tau, \sigma + 2\pi)|\ \rangle_0 = z^I(\tau, \sigma)|\ \rangle_0, \tag{3}$$

if $|\ \rangle_0$ is a state in $\mathcal{H}_0$, and

$$z^I(\tau, \sigma + 2\pi)|\ \rangle_k = e^{\frac{2\pi i k l}{n}} z^I(\tau, \sigma)|\ \rangle_k, \tag{4}$$

if $|\ \rangle_k$ is a state in $\mathcal{H}_k$.

The Hilbert space of the unoriented closed string on the Z_n orbifold can be obtained as the subspace of $\mathcal{H}$ which is invariant under the orientation reversing operation (flip) $\sigma \to 2\pi - \sigma$. To be more explicit, we should define the flip operator Ω, which satisfies

$$\Omega^{-1} z^I(\tau, \sigma)\Omega = z^I(\tau, 2\pi - \sigma). \tag{5}$$

Then $\frac{1+\Omega}{2}\mathcal{H}$ is what we want.

The action of Ω on the states in the untwisted sector is well-known. It corresponds to the interchange $\alpha_n^I \leftrightarrow \tilde{\alpha}_n^I$ of the mode expansion coefficients.

$$z^I(\tau, \sigma) = x^I + 2p^I\tau + 2w^I\sigma + i\sum_{n\neq 0}\frac{1}{n}[\alpha_n^I e^{-in(\tau+\sigma)} + \tilde{\alpha}_n^I e^{-in(\tau-\sigma)}]. \tag{6}$$

On the states in the twisted sectors, Ω acts in a nontrivial way. In general, Ω takes one Hilbert space to another, namely , if $|\ \rangle_k \in \mathcal{H}_k, \Omega|\ \rangle_k \in \mathcal{H}_{n-k}$. This fact can be seen, by examining the boundary conditions of z^I acting on $\Omega|\ \rangle_k$. From eq. (4) and eq. (5),

$$\begin{aligned} z^I(\tau, \sigma + 2\pi)\Omega|\ \rangle_k &= \Omega z^I(\tau, 2\pi - (\sigma + 2\pi))|\ \rangle_k \\ &= e^{-\frac{2\pi i k l}{n}} \Omega z^I(\tau, 2\pi - \sigma)|\ \rangle_k \\ &= e^{\frac{2\pi i (n-k) l}{n}} z^I(\tau, \sigma)\Omega|\ \rangle_k, \end{aligned} \tag{7}$$

which means $\Omega|\ \rangle_k \in \mathcal{H}_{n-k}$.

Therefore, in order to form flip invariant states, we have to consider the combination of two states in sectors $\mathcal{H}_k$ and $\mathcal{H}_{n-k}$. Unless n is even and $k = \frac{n}{2}$, these two sectors are different.

The partition function of an unoriented closed string theory can be decomposed into the torus partition function and the Klein bottle partition function, as follows.

$$Trq^{L_0}\bar{q}^{\bar{L}_0}\frac{1+\Omega}{2}P = \frac{1}{2}Trq^{L_0}\bar{q}^{\bar{L}_0}P + \frac{1}{2}Trq^{L_0}\bar{q}^{\bar{L}_0}\Omega P, \tag{8}$$

where P is the projection operator into the twist invariant states. In the case of the closed string on Z_n orbifolds, the traces in eq. (8) are given as the sums of traces over sectors. The Klein bottle partition function can be written as

$$\frac{1}{2}Trq^{L_0}\bar{q}^{\bar{L}_0}\Omega P = \frac{1}{2}\sum_{i=0}^{n-1} Tr_{\mathcal{H}_i}q^{L_0}\bar{q}^{\bar{L}_0}\Omega P. \tag{9}$$

If n is odd, only the trace over untwisted sector ($\mathcal{H}_0$) contributes to the above sum, because Ω takes one sector to another. If n is even, only the traces over $\mathcal{H}_0$ and $\mathcal{H}_{\frac{n}{2}}$ are nonzero.

The vanishing of the Klein bottle partition functions for twisted sectors can be explained from the path integral point of view. When we compute the Klein bottle amplitudes in the path integral formalism, we have to assign a Z_n twist to each homology cycle of the Klein bottle. Unless the twist in the space direction is of order 2, the assignment of twists is not compatible with the first homology group of the Klein bottle. Therefore, such kinds of amplitudes should vanish. The situation is somewhat similar to that of the non-abelian orbifolds [3].

Since we know the action of Ω on each sectors, we can now compute partition functions. From these partition functions, it is easy to see what kinds of crosscap states are present in the theory. Constructing the crosscap states is very important in open string model building.

Here, we will consider the two simple examples. One is bosonic strings on Z_2 orbifolds. This case is elaborated in [4]. The Klein bottle partition function consists of the following four types.

$$\begin{aligned}
&Tr_{\mathcal{H}_0}q^{L_0}\bar{q}^{\bar{L}_0}\Omega,\\
&Tr_{\mathcal{H}_0}q^{L_0}\bar{q}^{\bar{L}_0}\Omega\alpha,\\
&Tr_{\mathcal{H}_1}q^{L_0}\bar{q}^{\bar{L}_0}\Omega,\\
&Tr_{\mathcal{H}_1}q^{L_0}\bar{q}^{\bar{L}_0}\Omega\alpha,
\end{aligned} \tag{10}$$

each of which is characterized by the boundary conditions. In order to express these four as tree amplitudes with two crosscap insertions [4], two kinds

of crosscap states $|C_\pm\rangle$, which satisfy the following boundary conditions, are necessary.

$$\begin{cases} [z^I(\sigma) - z^I(\sigma+\pi) + 2\pi u^I]|C_+\rangle = 0 \\ [\partial_\tau z^I(\sigma) + \partial_\tau z^I(\sigma+\pi)]|C_+\rangle = 0, \end{cases} \tag{11}$$

$$\begin{cases} [z^I(\sigma) + z^I(\sigma+\pi) - 2\pi u^I]|C_-\rangle = 0 \\ [\partial_\tau z^I(\sigma) - \partial_\tau z^I(\sigma+\pi)]|C_-\rangle = 0, \quad \vec{u} \in \Lambda. \end{cases} \tag{12}$$

Eqs.(11) for $|C_+\rangle$ are the same as the conditions for the crosscap states of the strings in the flat background. In these, $z^I(\sigma)$ and $z^I(\sigma+\pi)$ are identified. Eqs.(12) are peculiar to Z_2 orbifolds. In this case $z^I(\sigma)$ and $-z^I(\sigma+\pi)$ are identified, because $z^I \sim -z^I$ on Z_2 orbifolds. Both of the crosscap states are included in the untwisted sector, because $z^I(\sigma+\pi) \sim \pm z^I(\sigma)$ implies $z^I(\sigma+2\pi) \sim z^I(\sigma)$.

Another example is bosonic strings on Z_3 orbifolds. In this case, the Klein bottle amplitudes are

$$\begin{aligned} &Tr_{\mathcal{H}_0} q^{L_0}\bar{q}^{\bar{L}_0}\Omega, \\ &Tr_{\mathcal{H}_0} q^{L_0}\bar{q}^{\bar{L}_0}\Omega\alpha, \\ &Tr_{\mathcal{H}_0} q^{L_0}\bar{q}^{\bar{L}_0}\Omega\alpha^2, \end{aligned} \tag{13}$$

without any contributions from the traces over twisted sectors. Proceeding in the same way as in the Z_2 cases, we can easily find that the following three kinds of crosscap states are necessary.

$$\begin{cases} [z^I(\sigma+\pi) - z^I(\sigma) + 2\pi u^I]|C_0\rangle = 0 \\ [\partial_\tau z^I(\sigma+\pi) + \partial_\tau z^I(\sigma)]|C_0\rangle = 0, \end{cases} \tag{14}$$

$$\begin{cases} [z^I(\sigma+\pi) - e^{-\frac{2\pi i l}{3}} z^I(\sigma) + 2\pi u^I]|C_1\rangle = 0 \\ [\partial_\tau z^I(\sigma+\pi) + e^{-\frac{2\pi i l}{3}}\partial_\tau z^I(\sigma)]|C_1\rangle = 0, \end{cases} \tag{15}$$

$$\begin{cases} [z^I(\sigma+\pi) - e^{\frac{2\pi i l}{3}} z^I(\sigma) + 2\pi u^I]|C_2\rangle = 0 \\ [\partial_\tau z^I(\sigma+\pi) + e^{\frac{2\pi i l}{3}}\partial_\tau z^I(\sigma)]|C_2\rangle = 0. \end{cases} \tag{16}$$

Eqs.(14) for $|C_0\rangle$ are the same as those in the flat background, and eqs.(15),(16) for $|C_1\rangle$, $|C_2\rangle$ are peculiar to Z_3 orbifolds. $|C_1\rangle$ and $|C_2\rangle$ are similar to $|C_-\rangle$ in the Z_2 orbifold case. $z^I(\sigma+\pi)$ and $e^{\pm\frac{2\pi i}{3}}z^I(\sigma)$ are identified, because $z^I \sim e^{\pm\frac{2\pi i}{3}}z^I$. The crucial difference between $|C_-\rangle$ and $|C_1\rangle, |C_2\rangle$ is that while $|C_-\rangle$ is included in the untwisted sector, $|C_1\rangle$ and $|C_2\rangle$ are included in the twisted sector. Indeed, for $|C_1\rangle$ and $|C_2\rangle$, $z^I(\sigma+\pi) \sim e^{\pm\frac{2\pi i}{3}}z^I(\sigma)$ and this implies $z^I(\sigma+2\pi) \sim e^{\mp\frac{2\pi i}{3}}z^I(\sigma)$.

The two examples above illustrates the general features of the crosscap states for the string theory on Z_n orbifolds. There exist crosscap states with boundary conditions $z^I(\sigma+\pi) \sim e^{\frac{2\pi i k l}{n}}z^I(\sigma)(k = 0, 1, \cdots, n-1)$. When n is

odd, only the crosscap state with $k=0$ belongs to the untwisted sector. When n is even, the crosscap states with $k=0$ and $k=\frac{n}{2}$ belong to the untwisted sector. This fact corresponds to the statement below eq. (9). The difference in nature of the crosscap states for n =even and odd, is very crucial to the open string model building.

Next let us consider open strings which are coupled to the unoriented closed strings on the Z_n orbifolds in the above. The procedure for constructing such models is given in [4]. There exist several types of boundary conditions for the crosscap. In order to incorporate open strings in the theory, we should consider the Riemann surfaces with boundaries. The first step to the open string model building will be taken by deciding what kinds of boundary conditions for the boundary (or equivalently, what kinds of boundary states) are possible.

The boundary state $|B\rangle$ of a consistent theory should satisfy the following massless tadpole cancellation condition.

$$|B\rangle_0 + |C\rangle_0 = 0, \tag{17}$$

where the subscript 0 indicates the massless part. This condition puts restrictions on the possible types of the boundary states.

Here let us consider the two examples, bosonic strings on Z_2 and Z_3 orbifolds again. In the Z_2 orbifold case [4], there exist two kinds of crosscap states $|C_\pm\rangle$, and $|C\rangle$ is a linear combination,

$$|C\rangle = N_{C+}|C_+\rangle + N_{C-}|C_-\rangle. \tag{18}$$

The explicit forms of $|C_\pm\rangle$ can be derived from eq. (11) and . (12), and the massless part of $|C_\pm\rangle$ and $|C\rangle$ becomes [4]

$$\begin{aligned}
|C_+\rangle_0 &= -(-\sum_\mu \alpha^\mu_{-1}\tilde{\alpha}^\mu_{-1} - \sum_I \alpha^I_{-1}\tilde{\alpha}^I_{-1} + \tilde{c}_{-1}b_{-1} + c_{-1}\tilde{b}_{-1})(c_0+\tilde{c}_0)c_1\tilde{c}_1|0\rangle, \\
|C_-\rangle_0 &= -(-\sum_\mu \alpha^\mu_{-1}\tilde{\alpha}^\mu_{-1} + \sum_I \alpha^I_{-1}\tilde{\alpha}^I_{-1} + \tilde{c}_{-1}b_{-1} + c_{-1}\tilde{b}_{-1})(c_0+\tilde{c}_0)c_1\tilde{c}_1|0\rangle, \\
|C\rangle_0 &= -[N_{C+}(-\sum_\mu \alpha^\mu_{-1}\tilde{\alpha}^\mu_{-1} - \sum_I \alpha^I_{-1}\tilde{\alpha}^I_{-1} + \tilde{c}_{-1}b_{-1} + c_{-1}\tilde{b}_{-1}) \\
&\quad + N_{C-}(-\sum_\mu \alpha^\mu_{-1}\tilde{\alpha}^\mu_{-1} + \sum_I \alpha^I_{-1}\tilde{\alpha}^I_{-1} + \tilde{c}_{-1}b_{-1} + c_{-1}\tilde{b}_{-1})](c_0+\tilde{c}_0)c_1\tilde{c}_1|0\rangle
\end{aligned} \tag{19}$$

In order to satisfy eq. (17), we should consider two kinds of boundary conditions for the boundary. One is the ordinary free boundary condition. The boundary state $|B_+\rangle$ for such boundary condition satisfies

$$\partial_\tau z^I(\sigma)|B_+\rangle = 0, \tag{20}$$

and the massless part of $|B_+\rangle$ becomes

$$|B_+\rangle_0 = [-\sum_\mu \alpha^\mu_{-1}\tilde{\alpha}^\mu_{-1} - \sum_I \alpha^I_{-1}\tilde{\alpha}^I_{-1} + \tilde{c}_{-1}b_{-1} + c_{-1}\tilde{b}_{-1}](c_0+\tilde{c}_0)c_1\tilde{c}_1|0\rangle. \quad (21)$$

Evidently another kind of boundary state is necessary to cancel $|C\rangle_0$ in eq. (19), unless $N_{C-} = 0$. Therefore we will introduce the boundary with the fixed boundary condition into the theory. On such a boundary, z^I is fixed at the fixed point of Z_2 action. The boundary state $|B_-\rangle$ for such boundary satisfies

$$\partial_\sigma z^I(\sigma)|B_-\rangle = 0, \quad (22)$$

and the massless part of $|B_-\rangle$ becomes

$$|B_-\rangle_0 = [-\sum_\mu \alpha^\mu_{-1}\tilde{\alpha}^\mu_{-1} + \sum_I \alpha^I_{-1}\tilde{\alpha}^I_{-1} + \tilde{c}_{-1}b_{-1} + c_{-1}\tilde{b}_{-1}](c_0+\tilde{c}_0)c_1\tilde{c}_1|0\rangle. \quad (23)$$

By making a linear combination of $|B_+\rangle$ and $|B_-\rangle$, it is possible to obtain a boundary state

$$|B\rangle = N_{C+}|B_+\rangle + N_{C-}|B_-\rangle, \quad (24)$$

which satisfies eq. (17). Having two kinds of boundary conditions, the open string sector of the theory consists of three kinds. Besides the ordinary open strings with the both ends free, we have the twisted open strings with one end free and the other end fixed, and with the both ends fixed. Such twisted open strings are considered in [5] [6]. The mode expansions of $z^I(\sigma)$ and the Fock space of the oscillators can easily be obtained. The first quantized Hilbert space for the open string sector of the theory is the α invariant subspace of the Fock space. Therefore when we compute the annulus partition function, we should insert the projection operator into the α invariant subspace. Hence the annulus partition function include the ones with a twist in the time direction. This fact forces us to consider the free and fixed boundary states which are included in the twisted closed string sector. The total boundary state of the theory is a certain linear combination of such boundary states and eq. (24). The coefficients of such a linear combination can be determined from the open string consistency conditions proposed in [4].

On the other hand, in the case of Z_3 orbifolds, there exist three kinds of crosscap states, $|C_0\rangle$, $|C_1\rangle$ and $|C_2\rangle$. The crosscap state of the theory is given as a certain linear combination of $|C_0\rangle$, $|C_1\rangle$ and $|C_2\rangle$.

$$|C\rangle = N_{C0}|C_0\rangle + N_{C1}|C_1\rangle + N_{C2}|C_2\rangle. \quad (25)$$

The boundary state of the theory should satisfy eq. (17). In this case, $|C_0\rangle$, $|C_1\rangle$ and $|C_2\rangle$ belong to the different sectors of the closed string Hilbert space.

Therefore eq. (17) suggests

$$|B_0\rangle_0 + |C_0\rangle_0 = 0, \tag{26}$$
$$|B_1\rangle_0 + |C_1\rangle_0 = 0, \tag{27}$$
$$|B_2\rangle_0 + |C_2\rangle_0 = 0, \tag{28}$$

where $|B_0\rangle \in \mathcal{H}_0$, $|B_1\rangle \in \mathcal{H}_1$, $|B_2\rangle \in \mathcal{H}_2$ and $|B\rangle = N_{C0}|B_0\rangle + N_{C1}|B_1\rangle + N_{C2}|B_2\rangle$. From eq. (14), $|C_0\rangle_0$ can be obtained explicitly as

$$|C_0\rangle_0 = -[-\sum_{\mu} \alpha^{\mu}_{-1}\tilde{\alpha}^{\mu}_{-1} - \sum_{I} \alpha^{I}_{-1}\tilde{\alpha}^{I}_{-1} + \tilde{c}_{-1}b_{-1} + c_{-1}\tilde{b}_{-1}](c_0 + \tilde{c}_0)c_1\tilde{c}_1|0\rangle. \tag{29}$$

Eq.(26) can be satisfied, if we take $|B_0\rangle$ to be the boundary state with the ordinary free boundary condition. Therefore in this case, eq. (17) does not require the existence of twisted boundary states and twisted open string sector.

The difference between the Z_2 and Z_3 orbifolds in the above arguments stems from the difference in the nature of the crosscap states . In Z_2 case, "the twisted crosscap state" $|C_-\rangle$ belongs to the untwisted closed string sector and this fact requires the existence of boundaries with the new boundary condition eq. (22). In Z_3 case, $|C_1\rangle$ and $|C_2\rangle$ belong to the twisted closed string sectors and this fact requires the existence of $|B_1\rangle$ and $|B_2\rangle$, which satisfy eq. (27) and eq. (28). However, since $|B_1\rangle$ and $|B_2\rangle$ are included in the twisted closed string sector, $|B_1\rangle$ and $|B_2\rangle$ need not be boundary states with new kinds of boundary conditions. For example, eq. (27) and eq. (28) may be satisfied, if we take $|B_1\rangle$ and $|B_2\rangle$ to be the free boundary states in the closed string twisted sectors, which are necessary as in the Z_2 case.

Here we will give an example of such open string models on Z_3 orbifold. Let us consider the type I superstring model on

$M_4 \times T^3/S$ $\quad M_4$; four-dimensional Minkowski space,

T^3 ; $= \mathbf{C}^3/\Lambda$ complex three dimensional torus such that $\{(z^1, z^2, z^3); z^i \sim z^i + r_i \sim z^i + r_i e^{\frac{2\pi i}{3}}\}$ (r_i; the radii of the torus),

S ; $\cong Z_3$ automorphism group of T^3 generated by $\alpha; (z^1, z^2, z^3) \mapsto e^{\frac{2\pi i}{3}}(z^1, z^2, z^3)$. (30)

In this model, we will assume the Chan-Paton factors of the model to be as follows. On the boundary, the Chan-Paton factors

$$\begin{array}{lll} A & ; & A = 1, \cdots, n_1, \\ a & ; & a = \cdots, n_2, \\ \overline{a} & ; & \overline{a} = 1, \cdots, \overline{n}_2, \quad \overline{a} \text{ is the complex conjugate of } a, \end{array} \tag{31}$$

are assigned, and as in the case of the heterotic string on orbifolds, we assume α acts on the Chan-Paton factors as

$$\begin{aligned} \alpha &= 1 \text{ on } A, \\ \alpha &= e^{\frac{2\pi i}{3}} \text{ on } a, \\ \alpha &= e^{-\frac{2\pi i}{3}} \text{ on } \overline{a}. \end{aligned} \tag{32}$$

In order for this model to be consistent up to one loop order, it should satisfy the conditions proposed in [4]. Those conditions are satisfied if we take $n_1 = 8$, $n_2 = 12$. Therefore, if we compactify the ordinary type I superstring theory on the torus T^3, and project the Hilbert space onto S invariant subspace, (taking the action of α on the Chan-Paton factors into account), we can obtain the Hilbert space of open superstring on T^3/S, which is consistent up to one loop order.

Let us count the massless particles included in this model. The Hilbert space of the open string sector is nothing but the S invariant subspace of the ordinary $SO(32)$ type I superstring theory. We should not forget the fact that α acts also on the Chan-Paton factors.

The massless particles in the type I superstring are in a vector multiplet in ten dimensional space-time. Compactifying the theory on the torus T^3, such multiplet will be decomposed into supermultiplets in four dimensional space-time. Ignoring the Chan-Paton factors, we can classify them by the eigenvalues of α and the helicities as follows.

$$\alpha = 1 \qquad (1, \frac{1}{2}) + (-1, -\frac{1}{2}), \tag{33}$$

$$\alpha = e^{\frac{2\pi i}{3}} \qquad (0, \frac{1}{2}) \times 3, \tag{34}$$

$$\alpha = e^{-\frac{2\pi i}{3}} \qquad (0, -\frac{1}{2}) \times 3. \tag{35}$$

In order to make an S invariant states, we should attach appropriate Chan-Paton factors to them. To the vector multiplets in eq. (33), we should attach the combinations AB or $a\bar{b}$. These Chan-Paton factors make the vector particles into the gauge bosons of the gauge group $SO(8) \otimes U(12)$. To the scalar multiplets in eq. (34), we should attach $A\bar{b}$ or ab. These particles give the matter multiplet which transform as $(8, \overline{12})$ and $(1, (12 \times 12)_{antisymm.})$ under $SO(8) \otimes U(12)$. The scalar multiplets in eq. (35) are the antiparticles of these matter particles.

If $U(12)$ is broken down to $SU(5)$, the representations $\overline{12}$ and $(12 \times 12)_{antisymm.}$ of $U(12)$ becomes

$$\begin{aligned} U(12) &\supset SU(5), \\ (\overline{12}) &= (\overline{5}) + (1) \times 7, \\ (12 \times 12)_{antisymm.} &= (5 \times 5)_{antisymm.} + (5) \times 7 + (1) \times 21. \end{aligned} \tag{36}$$

Therefore, considering this $SU(5)$ to be the gauge group of a grand unified theory, this model contains three generations of quarks and leptons, which transform as $\overline{5} + (5 \times 5)_{antisymm.}$ under $SU(5)$.

This model is the first example of four dimensional type I superstring models with chiral fermions. We should somehow deform the model in order to obtain phenomenologically more realistic models. This will be a future problem.

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Boundary and Crosscap States in Conformal Field Theories

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1. Introduction

In the past few years model building in four dimensional string theories has been extensively done by many authors. All the models in four dimensions with N=1 space-time supersymmetry are constructed by the use of the heterotic string theory or the type II superstring theory, which are close string theories. In constructing closed string models the "modular invariance condition" played an essential role. It is one of the criterion of the unitarity, the fineteness and the anomaly cancelltaion of the theory.

However in open string theory no one has ever constructed four dimensional models with N=1 space-time supersymmetry.* One of the main reason for this situation would be the technical complications as well as the lack of a simple criterion of the consistency in open superstring theories. In order to calculate the scattering amplitude in the perturbative expansion one needs much more diagrams in open superstring theory than in closed string theory. In fact the two dimensional surfaces corresponding to the string diagrams are orientable and non orientable surfaces with possible boundaries in the former case and orientable closed surfaces in the latter case. Therefore it is very important to construct a simple criterion for the consistency of the open string theory and apply it to the open string model building.

In this we present the consistency condition in open string theory, which is the open string analogue of the modular invariance condition in closed string theory. This consistency condition is also applicable to the classification of possible conformal field theories on a bordered surface. We will study the minimal models and determine the operator contents of the theories. Applictation to the open string model building will be given in N. Ishibashi's talk.

⋆ N.I. and T.O. have recently constructed a new open string model in four dimensions with N=1 space-time supersymmetry. This model is quite interesting : it contains the standard model and has three generations. (Cf. the talk by N. Ishibashi which is reported in this proceedings.)

One loop amplitudes in open string theories are given by sum of the path integral over the following four two-dimensional surfaces : the torus (T), the Klein bottle (KB), the annulus (A) and the Möbius strip (MS).The first two represent the one-loop amplitudes for the closed string states, while the latter two represent those for the open string states. Here in KB and MS amplitudes the string states are flipped during the propagation so that the surfaces are non-orientable.

Now the one-loop amplitudes contain another kind of physical processes, namely KB, A and MS amplitudes also represent the process in which closed string states appear from the vacuum and propagate for a certain time and then disappear again. It is most clearly understood by transforming the two dimensional surfaces as shown in fig.1. By cutting out the two dimensional surfaces and rejoining them again we find that KB, A and MS are respectively equivalent to cylinders with crosscaps for both ends, with boundaries for both ends, and with a crosscap and a boundary for each end. A crosscap is defined by the boundary of a disk with the identification of antipodal points. Therefore one can calculate the one-loop amplitudes either by taking traces in the open and closed string states or by computing the tree-level transition amplitudes. These two must give the same result since they correspond to the path integral on the same two dimensional surfaces. Let us call this duality of the one-loop amplitudes in open string theory as "the loop channel - tree channel duality".

This duality imply that there exits a close relationship in the operator content (in other words, the spectrum) of the open string sector and the closed string sector. In fact given the gauge group and the spectrum of the open string sector we can completely determine the spectrum of the closed string sector. On the other hand in order to define a finite amplitudes from the T amplitude the closed string sector must satisfy the the modular invariance condition just as in closed string theories. This condition is reflected to the open string sector so that one cannot take an arbitrary spectrum in the open string sector. This is a strong restriction in constructing consistent open string models.

The only missing point in the above argument is how we determine the gauge group. The answer is the gauge anomaly cancellation condition. In string theory anomalies appearing in the scattering amplitudes are reduced to the surface integral over the boundary of the moduli space. In the one-loop case, one can show that the possible source of the non-vanishing surface term is the tadpoles of massless fields.

To summarize the one-loop consistency conditions in open string theories are

1. the loop channel - tree channel duality (plus the modular invariance condition of the T amplitude)
2. tadpole cancellation

Note that in closed string theory only the modular invariance is sufficient to guarantee the consistency of the models. Moreover the concept of the modular invariance can be extended to higher loops. In the open string case, however we do not yet know how to extend the above conditions to higher loop cases. The consistency of the open string theories in all orders in the perturbation will be left as a future problem.

2. Boundary and Crosscap States in Confrmal Field Theories

The consistency conditions in the previous chapter is most effectively described by the boundary and the crosscap states $|B\rangle$ and $|\rangle$. The boundary and the crosscap states are the wave function which represent the boundary conditions the boundary and the crosscap.

The loop channel - tree channel duality condition as well as the modular

invariant condition are

$$
\begin{aligned}
\mathrm{Tr} e^{2\pi i\tau L_0^{\mathrm{open}}} &= \langle B \mid e^{-\frac{\pi i}{\tau}(L_0^{\mathrm{closed}}+\bar{L}_0^{\mathrm{closed}})} | B\rangle, \\
\mathrm{Tr} e^{2\pi i\tau L_0^{\mathrm{open}}} \mathrm{f} &= \langle C \mid e^{-\frac{\pi i}{4\tau}(L_0^{\mathrm{closed}}+\bar{L}_0^{\mathrm{closed}})} | B\rangle, \\
\mathrm{Tr} e^{2\pi i\tau (L_0^{\mathrm{closed}}+\bar{L}_0^{\mathrm{closed}})} \Omega &= \langle C \mid e^{-\frac{\pi i}{2\tau}(L_0^{\mathrm{closed}}+\bar{L}_0^{\mathrm{closed}})} | C\rangle, \\
\mathrm{Tr} e^{2\pi i\tau L_0^{\mathrm{closed}}} \mathrm{Tr} e^{-2\pi i\bar{\tau} \bar{L}_0^{\mathrm{closed}}} &= \mathrm{Tr} e^{-\frac{2\pi i}{\tau} L_0^{\mathrm{closed}}} \mathrm{Tr} e^{+\frac{2\pi i}{\bar{\tau}} \bar{L}_0^{\mathrm{closed}}},
\end{aligned}
\tag{2.1}
$$

where f and Ω are the flip operators.

The tadpole cancellation condition is

$$
\text{the massless part of}(\mid B\rangle + \mid C\rangle) = 0.
$$

In the rest of this talk we study the loop channel - tree channel duality condition more closely in simple examples. String theories in four dimensions are, in general, desribed by four pairs of free bosons and fermions for the propagation in the space-time and by general conformal field theories for the internal space. Thus we concentrate on the loop channel -tree channel duality condition of conformal field theories.

Apart from the motivation of open string model building, the duality condition is necessary in conformal field theories. In statistical systems with local interactions one can define the partition function (or the one-loop amplitude) just by the summation of the Boltzman weight. Here the partition function is also calculated by the Hamiltonian formulation using the transfer matrix, i.e. one chooses the time axis and space axis, specify the configuration of the field variables at a certain time, and calculate the partition function as a time evolution process. However the choice of the time axis is not unique. One can redefine one of the space axis as the time axis and set other directions as the space directions. If the local interactions of the system are symmetries under space time raotaion, the transfer matrix is independent of the choice of the space-time axis except as to the boundary conditions. Thus the duality condition is in this sense is a trivial

manifestation of the rotational symmetries of the underlying local interactions of the system.

However in general conformal field theories, since one does not necessarily know the Lagranians, the duality condition implies something non-trivial. Although one knows what primary fields could in principle exist in the theory, since one does not know the Lagrangian, one does not know which primary fields actuall appear with which multiplicities or what is the three point interaction terms of three primary fields. All these informations are determined from consistency.The partition function χ is a linear combination of the Virasoro characters χ in the Hilbert space of each primary field.

$$\chi = \sum_i a_i \chi_i,$$

where i runs over aill the primary fields. The duality condition is useful to determine the multiplicity a_i of the primary fields in the partition functions. If we know the multiplicities a_i of various conformal field theories on a bordered surface, we can use them as building blocks for open string model building. Now we consider the minimal conformal models found by Belavin, Polyakov, and Zamolodchikov. In these models Friedan, Qiu, and Shenker showed that the allowed values of c are quantized as

$$c = 1 - \frac{6}{m(m+1)} \; m = 3, 4, \ldots,$$

and there exists only a finite number of primary fields, whose conformal dimensions h_{pq} are,

$$h_{pq} = \frac{(p(m+1) - mq)^2 - 1}{4} \; 1 \leq q \leq p \leq m - 1$$

Exploiting the duality of the torus (in other words, modular invariance), Cardy[*] decided the operator content of the unitary theory in the periodic boundary

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conditions. He derived the sum rules for the multiplicities of the primary fields in the partition functions and obtained the solutiongs for the first few models in the series. These were useful in constructing closed string theories.

Following similar ideas, Cardy† also derived the sum rules for the multiplicities of the primary fields of the unitary theories in anti-periodic, free, and fixed boundary conditions. The latter two theories would be useful in constructing open sting theories.

However, since for free and fixed boundary conditions the sum rules contain a number of unknown constants characterizing the boundary conditions, it is not possible to determine the operator content without further informations of the theory. In fact Cardy imposed the discrete Z_n symmetry on the boundaries for specific models such as Ising model ($m = 3$), Tricritical Ising model ($m = 4$), and 3 state Potts model ($m = 5$). But he could not determine the operator contents for general m.

Recently one of us (N.I.)[2] has obtained the explicit form of these states in the case of current algebras, minimal conformal and minimal superconformal field theories with $N = 1, 2$. The boundary and the crosscap states in the minimal conformal field theories are the solution for the following equations

$$\begin{aligned}(L_0 - \tilde{L}_0) \mid B\rangle &= 0 \\ (L_0 - (-)^n \tilde{L}_0) \mid C\rangle &= 0.\end{aligned} \tag{2.2}$$

For general conformal field theories the above conditions are far too weak to decide $\mid B\rangle$ and $\mid C\rangle$. However, for minimal models the solutions to eqs. (2.2)are unique up to normalization constants in the following way (See [2]).

$$\begin{aligned}\mid B\rangle &= \sum_{pq} N_B^{pq} (\sum_n \mid n\rangle_{pq} \times \mid \tilde{U} \mid n\rangle_{pq}) \\ \mid C\rangle &= \sum_{pq} N_C^{pq} (\sum_n \mid n\rangle_{pq} \times \tilde{V} \mid \tilde{n}\rangle)\end{aligned} \tag{2.3}$$

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where $| n\rangle_{pq}$, $| \tilde{n}\rangle_{pq}$ are the complete set of L_0 , $\tilde{L}_0$ eigenstates which are the descendants of the primary fields ($h = h_{pq} = \tilde{h}_{pq}$), N_B^{pq} and N_C^{pq} are the normalization constants of each primary fields. U , V are anti-unitary operators with the following properties.

$$
\begin{aligned}
UL_nU^\dagger =&\tilde{L}_n \\
VL_nV^\dagger =&(-)^n\tilde{L}_n
\end{aligned}
$$

As will become clear later, there are more than one solutions to eqs. (2.2), hence we the possible values of the constants N_B^{pq} and N_C^{pq} are not unique.what does this mean ? It means that there are more than one types of boundary conditions.

Cardy argued that there are three generic types of boundary conditions : free, fixed, and Neumann boundary conditions , when the system has a single order parameter. Our results might suggest that there exist non generic types of boundary conditions or that the system has several order parameters and each parameter can independently take one of the three types of boundary conditions .

On the other hand, the traces in the left hand side of eqs. (2.1)are calculated up to the multiplicities of the primary fields. We denote those multiplicities for each primary fields with $h = h_{pq}$, $\tilde{h} = \tilde{h}_{pq}$ in periodic and free (or fixed) boundary conditions as $a_{pq,\bar{p}\bar{q}}$ and b_{pq} respectively. Now substituting eqs. (2.3)into eqs. (2.1)we obtain,

$$
\begin{aligned}
\sum_{p,q;\bar{p},\bar{q}} a_{p,q;\bar{p},\bar{q}}\chi_{p,q}(\tau)\bar{\chi}_{\bar{p},\bar{q}}(\bar{\tau}) &= \sum_{p,q;\bar{p},\bar{q}} a_{p,q;\bar{p},\bar{q}}\chi_{pq}(-\frac{1}{\tau})\bar{\chi}_{\bar{p}\bar{q}}(-\frac{1}{\bar{\tau}}) \\
\sum_{p,q} a_{p,q;p,q}\chi_{pq}(2\tau) &= \sum_{p,q}(N_B^{pq})^2\chi_{pq}(-\frac{1}{2\tau}) \\
\sum_{p,q} b_{pq}\chi_{pq}(\tau) &= \sum_{p,q}(N_B^{pq})^2\chi_{pq}(-\frac{1}{\tau}) \\
\sum_{p,q} b_{pq}\tilde{\epsilon}_{pq}e^{h_{pq}\pi i}\chi_{pq}(\tau+\frac{i}{2}) &= \sum_{p,q} N_B^{pq}N_C^{pq}e^{h_{pq}\pi i}\chi_{pq}(-\frac{1}{4\tau}+\frac{i}{2})
\end{aligned}
\tag{2.4}
$$

ϵ_{pq} and $\tilde{\epsilon}_{pq}$ represent the action of Ω and f on the state $|p,q;p,q\rangle$ as was mentioned

before. χ_{pq} is the partition function in the Hilbert space containing the state $|p;q\rangle$. The explicit form is given as

$$\chi_{pq}(\tau)=\prod_{n=1}^{\infty}(1-e^{2\pi i\tau n})^{-1}$$

$$\times\sum_{k=-\infty}^{\infty}((\exp(\frac{\pi i\tau}{2m(m+1)}[(2m(m+1)k+(m+1)p-mq)^2-1])-\{q\rightarrow -q\}) \tag{2.5}$$

Using Poisson summation formula and the modular transformation properties of Dedekind's eta function, we obtain the following sets of summation formulas

$$\sum_{p,q,\bar{p},\bar{q}} a_{p,q;\bar{p},\bar{q}}(-)^{(p+q)(p'+q')+(\bar{p}+\bar{q})(\bar{p}'+\bar{q}')}$$

$$\sin\frac{\pi pp'}{m}\sin\frac{\pi qq'}{m+1}\sin\frac{\pi\bar{p}\bar{p}'}{m}\sin\frac{\pi\bar{q}\bar{q}'}{m+1}=\frac{m(m+1)}{8}a_{p'q',\bar{p}'\bar{q}'}$$

$$\sum_{p,q}\epsilon_{pq}a_{p,q;p,q}(-)^{(p+q)(p'+q')}\sin\frac{\pi pp'}{m}\sin\frac{\pi qq'}{m+1}=\sqrt{\frac{m(m+1)}{8}}(N_C^{p'q'})^2$$

$$\sum_{p,q}b_{pq}(-)^{(p+q)(p'+q')}\sin\frac{\pi pp'}{m}\sin\frac{\pi qq'}{m+1}=\sqrt{\frac{m(m+1)}{8}}(N_B^{p'q'})^2$$

$$(\text{for}(m+1)p'-mq'=\text{odd})$$

$$\{\sum_{(m+1)p-mq=odd,pq=odd}\tilde{\epsilon}_{pq}b_{pq}\cos\frac{\pi p((m+1)p'-mq')}{2m}\cos\frac{\pi q((m+1)p'-mq')}{2(m+1)}$$

$$+\sum_{(m+1)p-mq=odd,pq=even}\tilde{\epsilon}_{pq}b_{pq}\sin\frac{\pi p((m+1)p'-mq')}{2m}\sin\frac{\pi q((m+1)p'-mq')}{2(m+1)}\}$$

$$=\sqrt{\frac{m(m+1)}{16}}N_B^{p'q'}N_C^{p'q'}$$

$$(\text{for}(m+1)p'-mq'=\text{even})$$

$$\sum_{(m+1)p-mq=even}\tilde{\epsilon}_{pq}b_{pq}\sin\frac{\pi p((m+1)p'-mq')}{2m}\sin\frac{\pi q((m+1)p'-mq')}{2(m+1)}$$

$$=\sqrt{\frac{m(m+1)}{16}}N_B^{p'q'}N_C^{p'q'} \tag{2.6}$$

The first and the third equations have already been obtained by Cardy, while the other equations are the duality conditions on non-orientable surfaces with crosscaps.

Solving eqs. (2.6), we can rederive the same results those obtained by Cardy as well as other new results, without imposing on discrete Z_n symmetries.

1. $m = 3$

When $a_{p,q;\bar{p},\bar{q}} = \delta_{p\bar{p}}\delta_{q\bar{q}}$ (Ising model)

We have only two solutions as listed in Table 1.

1) and 2) are just the solutions in the free and fixed boundary conditions.

2. $m = 4$

When $a_{p,q;\bar{p},\bar{q}} = \delta_{p\bar{p}}\delta_{q\bar{q}}$ (Tri-critical Ising model)

We have eight solutions as listed in Table 2.

3) and 4) are the solutions in the free boundary condition as obtained by Cardy. He could not obtain the solutions in the fixed boundary condition.

3. $m = 5$

When $a_{11,11} = a_{21,21} = a_{31,31} = a_{41,41} = 1$,

$a_{41,11} = a_{11,41} = a_{21,31} = a_{31,21} = 1$,

and $a_{33,33} = a_{43,43} = 2$, (3 state Potts model)

we have four solutions, which are listed in Table 3.

1) is the solution in fixed boundary condition,

and

2) and 4) are the solutions in the free boundary conditions in ref 4) when $K = 0$ and $K = 1$. Again $K = 1$ case was eliminated by Cardy because it has an additional operator.

For general m we have so far found 4 solutions in the case of the main sequence; i.e. $a_{p,q;\bar{p},\bar{q}} = \delta_{p\bar{p}}\delta_{p\bar{p}}$, though there might exist other solutions.

1. $\epsilon_{pq} = \tilde{\epsilon_{pq}} = 1$

$$b_{pq} = \begin{cases} 1, & \text{if p,q=odd} \\ 0 & \text{otherwise.} \end{cases}$$

2. $\epsilon_{pq} = \tilde{\epsilon_{pq}} = 1$

$$b_{pq} = \begin{cases} 1 & \text{if (m+1)p-mq=odd} \\ 0 & \text{if (m+1)p-mq=even} \end{cases}$$

3. $\epsilon_{pq} = \tilde{\epsilon_{pq}} = 1$

$$b_{pq} = \begin{cases} 1 & \text{if p=odd, q=1} \\ 0 & \text{otherwise} \end{cases}$$

4. $\epsilon_{pq} = \tilde{\epsilon_{pq}} = 1$

$$b_{pq} = \begin{cases} 1 & \text{if p=q=1 or p=m-1, q= m (mod2)} \\ 0 & \text{otherwise} \end{cases}$$

These include the solutions for m=3,4.

When m=3 (1) and (3) are 1) (fixed) ; (2) and (4) are 2) (free).

When m=4 (1) is 1); (2) is 4) (free K=1); (3) is 3) (free K=0) ;(4) is 2).

From the above results we conjecture that

for m=even

solutions in the free boundary conditions are (2) and (3),

and

solutions in the fixed boundary conditions are (1) and (4),

for m=odd

solutions in the free boundary conditions are (2) and (4),

and

solutions in the fixed boundary conditions are (1) and (3).

We are not yet clear whether the duality conditions are not only necessary but also sufficient conditions i.e. every solution to eqs. (2.6)corresponds to some boundary condition of some statistical system. Instead, it might be that some solutions to eqs. (2.6)are not physically acceptable, and we must impose further restrictions to obtain meaningful solutions. Nevertheless, we believe this work leads to considerable progress, since in any case the duality conditions seems to be restrictive enough to reduce the answers to a finite number of choices.

To find solutions for general m, in the case when $a_{pq,\bar{p}\bar{q}}$ are not in the main sequence or when $\epsilon_{pq}, \epsilon_{\tilde{pq}}$ are not all +1, is a future problem. It would also be possible to classify the operator contents in the unitary superconformal field theory on surfaces with boundaries and crosscaps. They are of great interest in constructing open string theories in curved space.

Table 1. The solutions of the duality conditions for m = 3

	ε_{11}	ε_{21}	ε_{22}	$\tilde{\varepsilon}_{11}$	b_{11}	$\tilde{\varepsilon}_{21}$	b_{21}	$\tilde{\varepsilon}_{22}$	b_{22}	
1)	+	+	±	+	1		0		0	[Fixed]
2)	+	+	±	+	1	±	1		0	[Free]

Table 2. The solutions of the duality conditions for m = 4

	ε_{11}	ε_{31}	ε_{32}	ε_{33}	ε_{21}	ε_{22}	$\tilde{\varepsilon}_{11}$	b_{11}	$\tilde{\varepsilon}_{31}$	b_{31}	$\tilde{\varepsilon}_{32}$	b_{32}	$\tilde{\varepsilon}_{33}$	b_{33}	
1)	+	+	+	+	+	+	+	1		0		0		0	
2)	+	+	+	+	+	+	+	1		0	+	1		0	
3)	+	+	+	+	+	+	+	1	+	1		0		0	[Free K=0]
4)	+	+	+	+	+	+	+	1	+	1	+	1	+	1	[Free K=1]
5)	+	+	+	+	−	−		0	+	1		0		0	
6)	+	+	+	+	−	−		0	+	1		0	+	1	
7)	+	+	+	+	−	−	+	1	−	1		0		0	
8)	+	+	+	+	−	−	+	1	−	1	+	1	−	1	

Table 3. The solutions of the duality conditions for m = 5

	ε_{pq}	$\tilde{\varepsilon}_{11}$	b_{11}	$\tilde{\varepsilon}_{21}$	b_{21}	$\tilde{\varepsilon}_{31}$	b_{31}	$\tilde{\varepsilon}_{41}$	b_{41}	$\tilde{\varepsilon}_{33}$	b_{33}	$\tilde{\varepsilon}_{43}$	b_{43}	
1)	+(for all)	+	1		0		0	+	1		0		0	[Fixed]
2)	+(for all)	+	1		0		0	−	1		0	+	2	[Free K=0]
3)	+(for all)	+	1	+	1	+	1	+	1		0		0	
4)	+(for all)	+	1	−	1	+	1	−	1	+	2	+	2	[Free K=1]

RENORMALIZATION GROUP FLOW and STRING DYNAMICS

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ABSTRACT

The renormalization group flow in the nonlinear sigma model approach is explicitly solved up to fourth order in the case of an open string propagating in the tachyon background. We show that its fixed point solution produces the tree-level 5-point tachyon amplitude. Furthermore this argument is extended to all orders.

1.Introduction.

One of the most mysterious features of a string theory is the connection between the two dimensional world sheet physics and the space-time physics. For example, N=2 world sheet supersymmetry implies N=1 space-time supersymmetry, the world sheet currents represent the space-time gauge symmetry, and the conformal invariance of the world sheet physics seems to be connected with the dynamics of a string, etc. Here we investigate the relationship between the renormalization group (RG) and string dynamics in the context of an open bosonic string theory. We concentrate on the tachyon mode for simplicity and complete the argument of Klebanov and Susskind (K-S). This report is based on the work of N.Nakazawa, K.Sakai and myself.

Nowadays it is widely believed that the equations resulting from setting all the β-functions of the two dimensional nonlinear σ-model to zero are equivalent to variational equations for a space-time functional, the effective action of string theory. That is, the equation

$$\beta^i = G^{ij}\frac{\delta I}{\delta g^j}$$

is expected to be valid where β^i is the β-function corresponding to the coupling g^i, which represents a component field of string excitations, and I is the string

effective action. At this moment, however, there appears to be no reliable proof of this important issue. K-S showed that the solutions of the renormalization group fixed-point equations generate open string scattering amplitudes. Their strategy is the following: First the most general RG equations are assumed to be

$$\beta^i = \frac{dg^i}{dt} = \lambda_i g^i + \alpha^i_{jk} g^j g^k + \gamma^i_{jkl} g^j g^k g^l + \delta^i_{jklm} g^j g^k g^l g^m + \cdots , \tag{1}$$

and RG flow is solved as follows

$$g^i(t) = \exp(\lambda_i t) g^i(0) + \{\exp[(\lambda_j + \lambda_k)t] - \exp(\lambda_i t)\} \frac{\alpha^i_{jk}}{\lambda_j + \lambda_k - \lambda_i} g^j(0) g^k(0) + \cdots \tag{2}$$

Comparing this solution eq.(2) with the perturbative calculations in the nonlinear σ-model, the coefficients of the β -function in eq.(1) are determined. On the other hand one finds a perturbative solution to the fixed point equation

$$\beta^i = \lambda_i g^i + \alpha^i_{jk} g^j g^k + \gamma^i_{jkl} g^j g^k g^l + \delta^i_{jklm} g^j g^k g^l g^m + \cdots = 0 , \tag{3}$$

as follows

$$g^i = g^i_0 - \frac{1}{\lambda_i} \alpha^i_{jk} g^j_0 g^k_0 + \frac{1}{\lambda_i} \left(\frac{2\alpha^i_{jm} \alpha^m_{kl}}{\lambda_m} - \gamma^i_{jkl} \right) g^j_0 g^k_0 g^l_0 + \cdots , \tag{4}$$

where $\lambda_i g^i_0 = 0$. If the coefficients of the solution reproduce the scattering amplitude of a string theory, we can give a strong evidence for the equivalence between the vanishing β-function and a true equation of motion (which must satisfy the appropriate properties, the one particle irreducibility, the finiteness and so on).

K-S showed that the above procedure really reproduces the amplitudes of a string theory up to third order in the coupling g^i and conjectured it up to all orders. In this note, we shall demonstrate the correctness of the procedure up to all orders.

The construction of this report is as follows: In sect.2 the calculations of K-S are repeated to explain our regularization. In sect.3 an explicit fourth order calculation is shown. In sect.4 we shall prove our claim. In the last section we conclude by some discussions.

2.Regularization.

Let us recall that an open bosonic string propagating in the tachyon background is represented by the following action

$$S = \frac{1}{4\pi}\int_{y>0} dxdy\eta^{ab}\partial_a X_\mu \partial_b X^\mu + \int_{-\infty}^{\infty} \frac{dx}{a}\int dkT(k)\exp(ikX).$$

Here the explicit short distance cut off parameter, a, is introduced. Let us expand X^μ into a classical part X_0^μ and a quantum part Y^μ , $X^\mu = X_0^\mu + Y^\mu$. The effective action is given by

$$S_{eff}(X_0) = -\log W(X_0),$$

$$W(X_0) = \exp[-S_0(X_0)]\int [DY]\exp\{-\frac{1}{4\pi}\int_{y>0} dxdy\eta^{ab}\partial_a Y_\mu \partial_b Y^\mu$$
$$-\int_{-\infty}^{\infty}\frac{dx}{a}\int dkT(k)\exp(ikX_0)\exp(ikY)\}.$$

As a two dimensional field theory, the nonlinear sigma model on a two-sphere is divergent at short distances. In the leading order we find the contribution to $W(X_0)$

$$-\int_{-\infty}^{\infty}\frac{dx}{a}\int dkT(k)\exp(ikX_0) < \exp(ikY) >$$
$$= -\int_{-\infty}^{\infty} dx\int dka^{k^2-1}T(k)\exp(ikX_0),$$

where we use the cut off a in the self contraction $< Y(x)Y(x) >= -2\log a$. Therefore,

$$S_{eff}(X_0) = S_0(X_0) + \int dx\int dka^{k^2-1}T(k)\exp(ikX_0) + \cdots .$$

Up to this order, the conformal invariance of the nonlinear sigma model gives the linearized equation of motion. In order to obtain the nonlinear terms, let us

calculate the higher order contribution to $W(X_0)$. The second order term in the expansion of $W(X_0)$ is

$$\int_{-\infty}^{\infty}\frac{dx_1}{a}\int_{-\infty}^{x_1}\frac{dx_2}{a}\int dk_1 T(k_1)\int dk_2 T(k_2)$$
$$\times \exp[ik_1X_0(x_1)+ik_2X_0(x_2)] < \exp[ik_1Y(x_1)]\exp[ik_2Y(x_2)] > .$$

The necessary integral is

$$\begin{aligned}\int_{-\infty}^{x_1}\frac{dx_2}{a^2} < \exp[ik_1Y(x_1)]\exp[ik_2Y(x_2)] > &= \int_{-\infty}^{x_1} dx_2 a^{k_1^2+k_2^2-2}(x_1-x_2)^{2k_1\cdot k_2}\\ &= \int_a^{\infty} dt a^{k_1^2+k_2^2-2}t^{2k_1\cdot k_2}\\ &= a^{(k_1+k_2)^2-1}\frac{-1}{2k_1\cdot k_2+1},\end{aligned}$$

where the transformation of variable, $t = x_1 - x_2$, is performed. We also assume the condition $2k_1\cdot k_2 + 1 < 0$. The third order term in $W(X_0)$ is

$$-\int_{-\infty}^{\infty}\frac{dx_1}{a}\int_{-\infty}^{x_1}\frac{dx_2}{a}\int_{-\infty}^{x_2}\frac{dx_3}{a}\int dk_1T(k_1)\int dk_2T(k_2)\int dk_3T(k_3)$$
$$\times \exp[ik_1X_0(x_1)+ik_2X_0(x_2)+ik_3X_0(x_3)]$$
$$\times < \exp[ik_1Y(x_1)]\exp[ik_2Y(x_2)]\exp[ik_3Y(x_3)] > .$$

The requisite integral is reduced by using the transformation, $t = x_1 - x_3$ and $t\nu = (x_1 - x_2)$, to

$$\begin{aligned}&-\int_{-\infty}^{x_1} dx_2\int_{-\infty}^{x_2} dx_3 a^{k_1^2+k_2^2+k_3^2-3}(x_1-x_2)^{2k_1\cdot k_2}(x_2-x_3)^{2k_2\cdot k_3}(x_1-x_3)^{2k_1\cdot k_3}\\ &= -a^{k_1^2+k_2^2+k_3^2-3}\int_a^{\infty} dt t^{\sum_{1\le i<j\le 3}2k_i\cdot k_j+1}\int_0^1 d\nu \nu^{2k_1\cdot k_2}(1-\nu)^{2k_2\cdot k_3}\\ &= \frac{a^{(k_1+k_2+k_3)^2-1}}{\sum_{1\le i<j\le 3}2k_i\cdot k_j+2}B(1+2k_1\cdot k_2, 1+2k_2\cdot k_3).\end{aligned}$$

In this calculation we use the cut off only in the t-integration. This regularization

procedure, which is slightly different from K-S, corresponds to picking up the singularity which originates in shrinking all the vertex operators together. Up to this order the renormalized coupling is

$$\begin{aligned}\tilde{T}(k) = a^{k^2-1}\{T(k) &+ \int dk_1 \int dk_2 \frac{T(k_1)T(k_2)}{2k_1\cdot k_2+1}\delta(k_1+k_2-k) \\ &- \int dk_1 \int dk_2 \int dk_3 T(k_1)T(k_2)T(k_3) \\ &\times \frac{B(1+2k_1\cdot k_2, 1+2k_2\cdot k_3)}{\sum_{1\le i<j\le 3} 2k_i\cdot k_j + 2}\delta(k_1+k_2+k_3-k)\}\,.\end{aligned} \tag{5}$$

Comparing the above result with RG flow in eq.(2), we can find

$$\begin{aligned}\alpha^k_{k_1k_2} &= -\delta(k_1+k_2-k)\,, \\ \gamma^k_{k_1k_2k_3} &= \delta(k_1+k_2+k_3-k)\{\frac{2}{2k_1\cdot k_2+2k_1\cdot k_3+1} \\ &\qquad + B(1+2k_1\cdot k_2, 1+2k_2\cdot k_3)\}.\end{aligned} \tag{6}$$

Substituting these results into eq.(4), we find that the coefficient of the solution of eq.(3) gives the correct 4-point amplitude.

3.Fourth order calculation.

It is not so trivial to see whether this success up to the third order continues to be valid or not. Here we analyze the fourth order case explicitly for the purpose of supporting the discussion in the next section. The fourth order term is

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{dx_1}{a}\int_{-\infty}^{x_1}\frac{dx_2}{a}\int_{-\infty}^{x_2}\frac{dx_3}{a}\int_{-\infty}^{x_3}\frac{dx_4}{a}&\int dk_1dk_2dk_3dk_4T(k_1)T(k_2)T(k_3)T(k_4) \\ &\times \exp[ik_1X_0(x_1)+ik_2X_0(x_2)+ik_3X_0(x_3)+ik_4X_0(x_4)] \\ &\times <\exp[ik_1Y(x_1)]\exp[ik_2Y(x_2)]\exp[ik_3Y(x_3)]\exp[ik_4Y(x_4)]>\,.\end{aligned}$$

The necessary integral is

$$\int_{-\infty}^{x_1} dx_2 \int_{-\infty}^{x_2} dx_3 \int_{-\infty}^{x_3} dx_4 a^{k_1^2+k_2^2+k_3^2-4}(x_1-x_2)^{2k_1\cdot k_2}(x_1-x_3)^{2k_1\cdot k_3}$$
$$\times (x_1-x_4)^{2k_1\cdot k_4}(x_2-x_3)^{2k_2\cdot k_3}(x_2-x_4)^{2k_2\cdot k_4}(x_3-x_4)^{2k_3\cdot k_4}$$

$$\begin{aligned}
&= a^{k_1^2+k_2^2+k_3^2-4}\int_a^{\infty} dt\, t^{\sum_{1\le i<j\le 4} 2k_i\cdot k_j+2.} \\
&\quad\times \int_0^1 d\nu_1 \int_0^{\nu_1} d\nu_2 \nu_1^{2k_1\cdot k_3}\nu_2^{2k_1\cdot k_2}(1-\nu_1)^{2k_3\cdot k_4}(1-\nu_2)^{2k_2\cdot k_4}(\nu_1-\nu_2)^{2k_2\cdot k_3}
\end{aligned} \tag{7}$$

$$= -\frac{a^{(k_1+k_2+k_3)^2-1}}{\sum_{1\le i<j\le 4} 2k_i\cdot k_j+3} V^{(5)},$$

where $V^{(5)}$ denotes the 5-point amplitude of the tachyon mode. The forth order coefficient of the solution of RG flow is given by

$$\begin{aligned}
d^i_{jklm}(t) = &\{\exp[(\lambda_j+\lambda_k+\lambda_l+\lambda_m)t]-\exp[\lambda_i t]\}\frac{1}{(\lambda_j+\lambda_k+\lambda_l+\lambda_m-\lambda_i)} \\
&\times [4\alpha^i_{jp}\alpha^p_{mn}\alpha^n_{kl}\frac{1}{(\lambda_k+\lambda_l+\lambda_m-\lambda_p)(\lambda_l+\lambda_m-\lambda_n)} \\
&+ 2\alpha^i_{jp}\gamma^p_{mkl}\frac{1}{(\lambda_k+\lambda_l+\lambda_m-\lambda_p)} + 3\gamma^i_{jkp}\alpha^p_{lm}\frac{1}{(\lambda_j+\lambda_m-\lambda_p)} \\
&+ \alpha^p_{jk}\alpha^i_{pn}\alpha^n_{ml}\frac{1}{(\lambda_j+\lambda_k-\lambda_p)(\lambda_l+\lambda_m-\lambda_n)} + \delta^i_{jklm}] + \cdots,
\end{aligned} \tag{8}$$

where we suppressed the irrelevant parts when put on shell. As we would like to show that our off-shell β-function can be used to obtain the on-shell amplitudes, the suppressed terms are indeed irrelevant after all for this purpose. The fourth order part of the fixed point solution is

$$\begin{aligned}
g^i_4 = -\frac{1}{\lambda_i}\{&4\alpha^i_{jp}\frac{1}{\lambda_p}\alpha^p_{mn}\frac{1}{\lambda_n}\alpha^n_{kl} - 2\alpha^i_{jp}\frac{1}{\lambda_p}\gamma^p_{mkl} \\
&+ \alpha^p_{jk}\frac{1}{\lambda_p}\alpha^i_{pn}\frac{1}{\lambda_n}\alpha^n_{ml} - 3\gamma^i_{jkp}\frac{1}{\lambda_p}\alpha^p_{lm} + \delta^i_{jklm}\}g^j_0 g^k_0 g^l_0 g^m_0 .
\end{aligned} \tag{9}$$

Comparing eq.(7) with eq.(8), we find

$$
\begin{aligned}
-V^{(5)i}_{jklm} =& 4\alpha^{i}_{jp}\alpha^{p}_{mn}\alpha^{n}_{kl}\frac{1}{(\lambda_k+\lambda_l+\lambda_m-\lambda_p)(\lambda_l+\lambda_m-\lambda_n)} \\
&+2\alpha^{i}_{jp}\gamma^{p}_{mkl}\frac{1}{(\lambda_k+\lambda_l+\lambda_m-\lambda_p)} \\
&+3\gamma^{i}_{jkp}\alpha^{p}_{lm}\frac{1}{(\lambda_j+\lambda_m-\lambda_p)} \\
&+\alpha^{p}_{jk}\alpha^{i}_{pn}\alpha^{n}_{ml}\frac{1}{(\lambda_j+\lambda_k-\lambda_p)(\lambda_l+\lambda_m-\lambda_n)} \\
&+\delta^{i}_{jklm}.
\end{aligned}
\tag{10}
$$

Substituting eq.(10) into eq.(9),

$$
g^i_4 = \frac{1}{\lambda_i} V^{(5)i}_{jklm} g^j_0 g^k_0 g^l_0 g^m_0 .
$$

Thus the fact that our off-shell β-function can be used to obtain the on-shell amplitudes is proved to be valid to the fourth order level. In the next section, the arguments in this section are generalized to all orders.

4.A proof to all orders.

In this section we show that the solutions of the renormalization group fixed point equations generate open string scattering amplitudes to all orders. First the fact that the renormalized couplings are factorized into the amplitude and the remaining parts is shown in our regularization procedure. What we should do is to solve the RG flow to derive the β-function. Setting $\beta = 0$, we recognize that the β-function produces the correct on-shell amplitudes to all orders.

The n-th order contribution to $W(X_0)$ is

$$
\begin{aligned}
&(-1)^n \int_{-\infty}^{\infty} \frac{dx_1}{a} \int_{-\infty}^{x_1} \frac{dx_2}{a} \cdots \int_{-\infty}^{x_{n-1}} \frac{dx_n}{a} \int dk_1 T(k_1) \cdots \int dk_n T(k_n) \\
&\times \exp[ik_1 X_0(x_1) + \cdots + ik_n X_0(x_n)] < \exp[ik_1 Y(x_1)] \cdots \exp[ik_n Y(x_n)] > .
\end{aligned}
$$

To see the factorization of amplitude the following integral is necessary

$$
\begin{aligned}
&(-1)^n \int_{-\infty}^{x_1} dx_2 \cdots \int_{-\infty}^{x_{n-1}} dx_n a^{\sum_{i=1}^n k_i^2 - n} \prod_{1\le i<j\le n} (x_i - x_j)^{2k_i\cdot k_j} \\
&= (-1)^n a^{\sum_{i=1}^n k_i^2 - n} \int_a^{\infty} dt t^{\sum_{1\le i<j\le n} 2k_i\cdot k_j + n-2} \int_0^1 d\nu_1 \int_0^{\nu_1} d\nu_2 \cdots \int_0^{\nu_{n-3}} d\nu_{n-2} \\
&\quad \times \prod_i \nu_i^{2k_1\cdot k_i} \prod_i (1-\nu_i)^{2k_2\cdot k_i} \prod_{i<j} (\nu_i - \nu_j)^{2k_i\cdot k_j} \\
&= -(-1)^n \frac{a^{(k_1+k_2+k_3)^2-1}}{2k_i\cdot k_j + n - 1} V^{(n+1)},
\end{aligned} \tag{11}
$$

where we used the transformation, $t = x_1 - x_n$ and $t\nu_{n-i} = (x_1 - x_i)$ for $2 \le i \le n-1$ and we note that the sign factor can be absorbed by the coupling redefinition.

Let us turn to the general argument. The most general RG equations are as follows

$$\frac{dg^i}{dt} = \beta^i(g) = \sum_{n=1}^{\infty} \lambda^i_{j_1\cdots j_n} g^{j_1}\cdots g^{j_n}, \quad \lambda^i_j = \lambda_i \delta^i_j . \tag{12}$$

Here the weak field expansion is performed and the anomalous dimension matrix is diagonalized. To solve the RG flow we set

$$g^i(t) = \sum_{n=1}^{\infty} \alpha^i_{j_1\cdots j_n}(t) g^{j_1}(0)\cdots g^{j_n}(0). \tag{13}$$

Substituting eq.(13) into eq.(12), we obtain

$$\dot{\alpha}^i_{j_1\cdots j_n}(t) = \lambda_i \alpha^i_{j_1\cdots j_n}(t) + \Lambda(\lambda,\alpha)^i_{j_1\cdots j_n}, \tag{14}$$

where Λ includes the lower order coefficients with respect to α. We split $\alpha^i_{j_1\cdots j_n}(t)$ into relevant parts and irrelevant parts $\tilde{\alpha}^i_{j_1\cdots j_n}(t)$ in the on shell limit as we did in

eq.(8),

$$\alpha^i_{j_1\cdots j_n}(t) = A^i_{j_1\cdots j_n}\exp[(\lambda_{j_1}+\cdots+\lambda_{j_n})t] + \tilde{\alpha}^i_{j_1\cdots j_n}(t)\,.$$

After inserting this expression into eq.(14), the functional form of Λ is shown to be invariant in the leading order. Then eq.(14) can be rewritten as follows

$$\dot{\alpha}^i_{j_1\cdots j_n}(t) = \lambda_i\alpha^i_{j_1\cdots j_n}(t) + \Lambda(\lambda,A)^i_{j_1\cdots j_n}\exp[(\lambda_{j_1}+\cdots+\lambda_{j_n})t] + \tilde{\Lambda}(\lambda,\tilde{\alpha})^i_{j_1\cdots j_n}\,,$$

which is easily solved as

$$\alpha^i_{j_1\cdots j_n}(t) = \frac{\Lambda(\lambda,A)^i_{j_1\cdots j_n}}{\lambda_{j_1}+\cdots+\lambda_{j_n}-\lambda_i}\{\exp[(\lambda_{j_1}+\cdots+\lambda_{j_n})t] - \exp[\lambda_i t]\} + \cdots\,. \tag{15}$$

This eq.(15) gives the new $A^i_{j_1\cdots j_n}$ as a function of the lower order coefficients of A.

On the other hand, the fixed point equation is solved perturbatively

$$g^i = \sum_{n=1}^{\infty}\gamma^i_{j_1\cdots j_n}g_0^{j_1}\cdots g_0^{j_n}\,, \tag{16}$$

where g_0^i represents the solution of linearized on shell condition. The result is

$$\gamma^i_{j_1\cdots j_n} = \frac{\Lambda(\lambda,\gamma)^i_{j_1\cdots j_n}}{-\lambda^i}\,, \tag{17}$$

where Λ is the same one as in eq.(14). Using the lower order results we can conclude the equivalence of the two expressions by induction

$$\Lambda^i_{j_1\cdots j_n}(\lambda,A) = \Lambda^i_{j_1\cdots j_n}(\lambda,\gamma) \tag{18}$$

at the on shell, $\lambda_{j_1} = \cdots = \lambda_{j_n} = 0$.

Comparing eq.(15) with eq.(11), we find that Λ gives rise to the correct amplitude.

5.Discussions.

If we consider a string theory as a unified theory including gravity, we must resolve the many vacuum problem. For this purpose, the string field theory has received much interest of physicists. In spite of many efforts, there exists no complete string field theory. In our view point we must search for other possibilities. As for such a candidate, the non-linear σ-model approach seems to be attractive. As a modest step we proved the necessary condition for the β- function to be the equation of motion for a string in the case of the tachyon background. Of course to prove the equivalence other conditions should be verified. Furthermore the systematic method to treat all the string modes should be found. At any rate further investigations are necessary.

Interacting models on the torus

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abstract

We formulate the U(N)WZW model constructed vertex operators on the torus. This formulation corresponds to the chiral bosonization. We give the 2n-point correlation function in this formulation.

1. Introduction

It has been learnt that the theory of Virasoro algebras[1] and affine Kac-Moody algebras[2] provides us an extremely powerful framework for studying models in two dimensions and string theories, because these algebras correlate in a uniform way diverse results of models.

In the previous paper[3], We have given a framework for studying interacting models. Restricting ourselves to representations of level one, we formulated the $U_L(N)\otimes U_R(N)$ symmetric Thirring model[4] and the associated Wess-Zumino-Witten(WZW) model[5] by the method of the vertex operator construction[6]. In this formulation we characterize

operator construction[6]. In this formulation we characterize the fermions of Thirring model and the field of the WZW model in terms of vertex operators, namely, by using the N-dimensional vectors of a root space. Further, as we can remove the singularities derived from interactions of the U(1) current by introducing regulators, we can construct the currents as the composite operators of the fields of these models. We also calculated 2n-point correlation functions by using a Fock space of bosons and Euclidean space-time being taken to be a sphere. We confirm that these correlation functions are the solutions of Knizhnik and Zamolodchikov equations[7], which should be satisfied when the Virasoro generator is expressed in quadratic forms of currents.

It has been studied that the bosonization of chiral fermion theories[8] and the WZW model[9] can be extended to the theories on arbitrary compact Riemann surfaces. So it is interesting to extend our scheme to the formulation on the torus and arbitrary compact Riemann surfaces, and to compare the results with the general properties. In this talk we will comment on the WZW model on the torus in the first place. We can obtain the 2n-point correlation function by the careful treatment of zero modes.

2. A left-moving boson and vertex operators

A left-moving boson is defined as

$$\phi(z) = \phi_0(z) + \phi_{osci}(z) \tag{2.1}$$

$$\phi_0 = q - ip\log z,$$

$$\phi_{osci} = i\sum_{n\neq 0} \frac{1}{n} J_n z^{-n},$$

$$[J_m, J_n] = m\, \delta_{m+n,0}, \quad [q, p] = i. \tag{2.2}$$

Here z is the complex number to express space-time. The conserved u(1) current is given by:

$$i\partial_z Q(z) = J(z) = \sum_{n=-\infty}^{\infty} J_n z^{-n-1}. \tag{2.3}$$

The vacuum of the current Fock space, $|0\rangle_c$ satisfies the condition:

$$J_n |0\rangle_c = 0 \quad n = 1,2,\cdots\cdots. \tag{2.4}$$

In order to determine the vacuum of the boson, we have to consider the condition on the zero mode. Since we assume that the boson exists on a circle of a unit radius, the states of zero mode are invariant under the shift of 2π on the circle:

$$e^{i2\pi p}|\text{state}\rangle_{zero\ mode} = |\text{state}\rangle_{zero\ mode}. \tag{2.5}$$

This condition restricts the eigenvalues of p as p = 0, ±1, ±2,⋯⋯. Then the states of zero mode are written as

$$|n\rangle_{zero\ mode} \quad n = 0, \pm 1, \pm 2,\cdots\cdots, \tag{2.6}$$

and

$$\langle n|m\rangle = \delta_{n,m}. \tag{2.7}$$

Here we interpret the above eigenvalues as winding numbers around

the circle. So we can choose the vacuum of the bosons as

$$|\mathrm{vac}\,\rangle_b = |0\,\rangle_{\mathrm{zero\ mode}} \otimes |0\,\rangle_c. \tag{2.8}$$

e^{iqm} is a operator to change the winding number:

$$e^{iqm}\,|n\,\rangle = |n + m\,\rangle, \tag{2.9}$$

where we use the the commutation relation $[\,p,\ e^{iqm}\,] = m\,e^{iqm}$.
We define a vertex operator by

$$V^{\alpha}(z) = :e^{i\alpha\cdot Q(z)}:$$

$$= z^{-\alpha^2/2}\; e^{i\alpha\cdot Q_<(z)}\; e^{i\alpha\cdot Q_0(z)}\, e^{i\alpha\cdot Q_>(z)} \tag{2.10}$$

$$Q_> = i\sum_{n>0}\frac{1}{n}\,J_n\,z^{-n}, \qquad Q_> = i\sum_{n<0}\frac{1}{n}\,J_n\,z^{-n},$$

and

$$Q_0(z) = q - ip\log z.$$

Here $:\cdots:$ denotes normal ordering with respect to J_n. The u(1) charge of $V^{\alpha}(z)$ is equal to α. One can calculate the vacuum expectation value of the products of the vertex operator:

$${}_b\langle\,\mathrm{vac}\,|\; V^{\alpha_1}(z_1)V^{\alpha_2}(z_2)\cdots\cdots\cdots V^{\alpha_n}(z_n)\;|\,\mathrm{vac}\,\rangle_b$$

$$= \begin{cases} \prod_{l<m}(z_l - z_m)^{\alpha_l\cdot\alpha_m} & \text{for } \sum_{j=1}^{n}\alpha_j = 0 \\ 0 & \text{for } \sum_{j=1}^{n}\alpha_j \neq 0 \end{cases} \tag{2.11}$$

and

$$|z_1| > |z_2| > |z_3| > \cdots\cdots > |z_n|.$$

As usual we choose a radial direction for time. Since the vertex operators are radial ordered, the vacuum expectation value(11) coincides with the correlation function of the vertex operators.

3. Vertex operator construction of the U(N)WZW model

We will briefly explain the WZW model in terms of the vertex operators. The Lagrangian of the U(N)WZW model is expressed as

$$
\begin{aligned}
L = & \frac{1}{4\lambda^2} \int d^2x \, \mathrm{Tr}(\partial^{\mu} U^{-1} \partial_{\mu} U) \\
& + \frac{k}{24\pi} \int_B d^3x \, \mathrm{Tr}(\varepsilon^{\lambda\mu\nu} U^{-1} \partial_{\lambda} U \, U^{-1} \partial_{\mu} U \, U^{-1} \partial_{\nu} U)
\end{aligned}
\tag{3.1}
$$

where U takes values of elements in the Lie group SU(N). Euclidean space-time is taken here to be a 2-dimensional sphere, S^2. The second term is expressed as a 3-dimensional integral over a ball region B, whose boundary is given by the S^2 mentioned above. For the special values of the coupling constant, $\lambda^2=4\pi/k$, with $k=1,2,\cdots$, the model remains conformally invariant up to the quantum corrections. The equations of motion are derived as

$$
\begin{aligned}
& \partial_{\bar{z}} \{U^{-1}(z,\bar{z})\partial_z U(z,\bar{z})\} = 0 \\
& \partial_{z} \{U(z,\bar{z})\partial_{\bar{z}} U^{-1}(z,\bar{z})\} = 0
\end{aligned}
\tag{3.2}
$$

Here we prepare a 2N-plet of left- and right- handed bosons, $\{\phi^j(z),\phi^j(w) | \ j=1,\cdots,N\}$ on the torus. We define the fields as

$$\Phi(z) = \sum_{i=1}^{N} \phi^j(z)\, e^j, \quad \bar{\Phi}(\bar{z}) = \sum_{i=1}^{N} \bar{\phi}^j(\bar{z})\, e^j, \tag{3.3}$$

Here $\{e^j\}(j=1,\cdots,N)$ is an orthonormal base of N-dimensional space. The vacuum is defined by the product of the vacuum $|vac>^j$ and $|vac>^j$ for $\phi^j(z)$ and $\phi^j(z)$ $(j=1,\cdots,N)$, respectively:

$$|vac> = \prod_{j=1}^{N} |vac>^j \otimes |vac>^j \tag{3.4}$$

The form of the fields $U(z,\bar{z})$ and $U^{-1}(z,\bar{z})$ are as follows

$$MU_j{}^i(z,\bar{z}) = \lambda^i : e^{i\{(a^i-b))\cdot\Phi(z)+(d^i-c)\cdot\bar{\Phi}(\bar{z})\}} : \lambda_j . \tag{3.5}$$

$$M(U)^{-1}{}_j{}^i(\bar{z},z) = \lambda^i : e^{-i\{(a^i-b))\cdot\Phi(z)+(d^i-c)\cdot\bar{\Phi}(\bar{z})\}} : \lambda_j .$$

Here a_i, b, c_i, d are

$$a^i = e^i - h_1 E \qquad b = k_1 E$$
$$c = h_2 E \qquad d^i = e^i - k_2 E \tag{3.6}$$

where E is a vector defined as

$$E = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} e^i \tag{3.7}$$

λ^j's $(j=1,\cdots,N)$ are the twisting operators, called Klein factor[10], to correct signs. These are written as

$$\lambda_-{}^j = \exp[i \frac{\pi}{2} \{(\sum_{k>j} - \sum_{k<j}) (a^k\cdot p - b\cdot \bar{p}) + \sum_k (-c\cdot p + d^k\cdot \bar{p})\}]$$
$$\lambda_+{}^j = \exp[-i \frac{\pi}{2} \{ \sum_k (a^k\cdot p - b\cdot \bar{p}) + (\sum_{k>j} - \sum_{k<j})(-c\cdot p + d^k\cdot \bar{p})\}] \tag{3.8}$$

where

$$p = \sum_{i=1}^{N} P_i \, e^i \qquad \text{and} \qquad \bar{p} = \sum_{i=1}^{N} \bar{P}_i \, e^i \tag{3.9}$$

M is a regulator to remove the divergence expressed as

$$M = N\,(z - w)^{(a-b)^2}(\bar{z} - \bar{w})^{(c-d)^2} \tag{3.10}$$

Using these expressions, one can realize the relations between the field $U(z,\bar{z})$ and the currents and derive the equations of motion for the WZW model. In this formulation we also obtain the equations from the operator product expansions between the field $U(z,\bar{z})$ and the Virasoro algebra:

$$\partial_z U(z,\bar{z}) = \frac{1}{N+1}\, \hat{:} J^a(z) t^a{}_L U(z,\bar{z}) \hat{:} + \{\frac{1}{\sqrt{N}} + (h_1 - h_2)\} \hat{:} I(z) U z,\bar{z}) \hat{:}$$

$$\partial_z U(z,\bar{z}) = \frac{1}{N+1}\, \hat{:} \bar{J}^a(\bar{z}) t^a{}_R U(z,\bar{z}) \hat{:} + \{\frac{1}{\sqrt{N}} - (k_1 - k_2)\} \hat{:} \bar{I}(\bar{z}) U z,\bar{z}) \hat{:} \tag{3.11}$$

Here $t^a{}_L$ and $t^a{}_R$ are the generators of $SU_L(N)$ and $SU_R(N)$, respectively, and $\hat{:}\cdots\hat{:}$ denotes the normal ordering with respect to the currents as

$$\hat{:} J^a(z) t^a{}_L U(z,\bar{z}) \hat{:} = J^a{}_<(z) U(z,\bar{z}) + U(z,\bar{z}) J^a{}_0 z^{-1} + U(z,\bar{z}) J^a{}_>(z),$$

$$J^a{}_> = \sum_{n=1}^{\infty} J^a{}_n \, z^{-(n+1)}, \qquad J^a{}_< = \sum_{n=1}^{\infty} \bar{J}^a{}_n \, \bar{z}^{-(n+1)}.$$

As discussed in the paper[3], the WZW model corresponds to the Thirring model, when the $U_L(1)$ and $U_R(1)$ charge are equal and the condition, $a^2 - b^2 = 1$ is satisfied.

4. The correlation function on the torus

In this section we give the two-point function of the left-moving boson on the torus. Using this function we will calculate the 2n-point correlation function of the WZW model.

The 2-point correlation function of the left-moving boson is defined as

$$\langle\phi(z)\phi(w)\rangle = \frac{\mathrm{Tr}\ q^{L_0}\ \phi(z)\phi(w)}{\mathrm{Tr}\ q^{L_0}} \tag{4.1}$$

where $q=\exp(2\pi i\tau)$, with $\mathrm{Im}(\tau)>0$. The expectation value of the oscillator mode is easily calculated in the standard coordinate $e(z)=\exp(2\pi iz)$:

$$\langle\phi_{osci}(e(z))\phi_{osci}(e(w))\rangle = -\log\frac{\theta_1(z-w|\tau)}{\eta(\tau)} + \log ie^{1/6\pi i\tau} + \pi i(z-w) \tag{4.2}$$

where $\theta_1(z|\tau)$ is a theta function with characteristic $\begin{pmatrix}1/2\\1/2\end{pmatrix}$ and $\eta(\tau)$ is Dedekind's function defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{4.3}$$

The expectation value of zero modes is well-defined only in the exponential:

$$\mathrm{Tr}\ q^{p^2/2}e^{\,i(\phi_0(z)-\phi_0(w))} = \mathrm{Tr}\ q^{p^2/2}\ e^{\,p\log(z/w)} \tag{4.4}$$

$$= \sum_{p=-\infty}^{\infty} q^{\,p^2/2}\ (z/w)^{\,p}$$

To find the two-point function $\langle J(z)J(w)\rangle$, one have to differentiate both side of (4.2) with respect to z and w:

$$(e(z)e(w))\langle J(e(z))J(e(w))\rangle = \frac{-i}{\pi}\,\partial_\tau \ln\theta_3(0|\tau) \tag{4.5}$$

$$- \frac{1}{\pi^2} \partial_z^2 \ln \theta_1(z-w|\tau) + \frac{5\, e^{2\pi i(z-w)}}{(e^{2\pi i(z-w)} - 1)^2}$$

where $\theta_3(z|\tau)$ is the theta function with characteristic $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We can calculate the 2n-point correlation function of the field defined by (2.4). The result are

$$\langle U^{i_1}{}_{j_1}(e(z_1), e(\bar{z}_1)) \cdots\cdots\cdots\cdots U^{i_n}{}_{j_n}(e(w_n), e(\bar{w}_n))$$

$$\times\ (U^{-1})^{k_n}{}_{l_n}(e(w_n), e(\bar{w}_n)) \cdots\cdots\cdots\cdots U^{k_1}{}_{l_1}(e(z_1), e(\bar{z}_1))\rangle$$

$$= \prod_{i=1}^{\infty} e^{2\pi i(N\alpha^2-\alpha)(z_i-w_i)} \{(1+q^{1/2})e^{-2\pi i\alpha(z_i-w_i)}\}^{-1} \{(1+q^{1/2})e^{2\pi i\alpha(z_i-w_i)}\}$$

$$\times \left[\frac{\prod_{i=1}^{N} \prod_{m=0}^{\infty} \{(1+q^{m+\alpha-1/2})e^{-2\pi i\alpha(z_i-w_i)}\}\{(1+q^{m+\alpha+1/2})\, e^{2\pi i\alpha(z_i-w_i)}\}}{\prod_{i=1}^{N} \prod_{m=0}^{\infty} \{(1+q^{m+\alpha-1/2})\}\{(1+q^{m+\alpha+1/2})\}} \right]^N$$

$$\times \prod_{i=1}^{\infty} e^{2\pi i(N\beta^2-\beta)(\bar{z}_i-\bar{w}_i)} \{(1+\bar{q}^{1/2})e^{-2\pi i\beta(\bar{z}_i-\bar{w}_i)}\}^{-1} \{(1+\bar{q}^{1/2})e^{2\pi i\beta(\bar{z}_i-\bar{w}_i)}\}$$

$$\times \left[\frac{\prod_{i=1}^{N} \prod_{m=0}^{\infty} \{(1+\bar{q}^{m+\beta-1/2})e^{-2\pi i\beta(\bar{z}_i-\bar{w}_i)}\}\{(1+\bar{q}^{m+\beta+1/2})\, e^{2\pi i\beta(\bar{z}_i-\bar{w}_i)}\}}{\prod_{i=1}^{N} \prod_{m=0}^{\infty} \{(1+\bar{q}^{m+\beta-1/2})\}\{(1+\bar{q}^{m+\beta+1/2})\}} \right]^N \tag{4.6}$$

$$\times \quad \frac{\prod_{r>s} E(z_r - z_s)^{e^{i_r}\cdot e^{i_s}} E(w_s - w_r)^{e^{l_r}\cdot e^{l_s}}}{\prod_{r,s} E(z_r - w_s)^{e^{i_r}\cdot e^{l_s}}}$$

$$\times \frac{\prod_{r>s} E(\bar{z}_r - \bar{z}_s)^{e^{j_r}\cdot e^{j_s}} E(\bar{w}_s - \bar{w}_r)^{e^{k_r}\cdot e^{k_s}}}{\prod_{r,s} E(\bar{z}_r - \bar{w}_s)^{e^{j_r}\cdot e^{k_s}}}$$

$$\times \frac{\prod_{r>s} E(z_r - z_s)\, E(w_s - w_r)}{\prod_{r,s} (z_r - w_s)} \frac{\prod_{r>s} \bar{E}(\bar{z}_r - \bar{z}_s)\, \bar{E}(\bar{w}_s - \bar{w}_r)}{\prod_{r,s} (\bar{z}_r - \bar{w}_s)},$$

where $\alpha = (h_1-h_2)/\sqrt{N}$ and $\beta = (k_1-k_2)/\sqrt{N}$, and $E(z-w)$ and $\bar{E}(\bar{z}-\bar{w})$ are defined as

and

$$E(z-w) = -\log\frac{\theta_1(z-w|\tau)}{\eta(\tau)} + \log i e^{1/6\pi i\tau})$$

$$\bar{E}(\bar{z}-\bar{w}) = -\log\frac{\bar{\theta}_1(\bar{z}-\bar{w}|\bar{\tau})}{\eta(\bar{\tau})} + \log i e^{1/6\pi i\bar{\tau}}), \quad (4.7)$$

Here the calculation of zero mode is done as follow

$$\langle e^{i\alpha(\phi_0(z)-\phi_0(w))}\rangle = \frac{\sum_{p\in\Lambda} q^{p^2/2}\, e^{i\alpha(\phi_0(z)-\phi_0(w))}}{\sum_{p\in\Lambda} q^{p^2/2}}, \quad (4.8)$$

where Λ is the lattice spanned by the vectors $\{a^i | i=1,\cdots,N\}$ defined by (2.6). The peculiarities of the correlation function are that not only the oscillator mode but also the zero one are factorized to the holomorhpic and antiholomorphic parts, and that there exists the freedom of the U(1) charges which corresponds to the ambiguity of the lattices.

5. Discussions

In this talk we extend the U(N) WZW model constructed by the vertex operators on the torus. We calculate the 2n-point

correlation function of the U(N)WZW model whose Virasoro central charge is equal to N. This scheme corresponds to the chiral bosonization on the torus because the correlation function is factorized to the holomorphic and antiholomorphic parts.

We should restrict our scheme to the systems which belong to representations of the Kac-Moody algebras of level one. Gepner [11] has described the vertex operator construction of current algebras at arbitrary level, arbitrary group. Then it is interesting to extend our scheme to the case of arbitrary level, arbitrary level. When one tries to calculate the Thirring model associated with WZW model, one has to consider the structure of spins, that is, how to sum the lattice points of the zero modes. Further it is possible to extend our scheme to arbitrary compact Riemann surface.

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E_7 type modular invariant Wess-Zumino theory and Gepner's string compactification

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1 Introduction

Conformal field theories in two dimensions have been actively investigated recently [2]. These theories are relevant to the two dimensional critical phenomena and string theory. In string theory, conformal field theories are building blocks of the perturbative vacuum.

In order to describe the four dimensional world in terms of string theory, some kind of compactification is necessary. Phenomenologically it has been argued that the internal six dimensional manifold should be a Calabi-Yau manifold. Gepner has pointed out that these internal manifolds may be realized in terms of conformal field theories. His construction makes use of minimal models of $N = 2$ superconformal theories [3] which are required to ensure $N = 1$ space time supersymmetry.

The advantage of constructing Calabi-Yau manifolds in terms of conformal field theories is that the correlation functions are now calculable. Therefore we can calculate the Yukawa couplings among massless particles of the theory in this construction.

In order to determine correlation functions of conformal field theories, it is necessary to patch holomorphic and antiholomorphic sectors together in a physically sensible way. It is known that the operator contents of the theory is constrained by the requirement of the modular invariance of the partition function [12]. The correlation functions are expected to reflect the operator contents of the theory. In diagonal modular invariant theories, the correlation functions have been studied in minimal models [6], Wess-Zumino-Witten models [8,9] and $N = 1$ superconformal models [11]. In particular

the operator product expansion coefficients (C's) in these models have been determined from the four point functions.

However the correlation functions in off-diagonal modular invariant theories have been studied little up to now. The complete classification of the modular invariant partition functions in Wess-Zumino-Witten theory is known. They are A_n, D_n, E_6, E_7 and E_8 type models [13], [14]. Of these only A_n type models are diagonally modular invariant. However D_{even}, E_6 and E_8 can be brought into the diagonal form in terms of super-characters of extended algebras [10].

In ref. [1] we study physical correlation functions in off-diagonal modular invariant theories in Wess-Zumino-Witten theories. Although our procedure works in this class of models in general, we study E_7 modular invariant theory in detail. We construct the four point functions in E_7 modular invariant theory and determine the C's of this theory.

Since the E_7 modular invariance has been used to construct a three generation model by Gepner [4], our result enables us to calculate the Yukawa couplings in such a model. Using the knowledge of the Yukawa couplings, we can study more detailed correspondences between conformal field theories and geometry. Needless to say that the Yukawa couplings are essential to discuss the phenomenology of the model. We discuss the phenomenological prospects of this model which is quite rich reflecting a very nontrivial internal manifold.

The E_7 modular invariant theory is also interesting in connection with the Verlinde's fusion rule [18]. He has proposed a general theory concerning the indicators of the C's. The indicator of C is zero or a positive integer depending on whether C vanishes or not. Since his idea requires some kind of factorization of C's into holomorphic and antiholomorphic parts, off-diagonal modular invariant theories require some further considerations. In particular a pair of operators which play a central role in his theory have not been well understood. Nevertheless we find that our results support his theory also in off-diagonal modular invariant theories.

This report is the concise version of ref. [1], and is organized as follows.

In section 2, we explain a general strategy to determine the operator product algebra in conformal field theories and state our results. In section 3, we discuss Gepner's three generation model making use of the results of the section 2. Section 4 consists of the investigation of Verlinde's fusion rule in E_7 modular invariant theory. We conclude in section 5.

2 The $SU(2) \times SU(2)$ Wess-Zumino-Witten theory and operator product algebra

In this section we briefly review the $SU(2) \times SU(2)$ Wess-Zumino-Witten theory and explain the way how OPE coefficients are calculated. We also state our results on C's in the E_7 type model.

The primary fields of WZW theory are the invariant tensors of $SU(2)_L \times SU(2)_R$. We denote by $\Phi^{(j,\bar{j})}_{(m,\bar{m})}$ ($m = -j, \ldots, j$; $\bar{m} = -\bar{j}, \ldots, \bar{j}$) the primary field with isospin j of $SU(2)_L$ and $\bar{j}$ of $SU(2)_R$. It is well known that the global $SL(2,C)$ invariance completely fix the form of two- and three-point functions up to normalizations. We normalize primary fields in such a way that

$$\begin{aligned} &\langle \Phi^{\dagger(j_1,\bar{j}_1)}_{(m_1,\bar{m}_1)}(z_1,\bar{z}_1)\Phi^{(j_2,\bar{j}_2)}_{(m_2,\bar{m}_2)}(z_2,\bar{z}_2)\rangle \\ &= \delta^{j_1,j_2}\delta^{\bar{j}_1,\bar{j}_2}\delta^{m_1,m_2}\delta^{\bar{m}_1,\bar{m}_2}(z_1-z_2)^{-2\Delta(j_1)}(\bar{z}_1-\bar{z}_2)^{-2\Delta(\bar{j}_1)}, \end{aligned} \tag{2.1}$$

where the conjugate field is defined by $\Phi^{\dagger(j,\bar{j})}_{(m,\bar{m})} = (-1)^{j-m}(-1)^{\bar{j}-\bar{m}}\Phi^{(j,\bar{j})}_{(-m,-\bar{m})}$.

$SU(2)$ symmetry fixes the form of the three point function as follows:

$$\begin{aligned} &\langle \Phi^{(j_1,\bar{j}_1)}_{(m_1,\bar{m}_1)}(z_1,\bar{z}_1)\Phi^{(j_2,\bar{j}_2)}_{(m_2,\bar{m}_2)}(z_2,\bar{z}_2)\Phi^{(j_3,\bar{j}_3)}_{(m_3,\bar{m}_3)}(z_3,\bar{z}_3)\rangle \\ &= C((j_1\bar{j}_1)(j_2\bar{j}_2)(j_3\bar{j}_3)) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \bar{j}_1 & \bar{j}_2 & \bar{j}_3 \\ \bar{m}_1 & \bar{m}_2 & \bar{m}_3 \end{pmatrix} \\ &\quad \times \left\{ \frac{(j_1+j_2+j_3+1)!(j_1+j_2-j_3)!(j_2+j_3-j_1)!(j_3+j_1-j_2)!}{(2j_1)!(2j_2)!(2j_3)!} \right\}^{1/2} \\ &\quad \times \left\{ \frac{(\bar{j}_1+\bar{j}_2+\bar{j}_3+1)!(\bar{j}_1+\bar{j}_2-\bar{j}_3)!(\bar{j}_2+\bar{j}_3-\bar{j}_1)!(\bar{j}_3+\bar{j}_1-\bar{j}_2)!}{(2\bar{j}_1)!(2\bar{j}_2)!(2\bar{j}_3)!} \right\}^{1/2} \\ &\quad \times (z_{12}^{-\Delta(j_1)-\Delta(j_2)+\Delta(j_3)} z_{23}^{-\Delta(j_2)-\Delta(j_3)+\Delta(j_1)} z_{31}^{-\Delta(j_3)-\Delta(j_1)+\Delta(j_2)}) \\ &\quad \times (\bar{z}_{12}^{-\Delta(\bar{j}_1)-\Delta(\bar{j}_2)+\Delta(\bar{j}_3)} \bar{z}_{23}^{-\Delta(\bar{j}_2)-\Delta(\bar{j}_3)+\Delta(\bar{j}_1)} \bar{z}_{31}^{-\Delta(\bar{j}_3)-\Delta(\bar{j}_1)+\Delta(\bar{j}_2)}), \end{aligned} \tag{2.2}$$

where $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the Wigner's $3j$ symbol [15].

In the case of $SU(2)$ WZW theory, we can show the truncated Clebsch-Gordan law [6]

$$C((j_1\bar{j}_1)(j_2\bar{j}_2)(j_3\bar{j}_3)) = 0$$
$$\text{unless } \begin{cases} |j_1 - j_2| \le j_3 \le \min(j_1 + j_2, k - j_1 - j_2) \\ |\bar{j}_1 - \bar{j}_2| \le \bar{j}_3 \le \min(\bar{j}_1 + \bar{j}_2, k - \bar{j}_1 - \bar{j}_2), \end{cases} \tag{2.3}$$

where k is the level of representations.

We want to determine all the OPE coefficients $C((j_1\bar{j}_1)(j_2\bar{j}_2)(j_3\bar{j}_3))$ of the WZW theory, especially in the case of off-diagonal E_7 type modular invariant model.

The general technique to calculate OPE coefficients is as follows [2,6,8]: Firstly, we construct so-called "conformal blocks" which are the solutions to Knizhnik-Zamolodchikov equation [5]. Then the solutions are patched together so that physical correlation functions are monodromy invariant. Finally, by factorizing the correlation function we obtain the operator product expansion coefficients. In ref. [8], this procedure is explicitly demonstrated in $SU(2)$ WZW theories of diagonal (A_n type) modular invariant.

In off-diagonal theories the procedure to calculate the OPE coefficients are essentially the same. As for the conformal blocks, nothing is changed because they are the solutions to Knizhnik-Zamolodchikov equation which regulates only holomorphic (chiral) behavior.

The difference between the diagonal and off-diagonal case occurs in patching the conformal blocks together. Let $\{I_k(z)\}$ be the conformal blocks, each of which have $z = 0, 1$ and ∞ as singular points. If they are analytically continued along a closed contour surrounding the point $z = 0$ or 1, they undergo the monodromy transformation, whose matrix we denote by g_0 and g_1. In order for the physical four point function

$$U(z, \bar{z}) = \sum_{k,\bar{k}} X_{k\bar{k}} I_k(z) \bar{I}_{\bar{k}}(\bar{z}) \tag{2.4}$$

to be monodromy invariant, $\{X_{k\bar{k}}\}$ must be the solution to the following

equations:

$$\sum_{k,\bar{k}}(g_0)_{kl}X_{k\bar{k}}(\bar{g}_0)_{\bar{k}\bar{l}} = X_{l\bar{l}}, \tag{2.5}$$

$$\sum_{k,\bar{k}}(g_1)_{kl}X_{k\bar{k}}(\bar{g}_1)_{\bar{k}\bar{l}} = X_{l\bar{l}}. \tag{2.6}$$

The monodromy matrix can be calculated from the connection matrix α_{kl} which relates two sets of fundamental solutions around $z \sim 0$ and $z \sim 1$.

In diagonal theories, one sets $X_{k\bar{k}} = X_k \delta_{k\bar{k}}$, which reduces the amount of computation a great deal. One can determine OPE coefficients without knowing all matrix elements α_{kl}. In fact, only one column data of α_{kl} is sufficient to determine the whole OPE coefficients [6].

In off-diagonal theories, however, off-diagonal components of $X_{k\bar{k}}$ can be non-zero provided corresponding intermediate state is allowed by the operator contents of the theory. Accordingly, almost all α_{kl} are needed. And also it is not so obvious that there is an unique solution to eqs.(2.5) and (2.6) for this class of models. However, with explicit calculations we checked that it is indeed the case for the E_7 type model and determined the structure of the OPE.

In table 1, we list all nontrivial OPE coefficients of E_7 type $SU(2)$ WZW theory together with their approximate numerical values. The relation to OPE coefficients of diagonal (A type) theory with the same central charge is also shown. We have not included trivial OPE coefficients which contain the identity operator. The remaining C's in the theory can be obtained by using "reflection symmetry":

$$C^2((j_1,\bar{j}_1)(j_2,\bar{j}_2)(j_3,\bar{j}_3)) = C^2((j_1,\bar{j}_1)(8-j_2,\bar{j}_2)(8-j_3,\bar{j}_3)). \tag{2.7}$$

This symmetry can be seen by comparing the integral representations of conformal blocks [8].

3 Yukawa couplings in Gepner's three generation model

Recently Gepner constructed string models with $N = 1$ space-time supersymmetry out of $N = 2$ superconformal theories. Such models may corre-

TABLE 1. OPE coefficients in E_7 type modular invariant $SU(2) \times SU(2)$ WZW theory. The right column shows the relation to OPE coefficients of diagonal theory.

$C^2((22)(22)(44))$	$=$	1.28	$=$	$\frac{1}{2}C^2(224)_A$
$C^2((22)(22)(33))$	$=$	12.5	$=$	$C^2(223)_A$
$C^2((22)(22)(22))$	$=$	13.3	$=$	$C^2(222)_A$
$C^2((22)(22)(14))$	$=$	2.56	$=$	$\frac{1}{2}C^2(221)_A$
$C^2((33)(33)(22))$	$=$	6.09	$=$	$\frac{1}{4}C^2(332)_A$
$C^2((33)(33)(26))$	$=$	6.09	$=$	$\frac{1}{4}C^2(332)_A$
$C^2((33)(33)(66))$	$=$	0.676	$=$	$\frac{1}{4}C^2(336)_A$
$C^2((33)(33)(33))$	$=$	49.8	$=$	$C^2(333)_A$
$C^2((33)(33)(35))$	$=$	0		
$C^2((33)(33)(55))$	$=$	21.2	$=$	$C^2(335)_A$
$C^2((33)(33)(44))$	$=$	24.7	$=$	$\frac{1}{2}C^2(334)_A$
$C^2((33)(33)(14))$	$=$	12.4	$=$	$\frac{1}{4}C^2(334)_A$
$C^2((44)(44)(22))$	$=$	28.5	$=$	$C^2(442)_A$
$C^2((44)(44)(44))$	$=$	190	$=$	$2C^2(444)_A$
$C^2((44)(44)(33))$	$=$	0		
$C^2((44)(44)(14))$	$=$	0		
$C^2((22)(33)(44))$	$=$	8.32	$=$	$\frac{1}{2}C^2(234)_A$
$C^2((22)(33)(41))$	$=$	4.16	$=$	$\frac{1}{4}C^2(234)_A$
$C^2((33)(44)(14))$	$=$	8.78	$=$	$4C^2(341)_A$
$C^2((14)(14)(14))$	$=$	23.8	$=$	$\frac{1}{4}C^2(444)_A$
$C^2((14)(14)(22))$	$=$	7.12	$=$	$\frac{1}{4}C^2(442)_A$
$C^2((14)(41)(33))$	$=$	1.10	$=$	$\frac{1}{2}C^2(143)_A$
$C^2((14)(41)(44))$	$=$	6.49	$=$	$C^2(144)_A$

spond to string compactification on Calabi-Yau manifolds.

Let us recapitulate his model. We consider the heterotic string theory propagating on some internal compact manifold K times Minkowski space M_4. Propagation on the internal space is described by some tensor product of $N=2$ minimal models. In order to have four dimensional space-time, the total trace anomaly of the $N=2$ minimal models must be $c=9$.

The trace anomaly (or central charge) of $N=2$ minimal models are given by $c=3k/(k+2)$, $(k=1,2,3,\ldots)$. For a fixed k, the primary fields (or irreducible representations) are labeled by three integers (l,q,s). l must be in the range $0 \leq l \leq k$. q and s are defined modulo $2(k+2)$ and 4 respectively. Two sectors $s=0,2$ constitute the Neveu-Schwarz sector and $s=1,3$ the Ramond sector. Hereafter we denote the corresponding primary field by $\Theta_{l,q,s;\bar{l},\bar{q},\bar{s}}$.

Gepner constructed a three generation model out of one $k=1$ and three $k=16$ $N=2$ minimal models. In his model, the E_7 type partition function is used in combining holomorphic and anti-holomorphic parts of the $k=16$ minimal model. As it is explained in ref. [4], this model has the discrete symmetry

$$G = (Z_3 \times S_3 \ltimes Z_9^3)/Z_9. \tag{3.1}$$

We denote an element of $Z_3 \times Z_9^3$ by $\{r_0, r_1, r_2, r_3\}$. A three generation model can be obtained by moding this 1 16^3 theory by the Z_3 symmetry generated by $g=\{0,3,6,0\}$ and further by the symmetry generated by h which cyclically permutes three $k=16$ theories. After these operations, we obtain 9 generations and 6 antigenerations. Three extra $U(1)$ gauge groups also have been reduced to single $U(1)$ in this operation. It is also possible to break E_6 gauge group down to $SU(3)^3$ using the Hosotani mechanism. We embed g in E_6 when we mod the original theory by g. In this case we have 9 generations of leptons and 6 antigenerations of leptons, 3 generations and no antigenerations of quarks.

It is known that these generations can be identified with the perturbations of the complex structure of a Calabi-Yau manifold. Such a manifold is defined

TABLE 2. List of generations in 1 16^3 model.
Φ and Θ denote $k=1$ and $k=16$ $N=2$ primary fields, respectively.

l_1	$\Phi_{121\,121}\ \Theta_{12131\,12131}\ \Theta^2_{011\,011}$	$z_0x_1^3e_2$
l_2	$\Phi_{121\,121}\ \Theta^2_{671\,671}\ \Theta_{011\,011}$	$z_0z_1z_2e_1$
l_3	$\Phi_{121\,121}\ \Theta^3_{451\,451}$	$z_0x_1x_2x_3e_2$
l_4	$\Phi_{011\,011}\ \Theta_{12131\,12131}\ \Theta_{671\,671}\ \Theta_{011\,011}$	$z_2x_1^3e_2$
$l_{\bar{4}}$	$\Phi_{011\,011}\ \Theta_{011\,011}\ \Theta_{671\,671}\ \Theta_{12131\,12131}$	$z_2x_3^3e_2$
l_5	$\Phi_{011\,011}\ \Theta_{10111\,10111}\ \Theta^2_{451\,451}$	$z_1x_1x_2x_3e_2$
l_6	$\Phi_{011\,011}\ \Theta^3_{671\,671}$	$z_1z_2z_3e_1$
l_{t1}	$\Phi_{011\,011}\ \Theta_{231\,891}\ \Theta_{891\,231}\ \Theta_{891\,891}$	
l_{t2}	$\Phi_{011\,011}\ \Theta_{231\,891}\ \Theta_{891\,891}\ \Theta_{891\,231}$	
q_{1L}	$\Phi_{121\,121}\ \Theta_{891\,891}\ \Theta_{451\,451}\ \Theta_{011\,011}$	$z_0x_1^2x_2e_2$
q_{1R}	$\Phi_{121\,121}\ \Theta_{011\,011}\ \Theta_{451\,451}\ \Theta_{891\,891}$	$z_0x_3^2x_2e_2$
q_{2L}	$\Phi_{011\,011}\ \Theta_{011\,011}\ \Theta_{891\,891}\ \Theta_{10111\,10111}$	$z_3x_2^2x_3e_2$
q_{2R}	$\Phi_{011\,011}\ \Theta_{10111\,10111}\ \Theta_{891\,891}\ \Theta_{011\,011}$	$z_1x_2^2x_1e_2$
q_{3L}	$\Phi_{011\,011}\ \Theta_{451\,451}\ \Theta_{671\,671}\ \Theta_{891\,891}$	$z_2x_3^2x_1e_2$
q_{3R}	$\Phi_{011\,011}\ \Theta_{891\,891}\ \Theta_{671\,671}\ \Theta_{451\,451}$	$z_2x_1^2x_3e_2$

by the following algebraic equations:

$$
\begin{aligned}
P_1 &= \frac{1}{3}(z_0^3+z_1^3+z_2^3+z_3^3)=0, \\
P_2 &= z_1x_1^3+z_2x_2^3+z_3x_3^3=0.
\end{aligned}
\tag{3.2}
$$

In table 2, we list all generations of the conformal field theory with corresponding perturbations of the complex structure. In this list, l_i, q_{iL} and q_{iR} are lepton, left-handed quark and right-handed quark generations respectively. The quantum numbers in this list correspond to space-time spinor and **16** of $SO(10)$ gauge group. On the right column, e_1 (e_2) represents a perturbation to the polynomial P_1 (P_2). We have listed representatives of the equivalence class under the symmetry h. Therefore the symmetrization of primary fields under the cyclic permutation h should be implicitly understood

throughout this paper.

In Gepner's model, every massless state corresponds to a product of $N = 2$ primary fields. Therefore the Yukawa coupling of this model is given by the product of the OPE coefficients of the $N = 2$ subtheories.

The $N = 2$ superconformal theory has close relationship with $SU(2)$ WZW theory. Let us briefly discuss this point. The $N = 2$ primary fields $\Theta_{l,q,s;\bar{l},\bar{q},\bar{s}}$ can be rewritten in terms of parafermionic fields $\psi^{l,\bar{l}}_{m,\bar{m}}$ and a free boson ϕ:

$$\Theta_{l,q,s;\bar{l},\bar{q},\bar{s}} = \psi^{l,\bar{l}}_{q-s,\bar{q}-\bar{s}} : \exp(i\alpha_q\phi + i\alpha_{\bar{q}}\bar{\phi}) :, \tag{3.3}$$

with

$$\alpha_q = \frac{q - s/2(k+2)}{\sqrt{2k(k+2)}}. \tag{3.4}$$

On the other hand, the parafermionic fields are related to the primary fields $\Phi^{(l,\bar{l})}_{(m,\bar{m})}$ of the $SU(2)$ WZW theory:

$$\Phi^{(l,\bar{l})}_{(m,\bar{m})} = \psi^{2l,2\bar{l}}_{2m,2\bar{m}} : \exp(\frac{im\phi}{\sqrt{k}} + \frac{i\bar{m}\bar{\phi}}{\sqrt{k}}) : . \tag{3.5}$$

Using these relations, we can study $N = 2$ superconformal theories through $SU(2)$ WZW theory.

In table 3, we list all nonzero Yukawa couplings in this model among generations. In Table 4, we also listed all antigenerations of this model and nonzero Yukawa couplings among them.

It is well known that the Yukawa couplings can be calculated by an algebraic method if we are given an algebraic manifold of the compactified space. However in the algebraic method the normalization of each generation is not fixed. In fact we find that by the appropriate rescaling of the fields, the both results agree. Let us recall that the same situation is also found in a four generation case [17]. Therefore our investigation provides a further support to the identification of Gepner's models with string propagation in Calabi-Yau manifolds.

It is also interesting to note that our pattern of the Yukawa couplings differs from the model which has been investigated by Candelas and Kalala

TABLE 3. Yukawa couplings among generations.
$\mu^2 = C^2((33)(33)(66))$, $\lambda^2 = C^2((14)(41)(33))$ and $\rho^2 = C^2((22)(22)(44))$.

$l_4 l_{\bar{4}} l_3$	$\mu = 0.822$	$l_3 q_{3L} q_{3R}$	$\mu\rho^2 = 1.05$
$l_6 l_6 l_3$	$\mu^3 = 0.556$	$l_2 q_{2L} q_{2R}$	$1/\sqrt{3} = 0.577$
$l_1 l_{\bar{4}} l_5$	$1/\sqrt{3} = 0.577$	$l_2 q_{3L} q_{3R}$	$\mu^2/\sqrt{3} = 0.390$
$l_1 l_4 l_5$	$1/\sqrt{3} = 0.577$	$l_4 q_{1L} q_{2R}$	$1/\sqrt{3} = 0.577$
$l_2 l_4 l_5$	$\mu/\sqrt{3} = 0.495$	$l_{\bar{4}} q_{1R} q_{2L}$	$1/\sqrt{3} = 0.577$
$l_2 l_{\bar{4}} l_5$	$\mu/\sqrt{3} = 0.495$	$l_5 q_{1L} q_{3R}$	$\rho^2/\sqrt{3} = 0.938$
$l_2 l_5 l_6$	$\mu^2 = 0.676$	$l_5 q_{1R} q_{3L}$	$\rho^2/\sqrt{3} = 0.938$
$l_{t1} l_{t2} l_2$	$\lambda^2/\sqrt{3} = 0.635$	$l_{\bar{4}} q_{1L} q_{3R}$	$\mu/\sqrt{3} = 0.495$
$q_{3L} q_{3L} q_{1L}$	$\rho\mu/\sqrt{3} = 0.536$	$l_4 q_{1R} q_{3L}$	$\mu/\sqrt{3} = 0.495$
$q_{3R} q_{3R} q_{1R}$	$\rho\mu/\sqrt{3} = 0.536$	$l_1 q_{2L} q_{3R}$	$1/\sqrt{3} = 0.577$
$q_{1L} q_{2L} q_{3L}$	$\rho/\sqrt{3} = 0.652$	$l_1 q_{2R} q_{3L}$	$1/\sqrt{3} = 0.577$
$q_{1R} q_{2R} q_{3R}$	$\rho/\sqrt{3} = 0.652$		

TABLE 4. Yukawa couplings among antigenerations.

$\bar{l}_4 \bar{l}_4 \bar{l}_2$	$\mu^3 = 0.556$
$\bar{l}_1 \bar{l}_3 \bar{l}_4$	$\lambda^3 = 1.15$
$\bar{l}_{t1} \bar{l}_{t2} \bar{l}_2$	$\mu = 0.822$

$\bar{l}_1$	$\Phi_{011\,011}$	$\Theta_{231\,891}$	$\Theta^2_{891\,14151}$	
$\bar{l}_2$	$\Phi_{121\,011}$	$\Theta^3_{451\,12131}$		
$\bar{l}_3$	$\Phi_{121\,121}$	$\Theta^2_{231\,891}$	$\Theta_{891\,14151}$	
$\bar{l}_4$	$\Phi_{011\,121}$	$\Theta^3_{631\,10111}$		
$\bar{l}_{t1}$	$\Phi_{011\,121}$	$\Theta_{011\,16171}$	$\Theta_{671\,10111}$	$\Theta_{12131\,451}$
$\bar{l}_{t2}$	$\Phi_{011\,121}$	$\Theta_{011\,16171}$	$\Theta_{12131\,451}$	$\Theta_{671\,10111}$

[16]. Their model is defined by

$$\begin{aligned}
P^1 &\doteq \frac{1}{3}(x_0^3 + x_1^3 + x_2^3 + x_3^3) = 0, \\
P^2 &= x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0, \\
P^3 &= \frac{1}{3}(y_0^3 + y_1^3 + y_2^3 + y_3^3) = 0,
\end{aligned} \tag{3.6}$$

moded by the Z_3 symmetry:

$$\begin{aligned}
(x_0, x_1, x_2, x_3) &\rightarrow (x_0, \alpha^2 x_1, \alpha x_2, \alpha x_3), \\
(y_0, y_1, y_2, y_3) &\rightarrow (y_0, \alpha y_1, \alpha^2 y_2, \alpha^2 y_3).
\end{aligned} \tag{3.7}$$

Although this manifold is known to be diffeomorphic to ours, the Yukawa couplings differ from one another. Therefore the physics of the Calabi-Yau three generation models does not seem to be unique.

Since the over all scale of our coupling constants is determined by the dilaton vacuum expectation value in string theory, a sensible thing to do is to normalize the Yukawa couplings by the gauge coupling. We find that the gauge coupling of this model to be 1. Therefore all Yukawa coupling are of the same order as the gauge coupling. This situation is also found in a four generation model.

Besides three generations of quarks, we still have 9 generations of leptons and the gauge group $SU(3)^3$ plus an extra nonanomalous $U(1)$. Therefore the phenomenology of this model depends how these higher symmetries and extra particles are got rid of. In what follows we describe a promising scenario. If we consider an energy scale considerably lower than the Planck mass, the physics can be described in terms of a renormalizable field theory. We know this theory exactly in the sigma model perturbation theory including nonperturbative effects. This is because the Yukawa couplings we have calculated in this paper determine the Lagrangian of it. However we have not calculated stringy loop effects. We look for a solution of this effective field theory which is phenomenologically attractive. This procedure can be justified if we consider physics considerably lower than the Planck mass scale.

We are interested in such solutions which possess the standard gauge group $SU_C(3) \times SU_L(2) \times U(1)$ and preserve SUSY. If such a solution exist, we can

break higher gauge symmetry down to the standard gauge group at very large energy scale. In this symmetry breaking process, we employ the conventional Higgs mechanism. Therefore we have to pay attention that the D and F terms do not destroy SUSY even if we give nonzero vacuum expectation values to some fields.

The observed particles can be assigned in the following representations of the gauge group $SU_C(3) \times SU_L(2) \times SU_R(2) \subset SU(3)^3$

$$\begin{aligned}
q_L &= (3,2,1) \in (3,3,1), \\
q_R &= (\bar{3},1,\bar{2}) \in (\bar{3},1,\bar{3}), \\
l_L &= (1,\bar{2},1) \text{ or } (1,\bar{2},\downarrow) \in (1,\bar{3},3), \\
l_R &= (1,1,\uparrow) \in (1,\bar{3},3), \\
H &= (1,\bar{2},2) \in (1,\bar{3},3),
\end{aligned} \tag{3.8}$$

where $q_L(q_R)$ denotes left handed (right handed) quarks, $l_L(l_R)$ denotes left handed (right handed) leptons and H denotes two Higgs boson multiplets which are required in SUSY models. The gauge group $SU(3)^3$ can be broken down to $SU_C(3)\times SU_L(2)\times U(1)$ by giving nonzero vacuum expectation values to $(1,1,1) \in l_5$ and $(1,1,\downarrow) \in l_{t1}$(or l_{t2}). At the same time it is necessary to give nonzero vacuum expectation values to antigenerations in the same representation of the gauge group in order to prevent the SUSY breaking due to the D term.

However we have to use appropriate antigenerations in this process. This is because the Yukawa couplings among $SU(3)^3$ singlets - generations - antigenerations put constraints on which antigenerations can be given nonzero vacuum expectation values while preserving SUSY. We recall that there are 61 singlets in this model.

In table 5 we list all combinations of this type which possess nonzero Yukawa couplings. From this table it is easy to see that if we use suitable antigenerations such as $\bar{l}_4$, we can obtain the standard gauge group while preserving SUSY. The extra $U(1)$ gauge group can also be broken by utilizing two $SU(3)^3$ singlets which possess opposite $U(1)$ charges in order to cancel the D term contribution. If we consider possibilities to give nonzero vacuum expec-

TABLE 5. Combinations of singlets, generations and antigenerations which possess nonzero Yukawa couplings.

$l_{\bar{4}}\bar{l}_{t2}\phi_{16}$	$l_{\bar{4}}\bar{l}_3\phi_3$	$l_4\bar{l}_{t1}\phi_{17}$	$l_4\bar{l}_3\phi_3$	$l_{t1}\bar{l}_1\phi_{18}$	$l_{t1}\bar{l}_2\phi_{13}$	$l_{t1}\bar{l}_3\phi_1$
$l_{t1}\bar{l}_3\phi_6$	$l_{t2}\bar{l}_1\phi_{19}$	$l_{t2}\bar{l}_2\phi_{14}$	$l_{t2}\bar{l}_3\phi_2$	$l_{t2}\bar{l}_3\phi_5$	$l_5\bar{l}_1\phi_{20}$	$l_5\bar{l}_2\phi_{12}$
$l_5\bar{l}_3\phi_4$	$l_6\bar{l}_4\phi_{15}$	$l_6\bar{l}_3\phi_3$	$l_1\bar{l}_1\phi_3$	$l_1\bar{l}_{t1}\phi_{11}$	$l_1\bar{l}_{t2}\phi_{10}$	$l_2\bar{l}_1\phi_3$
$l_2\bar{l}_{t1}\phi_8$	$l_2\bar{l}_{t2}\phi_9$	$l_2\bar{l}_4\phi_7$	$l_3\bar{l}_1\phi_4$			

ϕ_1	$\Phi_{011\,000}$	$\Theta_{011\,000}$	$\Theta_{671\,600}$	$\Theta_{12131\,400}$
ϕ_2	Φ_{011000}	$\Theta_{011\,000}$	$\Theta_{12131\,400}$	$\Theta_{671\,600}$
ϕ_3	$\Phi_{011\,000}$	$\Theta_{231\,8142}$	$\Theta^2_{891\,220}$	
ϕ_4	$\Phi_{011\,000}$	$\Theta^2_{451\,4340}$	$\Theta_{10111\,640}$	
ϕ_5	$\Phi_{011\,000}$	$\Theta_{671\,600}$	$\Theta_{671\,660}$	$\Theta_{671\,6300}$
ϕ_6	$\Phi_{011\,000}$	$\Theta_{671\,600}$	$\Theta_{671\,6300}$	$\Theta_{671\,660}$
ϕ_7	$\Phi_{011\,022}$	$\Theta^2_{451\,400}$	$\Theta_{10111\,660}$	
ϕ_8	$\Phi_{011\,022}$	$\Theta_{451\,400}$	$\Theta_{451\,4302}$	$\Theta_{10111\,6300}$
ϕ_9	$\Phi_{011\,022}$	$\Theta_{451\,400}$	$\Theta_{10111\,6300}$	$\Theta_{451\,4302}$
ϕ_{10}	$\Phi_{011\,022}$	$\Theta_{451\,462}$	$\Theta_{451\,4302}$	$\Theta_{10111\,660}$
ϕ_{11}	$\Phi_{011\,022}$	$\Theta_{451\,462}$	$\Theta_{10111\,660}$	$\Theta_{451\,4302}$
ϕ_{12}	$\Phi_{011\,042}$	$\Theta_{231\,8122}$	$\Theta^2_{891\,200}$	
ϕ_{13}	$\Phi_{011\,042}$	$\Theta_{451\,420}$	$\Theta_{451\,43}$[illegible]	$\Theta_{10111\,6320}$
ϕ_{14}	$\Phi_{011\,042}$	$\Theta_{451\,420}$	$\Theta_{10111\,6320}$	$\Theta_{451\,4320}$
ϕ_{15}	$\Phi_{121\,000}$	$\Theta^3_{451\,400}$		
ϕ_{16}	$\Phi_{121\,000}$	$\Theta_{451\,400}$	$\Theta_{451\,462}$	$\Theta_{451\,4302}$
ϕ_{17}	$\Phi_{121\,000}$	$\Theta_{451\,400}$	$\Theta_{451\,4302}$	$\Theta_{451\,462}$
ϕ_{18}	$\Phi_{121\,042}$	$\Theta_{011\,000}$	$\Theta_{671\,600}$	$\Theta_{671\,6300}$
ϕ_{19}	$\Phi_{121\,042}$	$\Theta_{011\,000}$	$\Theta_{671\,6300}$	$\Theta_{671\,600}$
ϕ_{20}	$\Phi_{121\,042}$	$\Theta^3_{451\,4340}$		

tation values to singlets, we have to pay attention to the Yukawa couplings among three singlet fields in addition. This is because such couplings put further constraints on the supersymmetric solutions of the effective Lagrangian. There are 136 combinations of three $SU(3)$ singlets which possess nonzero Yukawa couplings in this model. We have checked that if we use appropriate singlet fields in this process, the F terms do not spoil SUSY. We can also pair up lepton generations and antigenerations to become massive by giving nonzero vacuum expectation values to suitable singlet fields. For example if we use ϕ_3, ϕ_{16} and ϕ_{17} in table 5, four pairs of unnecessary generations and antigenerations become massive. The pattern of the Yukawa couplings among three singlets has shown that such a solution does not break SUSY.

We assign $H \in l_3$ and $(t, b) \in q_3$. Then the top quark mass is predicted to be equal to the W boson mass to the first approximation. This prediction is an universal one irrespective of a particular scenario as long as it is phenomenologically sensible. Let us assign the known leptons as follows:

$$\begin{aligned} (e, \mu, \tau)_L &\in (1, \bar{2}, \downarrow), \\ (e, \mu, \tau) &\in l_{t1}, l_{t2} \text{ and } l_5. \end{aligned} \tag{3.9}$$

If we break the gauge group $SU(3)^3$ down to the standard $SU_C(3) \times SU_L(2) \times U(1)$ at sufficiently high energy scale in our scenario, acceptable proton life time can also be obtained.

4 Verlinde's fusion rule and E_7 modular invariant theory

Verlinde's fusion algebra [18,19] is defined by

$$\phi_{[i]} \times \phi_{[j]} = N_{[i][j]}^{[k]} \phi_{[k]}, \tag{4.1}$$

where $\phi_{[i]}$'s are the irreducible representations of the chiral algebra. The nonnegative integer $N_{[i][j][k]}$ is the number of independent nonzero couplings of the type $([i], [j], [k])$. Since the above fusion rule is given in the holomorphic sector only, some kind of factorization of C's into left and right sectors is necessary. It is remarkable that $N_{[i][j][k]}$ is determined by the modular transformation

matrix alone:

$$N_{[i][j][k]} = \sum_{[n]} \frac{S_{[i][n]}S_{[j][n]}S_{[n][k]}}{S_{[0][n]}}. \tag{4.2}$$

Furthermore it is pointed out that the existence of off-diagonal modular invariant partition functions implies the existence of a nontrivial automorphism of the fusion algebra. In ref. [19], the E_7 type model has been investigated and how his fusion rule should work in this model becomes clear. In what follows we provide further support for his theory using our results.

In order to prove eq. (4.2), it is necessary to construct a set of operators $\phi_i(\mathrm{a})$ and $\phi_i(\mathrm{b})$ which act on the characters [18]. In ref. [20], these operators are explicitly constructed for diagonal modular invariant models. In the remainder of this section we construct Verlinde's operators for the E_7 modular invariant model. The same procedure should work for the D_{odd} type theories.

In E_7 type theory, a naive factorization of operator product expansion coefficient C is not valid. The prescription is as follows. Let us consider $\phi_{[2]} \times \phi_{[2]} \to \phi_{[2]}$ channel. This channel contains the following $SU(2)$ primary fields:

$$\begin{aligned}
\Phi^{(2,2)} \times \Phi^{(2,2)} &\rightarrow \Phi^{(2,2)}, \\
\Phi^{(2,6)} \times \Phi^{(2,2)} &\rightarrow \Phi^{(2,6)}, \\
\Phi^{(2,6)} \times \Phi^{(2,6)} &\rightarrow \Phi^{(2,2)}, \\
&\vdots \\
\Phi^{(6,6)} \times \Phi^{(6,6)} &\rightarrow \Phi^{(2,2)}.
\end{aligned} \tag{4.3}$$

As we have shown, all the coupling constants for those channels are equal, $C^2((22)(22)(22))$. Therefore it is consistent to define the factorized coupling $C([2][2][2])$ as follows:

$$C^2([2][2][2]) = C((22)(22)(22)). \tag{4.4}$$

For the couplings like $C^2((33)(33)(41))$ and $C^2((33)(33)(44))$, we can define the factorized couplings in the following way:

$$C^2([3][3][1]) = C^2([3][3][4^+]) = C((33)(33)(41)),$$

$$C^2([3][3][4^-]) \quad = \quad C((33)(33)(44)). \tag{4.5}$$

In the $\phi_{[3]} \times \phi_{[3]} \to \phi_{[2]}$ channel, there are two different couplings

$$C^2((33)(33)(22)) = C^2((33)(33)(26) = C^2((33)(33)(62)),$$

and

$$C^2((33)(33)(66)).$$

Therefore it is not possible to define single factorized coupling for this channel. So we introduce new labels α and β in order to distinguish different couplings

$$\begin{aligned}
\{C_{332}^{\alpha\alpha}\}^2 &= C((33)(33)(22)), \\
\{C_{332}^{\alpha\beta}\}^2 &= C((33)(33)(26)), \\
\{C_{336}^{\beta\alpha}\}^2 &= C((33)(33)(62)), \\
\{C_{336}^{\beta\beta}\}^2 &= C((33)(33)(66)).
\end{aligned} \tag{4.6}$$

We can maintain the relationship

$$\{C_{i,j,k}^{\gamma\gamma'}\}^2 = \{C_{i,8-j,8-k}^{\gamma\gamma'}\}^2 \tag{4.7}$$

in this factorization procedure, since

$$C^2((j_1,\bar{j}_1)(j_2,\bar{j}_2)(j_3,\bar{j}_3)) = C^2((j_1,\bar{j}_1)(8-j_2,\bar{j}_2)(8-j_3,\bar{j}_3)). \tag{4.8}$$

All the other couplings of the theory can be factorized in an analogous fashion.

In ref. [20], Verlinde's operators are constructed by starting from the quantity $\Gamma_{i[j]}(z,q)$ which is the normalized combination of holomorphic conformal blocks for the two point functions on the torus. We generalize their construction as follows:

$$\begin{aligned}
&\Gamma_{i[j]}(z,q) \\
&= \sum_{\gamma,\gamma'=\alpha}^{\beta} \sum_{j\in[j]} \sum_{[k]\subset[i]\times[j]} Y_{[i]^*[j]^*[k]^*}^{\gamma\gamma'} \left[\sum_{k\in[k]} (C_{ijk}^{\gamma'\gamma})^2 \Gamma_{ijk}(z,q) \right],
\end{aligned} \tag{4.9}$$

where Γ_{ijk}'s are holomorphic conformal blocks for the two point functions on the torus. Here $q = e^{2\pi i\tau}$ (τ is the modulus of the torus) and $z = e^{2\pi i w}$ (w

is the coordinate difference of the two operators). $[i]^* = [i]$ for all i except $[1]^* = [4^+]$ and $[4^+]^* = [1]$. The coefficient $Y_{[i][j][k]}$ is defined by

$$Y^{\gamma\gamma'}_{[i][j][k]} = \sum_{k\in[k]} (C^{\gamma\gamma'}_{ijk})^2 X^{ij}_{k0}, \tag{4.10}$$

where X^{ij}_{k0}'s are the normalized connection matrix [20].

Verlinde's operators $\phi_i(\mathbf{b})$ and $\phi_i(\mathbf{a})$ can be constructed just like in the diagonal modular invariant theories [20], if we start from the quantity $\Gamma_{i[j]}(z,q)$ defined in eq. (4.9). In particular we have found the following relation

$$N_{[i][j][k]} = \frac{\sum^{\beta}_{\gamma,\gamma'=\alpha} Y^{\gamma\gamma'}_{[i]^*[j]^*[k]^*} Y^{\gamma'\gamma}_{[i][k][j]}}{Y_{[i][i][0]}}. \tag{4.11}$$

In ref. [19], the fusion rule for the E_7 modular invariant theory is obtained from the modular transformation matrix using the relation (4.2). Once we have constructed a pair of operators $\phi_i(\mathbf{b})$ and $\phi_i(\mathbf{a})$, eq. (4.2) can be proven by noticing that the modular transformation diagonalizes the fusion algebra. In fact we have numerically verified that eq. (4.11) agrees with eq. (4.2) using the C's we have determined in this paper. What is not obvious is that the $N_{[i],[j],[k]}$'s are integers which count the number of the independent couplings of the corresponding channel. The proof of this fact requires the so-called 'pentagon' identity [21]. We have also checked such identities using our results.

5 Conclusions

We have developed a general procedure to study the structure of operator algebra in off-diagonal modular invariant theories. In particular we have carried out this procedure in E_7 type modular invariant Wess-Zumino-Witten theory and explicitly checked the closure of operator product algebra, which is required for any consistent conformal field theory. In the literature C's in diagonally modular invariant theories are fairly extensively studied. However in this respect off-diagonally modular invariant theories have been studied little. Therefore our work should be useful toward the determinations of the correlation functions in all conformal field theories.

The conformal field theory has been utilized to construct perturbative vacuums in string theory. Apparently quite nontrivial vacuums can be constructed out of minimal models of the $N = 2$ superconformal theory. We have studied the Yukawa couplings of such a model which uses E_7 type off-diagonal modular invariance. The Yukawa couplings have enabled us to study detailed correspondences between conformal field theory constructions and geometry. We have also discussed phenomenological prospects of this model. Since our work should also enable us to calculate the Yukawa couplings of this class of models in general, it is straightforward to study general models in a similar manner. Although off-diagonal modular invariant theories are rather special, realistic models seem to require very special manifolds. Therefore they may enhance the viability of string theory to describe real world.

We have also studied Verlinde's fusion algebra in E_7 modular invariant theory. It is a general theory concerning the indicators of C. The remarkable fact is that it is determined in the holomorphic sector only. Furthermore the indicator is given by the modular transformation matrix. A pair of operators which operate on the characters play a crucial role in his theory. Up to now the precise nature of these operators in off-diagonal modular invariant theories is not clear. We have explicitly constructed these operators in this model. Our investigations support Verlinde's theory also in off-diagonal modular invariant theories.

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Covariant Operator Formalism of Bosonic String Theory*

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ABSTRACT

Covariant operator formalism of bosonic string theory based on the the BRST symmetry is introduced. The duality of the ghost string vertex is explored and recovered in the N ghost string vertex. Duality guarantees that any amplitudes are reduced to the combination of a few primitive operators. We demonstrate this point in the planar case, showing the construction of the multi-loop operator from a set of one-loop tadpole operators. The result has a correct measure due to the ghost contribution.

* This talk is based on the works [1,2,3]. The works [1,2] is done in collaboration with T.Kobayashi and T.Suzuki.

1. Introduction

In the last few years the quantum properties of string theories are crilified completely in the one-loop order and partially in the higher order. In that analysis, the operator formalism is appeared to be a useful tool for the rigorous and simple calculation of amplitudes except for the point that the integration region should be restricted by hand. The extension of the formalism to the form, by which the multi-loop amplitudes can be calculated, was already considered in the eighteen years ago[4]. It however is incomplete in the point of the elimination of unphsical modes. The developement of the BRST formulation of string theory has a key to settle this problem. Freeman and Olive [5] showed this point calculating the one-loop vacuum anmplitude. Then the BRST invariant three dual string vertex was derived by Neveu and West[6] ,and was extended to the N-point case by DiVecchia, Frau, Lerda and Sciuto[7]. The investigation of the covariant multi-loop calculation was done by DiVecchia et.al. and the Napoli group[8] and by us[1,2,3]† . The one of the remakable feature in the new formalism is the ghost factor associated with the twisted or untwisted propagator[2,7] which carries ghost number minus one. This factor, which was determined by the requirement of the BRST invariance, yields systematically a proper measurer factor and ghost number to any operator in the formalism.

In this note, we introduce the multi-loop operator calculus in the BRST invariant operator formalism of bosonic string . Our basic idea is in the use of the primitive operators[9] in the calculation. For this purpose, we will first give an improvement of the N-dual string vertex so that the off-shell duality is manifest. Then the uniqueness of the multi-loop operator derived by the iterative use of the primitive operator is guaranteed. We next demonstrate the multi-loop calculation. It turns out that the measure factor for the moduli integration is derived explicitely in the Schottky parametrization. It is also shown that

† In Ref.[1] another scheme of covariant operator formalism based on the propagator $\int_0^1 \frac{dx}{x} x^{L_0}$ and the ghost insertion b_0 was also proposed. We here however follow the Lovelace scheme for the convenience of dealing with the duality.

the bilinear coefficients in the ghost multi-loop operator are identified with the differential coefficients of a certain automorphic form. Hence the resultant multi-loop operator is very similar with the one discussed in the operator formalism based on the KP-hierarchy[10] .

2. BRST Invariant N String Vertex and Duality

Let us begin by introducing our building blocks. The first one is the BRST invariant three dual string vertex given by

$$
\begin{aligned}
< V_{123}| =< 0_a; q = 3|exp\{ \sum_{r,s=1(r\neq s)}^{3} [\frac{1}{2} \sum_{n,m=0}^{\infty} a_n^r D^{(0)}(U_r V_s)_{nm} a_m^s \\
- \sum_{n=2}^{\infty} \sum_{m=-1}^{\infty} c_n^r D^{(1)}(U_r V_s)_{nm} b_m^s]\} \\
\times \prod_{n=0,\pm 1} (\sum_{r=1}^{3} \sum_{m=-1}^{1} D^{(1)}(V_r)_{nm} b_m^r),
\end{aligned}
\tag{2.1}
$$

where $D^{(0)}_{nm} \equiv \sqrt{\frac{m!\Gamma(n+\varepsilon)}{n!\Gamma(n+\varepsilon)}}\mathcal{D}^{(-\varepsilon/2,0)}_{nm}$ and $D^{(1)}_{nm} \equiv \mathcal{D}^{(1,0)}_{nm}$ are the normalized $(-\varepsilon/2, 0)$ and the unnormalized (1,0) representation matrix elements of SL(2,R);

$$
\mathcal{D}^{(J,p-J)}(\hat{\Lambda})_{nm} = \frac{1}{(m+p)!} \frac{d^{m+p}}{dz^{m+p}} [[\hat{\Lambda}(z)]^{n+p} \Big/ \left(\frac{d\hat{\Lambda}(z)}{dz}\right)^J]_{z=0}.
$$

The second one is the twisted propagator

$$
T = (b_0 - b_1) \int_0^1 \frac{dx}{x(1-x)} \wp(x), \qquad \wp(x) = x^{L_0} \Omega (1-x)^W, \tag{2.2}
$$

which satisfies $\{T, Q_B\} = 0$ with the BRST cherge Q_B. The untwisted counterpart of the propagator is given by the change $\wp(x)$ to $\tilde{\Omega}^\dagger \wp(x)$, $\tilde{\Omega}^\dagger = e^{-L_1}\Omega^\dagger$. The final one is the reflection operator

$$
|R_{12}\rangle = \prod_{n=0,\pm 1} (b_n^1 - b_{-n}^2) exp[\sum_{n=1}^{\infty} a_n^{1\dagger} a_m^{2\dagger} + \sum_{n=2}^{\infty} (c_n^{1\dagger} b_n^{2\dagger} + c_n^{2\dagger} b_n^{1\dagger})] |0_a; q = 3>, \tag{2.3}
$$

which transforms the bra-state to its ket counterpart in the BRST invariant way.

The N-point extension of (2.1) is constructed by the combination of (2.1)~ (2.3) according to the sewing rule of Lovelace[11] . The result is

$$< V_{1\ldots N}| = \int \prod_{r=1}^{N} \frac{dz_r \theta(z_{r+1}-z_r)}{dV_{abc}(z_{r+1}-z_r)} < W_N^{old}|\Delta_{1\ldots N} \tag{2.4}$$

where

$$< W_N^{old}| = \int d^D\hat{x} \prod_{r=1}^{N} \Big[< 0_a; q=3|\Big] exp\{ \sum_{r,s=1(r\neq s)}^{N} [\frac{1}{2} \sum_{n,m=0}^{\infty} a_n^r D^{(0)}(U_r V_s)_{nm} a_m^s$$

$$-\sum_{n=2}^{\infty} \sum_{m=-1}^{\infty} c_n^r D^{(1)}(U_r V_s)_{nm} b_m^s]\}$$

$$\times \prod_{n=0,\pm1} (\sum_{r=1}^{N} \sum_{m=-1}^{1} D^{(1)}(V_r)_{nm} b_m^r) \tag{2.5}$$

$$\Delta_{1\ldots N} = \prod_{s=1}^{N-3} \Big[\sum_{r=1}^{s+1} \sum_{n=0\pm1} b_n^r (D^{(1)}(U_E, V_r)_{0n} - D^{(1)}(U_E, V_r)_{1n})\Big]. \tag{2.6}$$

The vertex (2.4) is BRST invariant but not dual in the ghost zero modes sector. As suggested in Ref.[7] ,however it recover the duality when one attaches the constraints $b_0 - b_1$ on all the legs. We found that under this constraint the ghost sector of (2.4) reduces to the following cyclic symmetric form[2] :

$$< \tilde{V}_{1..N}^{gh}| \equiv < V_{1..N}^{gh}|\Big|_{b_0^r=b_1^r}$$

$$= \prod_{r=1}^{N} \Big[< q=3|\Big] exp\{ \sum_{r,s=1(r\neq s)}^{N} [-\sum_{n=2}^{\infty} \sum_{m=-1}^{\infty} c_n^r D^{(1)}(U_r V_s)_{nm} b_m^s]\}$$

$$\times \prod_{r=1}^{N} [b_{-1}^r - b_1^r - D^{(1)}(U_r V_{r-1})_{11} b_1^{r-1}] \prod_{r=1}^{N} \Big(\frac{z_{r+1}-z_{r-1}}{z_r - z_{r-1}}\Big). \tag{2.7}$$

Note that the last factor of (2.7) is just the measure factor which Lovelace introduced by hand to his original N-reggeon vertex to obtain the correct Koba-Nielsen

amplitude. It should also be noted that the reduced vertex (2.7) is not BRST invariant but recover the invariance by attaching the propagators $\int_0^1 \frac{dx}{x(1-x)}\wp(x)$ on the legs without los of consistency with the sewing rule above mentioned.

3. The Multi-Loop Calculus

The duality of the vertex encourages us to calculate the primitive operators. In the planar case, the primitive operator is only a one-loop tadpole. Let us begin by this calculation. The calculation of the orbital sector was already done by Gross and Schwarz[12] . The ghost counterpart was considered by us and the authors of Ref.[8]. The result is given by

$$< T_1| = \int_0^1 \frac{dw}{w} \int \frac{d^D k}{(2\pi)^D} < T_1^X| < T_1^{gh}|, \tag{3.1}$$

$$\begin{aligned} < T_1^X| = w^{-\frac{1}{2}k^2} \prod_{n=1}^{\infty}(1-w^n)^{-D} < 0_a|exp\{(a|[|\frac{1}{\beta}) - |\frac{1}{\alpha})]k \\ + \frac{1}{2}(a|\Gamma^{(0)}(\mathcal{P}^{(0)} - I)|a)\}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} < T_1^{gh}| = \frac{1}{w(1-w)} \prod_{n=1}^{\infty}(1-w^n)^2 < q = 3|exp\{-[c|\Gamma^{(1)}(\mathcal{P}^{(1)} - I)|b]\} \\ \times (b_{-1} - (1-w)b_0). \end{aligned} \tag{3.3}$$

Here k is a loop momentum, $\mathcal{P}^{(J)} = \sum_{n=-\infty}^{\infty}(P^{(J)})^n, (P^{(J)})_{nm} \equiv D^{(J)}(\hat{P})_{nm}$,

$\hat{P}(z) = wz + 1, \quad \hat{\Gamma}(z) = \frac{1}{z}, \quad (a|\alpha) = \sum_{n=1}^{\infty} a_n \frac{1}{\sqrt{n}} \alpha_n,$

$(a|A|a) = \sum_{n,m=1}^{\infty} a_n A_{nm} a_m, \quad [c|A|b] = \sum_{n,m=2}^{\infty} c_n A_{nm} b_m.$

We also took w, α, β as the multiplier and the two fixed points of $\hat{P}$. The two powers of the partition function in (3.3) is a desired result. In addition, the factor $\frac{1}{(1-w)}$ in (3.3) is remarkable. This cancells the jacobian factor appearing when one attaches a N-particle state to (3.1) and transforms it to the standard form[13] .

Now we go on to consider the multi-loop construction. By glueing the tadpole operators (3.1) iteratively with the string vertex (2.1) through the propagator (2.2) , we found that the M-loop planar ghost operator is given by[3]

$$< T_M^{gh}| = \frac{F^{(2)}[G^{(M)}]^2}{\prod_\alpha{}'(1-w_\alpha)}\prod_{i=0}^{M-1}(\frac{1-w_i}{w_i})\prod_{i=1}^{M-1}(\frac{1-x_i}{x_i}\frac{1-y_i}{y_i})$$

$$\times < q=3|_{12}exp\{-[c|\Gamma(\mathcal{P}^{(M)}-I)|b]\}$$

$$\times(b_{-1}-b_0+\prod_{i=0}^{M-1}w_i\prod_{i=1}^{M-1}(\frac{x_i}{1-x_i}\frac{y_i}{1-y_i})^2 b_0), \tag{3.4}$$

where $F^{(2)}[G^{(M)}]$ is the usual pratition function

$$F^{(s)}[G^{(M)}] = \prod_\alpha{}'\prod_{n=0}^{\infty}(1-w_\alpha^{n+s}) \tag{3.5}$$

associated with the Schottky group $G^{(M)}$ generated by M basic projective transformations $P_i^{(M)}$ (i=0,1,..M-1) with fixed points $\alpha_i^{(M)},\beta_i^{(M)}$ and multipier w_i. Each $P_i^{(M)}$ corresponds to the one loop tadpole. $\mathcal{P}^{(M)}$ is the sum of all the elements of $G^{(M)}$. The product $\prod'_\alpha$ is carried out over all primitive elements of $G^{(M)}$.

The bilinear coefficients in the exponent may be identified with the differential coefficients of the ghost Green function defined on the closed Riemann surface i.e. the double of the open Riemann surface corresponding to the dual diagram.

By definition

$$(\Gamma(\mathcal{P}^{(M)}-I))_{nm} = \sum_{P_\alpha\in G^{(M)}\setminus\{I\}} D^{(1)}(\Gamma\hat{P}_\alpha)_{nm}$$

$$= \frac{1}{(n-2)!}\frac{1}{(m+1)!}\Big(\frac{\partial}{\partial x}\Big)^{n-2}\Big(\frac{\partial}{\partial y}\Big)^{m+1}\Big[\mathcal{G}_M(x,y)-\frac{1}{x-y}\Big]_{x=y=0}, \tag{3.6}$$

where

$$\mathcal{G}_M(x,y) = \sum_{P_\alpha \in G^{(M)}} \left(\frac{d\hat{P}_\alpha(y)}{dy}\right)^{-1} \frac{1}{x - \hat{P}_\alpha(y)}$$

$$or \sum_{P_\alpha \in G^{(M)}} \{-\left(\frac{d\hat{P}_\alpha(x)}{dx}\right)^2 \frac{1}{y - \hat{P}_\alpha(x)} + \sum_{m=0,\pm 1} C_m^\alpha(x) y^{m+1}\}. \qquad (3.7)$$

This indicates that the bilinear coefficients $(n, m \geq 2)$ are simultaneously the differential coefficients of the automorphic form of weight -1 in y and 2 in x.

In the proof of (3.4), the following recursion formulae among the quantities asociated with the Schottky group is essential: in the suitable representation,

$$F^{(s)}[G^{(M+N)}] = F^{(s)}[G^{(M)}]F^{(s)}[G^{(N)}]det(1 - (\mathcal{P}^{(M)} - 1)(\mathcal{P}^{(N)} - 1))^{\frac{1}{2}} \qquad (3.8)$$

and

$$\mathcal{P}^{(M+N)} = \mathcal{P}^{(N)} + \mathcal{P}^{(N)}(\mathcal{P}^{(M)} - I)\sum_{l=0}^{\infty}[(\mathcal{P}^{(M)} - I)(\mathcal{P}^{(N)} - I)]^l \mathcal{P}^{(N)}. \qquad (3.9)$$

Now the covariant M-loop tadpole operator is given by joining our results (3.4) and the orbital part[14] :

$$< T_M^X| = \int \prod_{i=1}^{M-1} \{\frac{dx_i}{x_i(1-x_i)} \frac{dy_i}{y_i(1-y_i)}\} \int \prod_{i=0}^{M-1} \frac{dw_i}{w_i} F^{(1)}[G^{(M)}]^{-D}$$

$$\times \int \prod_{i=0}^{M-1} \frac{d^D k_i}{(2\pi)^D} exp\{\frac{1}{2}\sum_{i,j=0}^{M-1} k_i \cdot k_j 2\pi Im\tau_{ij}\}$$

$$\times < 0_a|exp\{\sum_{n=1}^{\infty} a_n \frac{\sqrt{n}}{n!}\frac{\partial^n}{\partial z^n}\phi_j(z)\Big|_{z=0} k_j$$

$$+\frac{1}{2}\sum_{n,m=1}^{\infty} a_n \frac{\sqrt{nm}}{n!m!}\frac{\partial^n}{\partial z^n}\frac{\partial^m}{\partial w^m} ln\frac{E(z,w)}{z-w}\Big|_{z=w=0} a_m\}, \qquad (3.10)$$

where τ_{ij}, $\phi_j(z)$ and $E(z,w)$ are the period matrix, the first Abelian integrals and the prime form associated with the Riemann double, respectively.

The totality satisfies the BRST invariance before neglection of the cross terms[3], if one attaches the twisted propagator $\int_0^1 \frac{dx}{x(1-x)}\wp(x)$. It also has a correct ghost number enough to obtain a non-zero answer when one attaches the physical external states.

The M-loop amplitude with any number of external particle states is easily derived by attaching a suitable particle state. For example, the M-loop N tachyon amplitude is derived as follows

$$\mathcal{A}^{(M)}(p_1, p_2, \ldots, p_N)$$

$$= \int_F \frac{\prod_{i=1(\neq F)}^{N+1} dz_i}{dV_{abc}} \prod_{j=0}^{M-1} [\frac{dw_j(1-w_i)^2 d\alpha_j^{(M)} d\beta_j^{(M)}}{w_j(\alpha_j^{(M)} - \beta_j^{(M)})^2}] F^{(1)}[G^{(M)}]^{-D} F^{(2)}[G^{(M)}]^2$$

$$\times (detIm\tau)^{-D/2} \prod_{r<s} \Big[E(z_s, z_r) exp(-\pi \int_{z_r}^{z_s} d\phi_i (Im\tau)_{ij}^{-1} \int_{z_r}^{z_s} d\phi_j)\Big]^{p_r \cdot p_s}. \tag{3.11}$$

Here the integration region F for w, α, β is restricted by hand to a fundamental region of the Modular transformation. This restriction is done due to the expectation of the modular invariance of the integrant. We will discuss this point in the next section.

4. Summary and Discussions

We show the construction of the covariant multiloop planar operator in the manifestly factorizable way. We also relate the bilinear coefficients of the ghost loop operator to the automorphic form of certain weight. It is found that the ghost part correctly contributes to the measure factor.

The multi-loop amplitude agrees with the one obtained by DiVecchia et.al.[7], sewing M-pairs of legs of the N+2M point vertex with setting N external states on-shell. The most remarkable feature of the result is the fact that in more than

two loop order the ghost contribution to the partition function appeares only in the $n \geq 2$ modes, on the contrary to the Mandelstam's conjecture in the light cone gauge calculation[15]. The reason why this happen is in the absence of the ghost zero modes on the Riemann surface of genus more than two. One can find the analogous feature in the results of the Polyakov approach. The partition function may be identified with the determinant of the Laplacian and the Fadeev-Popov determinant. Using the Selberg trace formula, these determinants are expressed as

$$(det\Delta_{\hat{g}})^{-13}(det' P_1^{\dagger} P_1)^{\frac{1}{2}} \sim Z'(1)^{-13} Z(2)$$

where in particular $Z(2) = \prod_{\{p\}} \prod_{n=2}^{\infty} \{1 - exp(-nl_p)\}$ is a ghost contribution.

In addition, as conserns the modular invariance, the equivalence of the three-loop vacuum amplitude between the results in the method of the complex analytic function theory and in ours have been shown[16]. As one can see in the proof of (3.4) and (3.10) which is done by the mathematical induction, there is no essential difference between the amplitudes in the less and more than three loop order. We therfore expect the modular invariance of our results in arbitrary order. To prove this point is a future problem.

The author would like to thank T.Kobayashi and T. Suzuki for fruitful collaboration.

Note Added.

After completing this work, I found a paper by G.Cristofano, R.Musto, F.Nicodemi and R.Pettorino in Phys.Lett.B211(1988)417, in which the same result is obtained.

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Parametrization of Super Light-Cone Diagrams with g-Loops.

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ABSTRACT

We review the recent deveropments of the light-cone gauge formulation of fermionic string of Mandelstam. To formulate the fermionic string in a superspace, we use the theory of super Riemann surfaces (SRS). We define the Neumann functions and the Mandelstam mappings in a superspace by means of the so-called abelian differentials of the first and the third kinds on SRS. The functional integral measure of a super light-cone diagram, which consists of $6g-6+2N$ even moduli parameters and $4g-4+2N$ odd ones, are specified in our formulation.

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1. Introduction

At the present time, there exist two types of functional integral methods for the calculation of the string scattering amplitudes. One of them is the light-cone gauge formulation of Mandelstam[1,2,3], in which the unitarity is manifest since one treats only physical states, while the Lorentz covariance is not. The other is the Polyakov string theory[4] that is a manifestly Lorentz covariant approach. For a long time, there have been a question whether these two functional methods are in fact equivalent. Recently Giddings *et al.*[5] proved that for the bosonic string they are indeed equivalent.

In this talk, we attempt to extend the argument to the case of fermionic strings. For fermionic strings, we have two types of light-cone gauge formulations. One is the Neveu-Schwarz-Ramond (N-S-R) formulation, which involves the world-sheet spinors as the fermionic partners to space-time coordinates. The other is the manifestly space-time supersymmetric formulation developed by Green and Schwarz.[6] However both of these two formulations have a disadvantage: To preserve the Lorentz covariance, we must introduce the non-trivial vertex operators at interaction points. In general, their dependence on interaction points prevents us from calculating S-matrix elements explicitly.

In the N-S-R formulation, fortunately, this disadvantage can be removed by going to a superspace, or a supersheet. Berkovits[7,8] showed that at the tree level, if one writes the action as an integral over superspace, the non-trivial vertex operators do not appear. Hence one finds that the functional method for the fermionic string becomes extremely tractable in the superspace formulation. In this paper we will extend his formalism to the higher loop cases.[9]

In formulating the supersheet functional integral, we naturally come to the notion of super Riemann surfaces (SRS).[10,11,12] A SRS is a complex manifold of dimension $1 \mid 1$ with a superconformal structure. For SRS of genus g with even spin structures as well as the classical theory, there are g even holomorphic $\frac{1}{2}$-superdifferentials which are so-called abelian differentials of the first kind on

SRS. There also exist the abelian differentials of the third kind which have simple poles with residues $+1$ and -1. Using the integrals of these superdifferentials, we define the Neumann functions on *SRS* and also the Mandelstam mappings on the supersheets. For odd spin structures there are the pathologies coming from the existence of the Dirac zero modes,[13] so that we only consider the even spin structures. At present, for odd spin structures we do not know how to get the Neumann functions on *SRS* of genus $g(\geq 2)$. Only in supertori it is available, then we can obtain the Neumann functions for all spin structures.

This talk is based on the work in collaboration with M. Takao and our results have been published elsewhere.[9,14,15]

2. Superconformal Mappings and Automorphic Functions on *SRS*

A one-dimensional complex supermanifold is locally described by a complex super coordinate $\mathbf{Z} = (z, \theta)$, where z is an ordinary complex coordinate and θ is an anti-commuting coordinate. First, let us consider a mapping on the supermanifold.[10] A superanalytic mapping $\tilde{\mathbf{Z}} = f(\mathbf{Z})$ is subject to the constraint $D_{\bar{\theta}} f(\mathbf{Z}) = 0$, where D_θ is the supercovariant derivative $D_\theta = \partial_\theta + \theta\partial_z$ and the bar denotes the complex conjugation.

Under a superanalytic mapping $\tilde{\mathbf{Z}} = (\tilde{z}(z,\theta), \tilde{\theta}(z,\theta))$, the supercovariant derivative D_θ is transformed as

$$D_\theta = (D_\theta \tilde{\theta})\tilde{D}_{\tilde{\theta}} + (D_\theta \tilde{z} - \tilde{\theta} D_\theta \tilde{\theta})\tilde{D}^2_{\tilde{\theta}} \ . \tag{2.1}$$

A superconformal mapping is a superanalytic mapping which is subject to the further constraint

$$D_\theta \tilde{z} = \tilde{\theta} D_\theta \tilde{\theta} \ , \tag{2.2}$$

so that the supercovariant derivative D_θ is mapped homogeneously. A superconformal mapping is necessary to describe *SRS*, which is a supermanifold possessing such a mapping as a transition function.

Integration over the Grassmann coordinate θ is given by

$$\int d\theta\theta = 1\,, \quad \int d\theta 1 = 0\,. \tag{2.3}$$

The super contour integral is then defined by

$$\oint d\mathbf{Z} f(\mathbf{Z}) = \oint dz \int d\theta f(\mathbf{Z})\,. \tag{2.4}$$

On the other hand, the line integral

$$F(\mathbf{Z}_1, \mathbf{Z}_2) = \int_{\mathbf{Z}_2}^{\mathbf{Z}_1} d\mathbf{Z}\; f(\mathbf{Z}) \tag{2.5}$$

is defined so that

$$F(\mathbf{Z}_1, \mathbf{Z}_1) = 0\,, \quad D_{\theta_1} F(\mathbf{Z}_1, \mathbf{Z}_2) = f(\mathbf{Z}_1)\,, \tag{2.6}$$

up to non-contractible cycles.†

As well as the classical theory, SRS of genus g has g so-called abelian differentials of the first kind $d\phi_i(\mathbf{Z})$ ($i = 1, \cdots, g$), which are even holomorphic $\frac{1}{2}$-superdifferentials.[13] We normalize them by demanding

$$\oint_{A_j} d\phi_i(\mathbf{Z}) = 2\pi i \delta_{ij}. \tag{2.7}$$

The super period matrix is then

$$\tau_{ij} = \frac{1}{2\pi i} \oint_{B_j} d\phi_i(\mathbf{Z})\,. \tag{2.8}$$

Here A_i and B_i are canonical homology cycles of SRS. To define the Mandelstam mappings and the Neumann functions on SRS, we introduce the abelian

† Recently the line integral on SRS is discussed in Ref.[14].

differentials of the third kind $d\omega_{\mathrm{AB}}(\mathbf{Z})$, which have simple poles at $\mathbf{A} = (a, \alpha)$ and $\mathbf{B} = (b, \beta)$ with residues 1 and -1 respectively. They are uniquely defined by requiring that its periods around the cycles A_i vanish:

$$\oint_{A_i} d\omega_{\mathrm{AB}}(\mathbf{Z}) = 0 \ . \tag{2.9}$$

There also exist other abelian differentials of the third kind $d\Omega_{\mathrm{AB}}(\mathbf{Z})$ with simple poles of residues 1 and -1 at $\mathbf{A}$ and $\mathbf{B}$, which have the pure imaginary periods on any homology cycle:

$$Re \oint_{A_i} d\Omega_{\mathrm{AB}}(\mathbf{Z}) = Re \oint_{B_i} d\Omega_{\mathrm{AB}}(\mathbf{Z}) = 0 \ , \quad i = 1, \cdots, g \ . \tag{2.10}$$

The difference between $d\Omega_{\mathrm{AB}}$ and $d\omega_{\mathrm{AB}}$ is a linear combination of $d\phi_i$'s:

$$d\Omega_{\mathrm{AB}}(\mathbf{Z}) = d\omega_{\mathrm{AB}}(\mathbf{Z}) - \frac{1}{2\pi} \sum_{i,j=1}^{g} d\phi_i(\mathbf{Z}) (\mathrm{Im}\tau)^{-1}_{ij} Re(\phi_j(\mathbf{A}) - \phi_j(\mathbf{B})) \ . \tag{2.11}$$

where ϕ_i is the integral of $d\phi_i$, which is called the abelian integral of the first kind. Since the superdifferentials $d\phi_i$ is holomorphic on SRS , the last term of RHS of eq.(2.11) does not change the singularity structure.

Using the integral of the abelian differential of third kind Ω_{AB}, we can construct the Neumann function for the super Laplacian on SRS. The Neumann function has a logarithmic singularity near $\mathbf{Z} \sim \mathbf{Z}'$, or

$$N(\mathbf{Z}, \mathbf{Z}') = \ln|z - z' - \theta\theta'| + \text{ regular terms.} \tag{2.12}$$

The regular terms make up for the single-valuedness of the Neumann function. Since the differential $d\Omega_{\mathbf{Z}'\mathbf{Z}_0}(\mathbf{Z})$ behaves like $d\mathbf{Z}\frac{\theta - \theta'}{z - z' - \theta\theta'}$ near $\mathbf{Z} \sim \mathbf{Z}'$, the real

part of the abelian integral of the third kind $\Omega_{Z'Z_0}(\mathbf{Z})$ becomes the Neumann function:[9]

$$N(\mathbf{Z},\mathbf{Z}') - N(\mathbf{Z},\mathbf{Z}_0) = \mathrm{Re}\Omega_{Z'Z_0}(\mathbf{Z}) \ . \tag{2.13}$$

We also construct the Mandelstam mapping from SRS to a super light-cone diagram in terms of the abelian integral of the third kind. It is defined in section 3.

3. S-Matrix Elements and Mandelstam Mappings

In this section, we present the light-cone gauge formulation of Mandelstam extended to a superspace.[7,8] This formulation gives the physical picture that strings propagate in the space-time along the (light-cone) time $\tau = \mathbf{X}^+$ and undergo occasional interactions. Fermionic string S-matrix elements are described as a sum over all super light-cone diagrams joining incoming and outgoing strings weighted by $\exp(-I_{LC})$. The generating function is given by

$$S = \int [d\mu] \int D\mathbf{X}^i \exp(-I_{LC}(\mathbf{X}^i))\Phi_f^\dagger(\mathbf{X}(\tau=\tau_f))\Phi_i(\mathbf{X}(\tau=\tau_i)) \ , \tag{3.1}$$

where the integration is over the parameters μ describing the shape of the super light-cone diagram $\mathbf{R} = (\rho(=\tau + i\sigma), \psi)$. The string superfields $\mathbf{X}^i(i = 1, \cdots, 8)$ on the super light-cone diagram are†

$$\mathbf{X}^i(\mathbf{R},\bar{\mathbf{R}}) = X^i(\rho,\bar{\rho}) + \psi S_1^i(\rho,\bar{\rho}) + \bar{\psi} S_2^i(\rho,\bar{\rho}) \ , \tag{3.2}$$

and I_{LC} is a light-cone gauge string action

$$I_{LC} = \frac{1}{\pi}\int d\mathbf{R}d\bar{\mathbf{R}}\bar{D}_{\bar{\psi}}\mathbf{X}^i D_\psi \mathbf{X}^i \ . \tag{3.3}$$

The functional $\Phi(\mathbf{X}^i)$ is a wave function for external states satisfying the equation of motion $(P^- + \partial_\tau)\Phi = 0$.

† We set the auxiliary fields to zero, which is possible in this case.

In the light-cone gauge, the Lorentz invariant measure becomes $d^8 p_r dp_r^+/\alpha_r$ (with $p_r^- = (p_r^{i^2} + m_r^2)/\alpha_r$), where α_r's are string length parameters of external states which are identified with $2p_r^+$ and satisfy the condition $\sum_{r=1}^N \alpha_r = 0$. Therefore one has the following relation between an S-matrix and an amplitude A†

$$S = \prod_{r=1}^{N} \mid \alpha_r \mid^{-1/2} A \,. \tag{3.4}$$

Hence, if one wants to obtain a scattering amplitude, one must multiply the S-matrix element by $\prod_{r=1}^N \mid \alpha_r \mid^{1/2}$.

If the external states are open Neveu-Schwarz strings, the superfields $\mathbf{X}^i(\mathbf{R})$ obey the following boundary condition

$$\begin{aligned} (D_\psi - \bar{D}_{\bar{\psi}})\mathbf{X}^i = 0 \qquad & \text{at } \sigma = 0 \text{ and } \psi = \bar{\psi} \ , \\ (D_\psi + \bar{D}_{\bar{\psi}})\mathbf{X}^i = 0 \qquad & \text{at } \sigma = \pi \text{ and } \psi = -\bar{\psi} \ . \end{aligned} \tag{3.5}$$

For the case of closed strings, the boundary condition is

$$\mathbf{X}^i(\rho, \psi) = \mathbf{X}^i(\rho + 2\pi i, -\psi) \ . \tag{3.6}$$

The Mandelstam mapping from SRS to a super light-cone diagram $\mathbf{R} = (\rho(\mathbf{Z}), \psi(\mathbf{Z}))$ is given in terms of the abelian integral of the third kind by

$$\rho(\mathbf{Z}) = \sum_{r=1}^{N} \alpha_r \, \Omega_{\mathbf{Z}_r \mathbf{Z}_o}(\mathbf{Z}) \ , \qquad \psi(\mathbf{Z}) = (\partial_z \rho)^{-\frac{1}{2}} D_\theta \rho(\mathbf{Z}) \ , \tag{3.7}$$

where $\mathbf{Z}_o$ is any point on SRS. Note that the superconformality condition $D_\theta \rho = \psi D_\theta \psi$ makes the fermionic part of the mapping $\psi(\mathbf{Z})$ represented by the bosonic coordinate $\rho(\mathbf{Z})$.

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† Strictly speaking, the relation between the S-matrix and the amplitude is given by

$$S = (2\pi)^{10} i \delta^{10}(\Sigma_r p_r^\mu) \Pi_{r=1}^N \mid \alpha_r \mid^{-1/2} A \,.$$

Here the momentum conservation $\delta(\Sigma_r p_r^i)$ is derived from the functional integration over the constant mode of $\mathbf{X}$ and the conservation of p^+ and p^- is given by the relation $\Sigma_{r=1}^N \alpha_r = 0$ and the on-shell condition, respectively.

Interaction points ($\tilde{\rho}_s = \rho(\tilde{\mathbf{Z}}_s), \tilde{\psi}_s = \psi(\tilde{\mathbf{Z}}_s)$) are determined from the conditions that the transition function of the Mandelstam mapping becomes singular at $\tilde{\mathbf{Z}}_s = (\tilde{z}_s, \tilde{\theta}_s)$, *i.e.*

$$D_\theta \psi \,|_{\mathbf{Z}=\tilde{\mathbf{Z}}_s} = 0 \quad (s = 1, \cdots, N + 2g - 2) \; . \tag{3.8}$$

We find from eqs.(3.7) and (3.8) that the parameters μ specifying a super light-cone diagram of closed strings are

$$\begin{aligned}
\tilde{\rho}_s \text{ (complex; } s = 1, \cdots, N + 2g - 3) &: \quad \text{interaction times and twist angles,} \\
\tilde{\psi}_s \text{ (complex; } s = 1, \cdots, N + 2g - 2) &: \quad \text{super partners of above,} \\
\alpha_i \text{ (real; } i = 1, \cdots, g) &: \quad \text{internal string length parameters,} \\
\beta_i \text{ (real; } i = 1, \cdots, g) &: \quad \text{twist angles,} \\
\alpha_r \text{ (real; } r = 1, \cdots, N) &: \quad \text{external string length parameters.}
\end{aligned} \tag{3.9}$$

Thus the total numbers of the even and the odd parameters are $6g - 6 + 2N$ and $4g - 4 + 2N$ in real dimensions respectively. These coincide with the correct numbers of the moduli parameters on SRS. For open strings all twist angles and its super partners are zero so that the number of the parameters becomes half of the closed ones. The absence of the super partner to the internal length and twist angle parameters α_i and β_i is explained as follows. The internal parameters measure the changes of the Mandelstam mapping $\rho(\mathbf{Z})$ when traveling along A_i and B_i cycles on SRS . On the other hand, from eq.(3.7) we find that the fermionic coordinate $\psi(\mathbf{Z})$ returns without change after the travel along any homology cycle. So there is no Grassmann odd internal moduli parameter.

As shown by Berkovits,[7,8] $\tilde{\psi}_r$ is singular so that we use the rescaled $\tilde{\psi}_r$, referred to as $\tilde{\tilde{\psi}}_r$:

$$\tilde{\tilde{\psi}}_r \equiv \lim_{\rho \to \tilde{\rho}_r} (\rho - \tilde{\rho}_r)^{1/4} \tilde{\psi}_r \; . \tag{3.10}$$

In calculating the S-matrix elements one must integrate over all these parameters specifying the super light-cone diagram. Especially the integration over

the rescaled odd parameters $\hat{\hat{\psi}}_s$ are necessary in order to preserve the Lorentz covariance.

4. Concluding Remarks

In this talk, we have discussed the light-cone gauge formulation of fermionic string on a supersheet with loops. It has been found that for even spin structures the abelian differentials of the first and the third kinds extended to a superspace work well in deriving the Neumann functions on SRS and the Mandelstam mappings from SRS to super light-cone diagrams. In Ref.[14,15], using the super Schottky parametrization, we explicitly construct these superdifferentials in term of the Poincaré theta series of superconformal weight $1/2$ at the multi-loop level. We have also constructed the measure of a super light-cone diagram which is correctly specified by $6g-6+2N$ even parameters and $4g-4+2N$ odd ones in real dimensions.

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Path Integral and Operator Formalism on Bordered Riemann Surfaces

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Abstract

The operator formalism has been developed on the once punctured Riemann surfaces based on a generalized ground state condition in Refs. 6-9. Solving the boundary value problems on bordered Riemann surfaces, we show that the similar conditions are satisfied by a string state defined by the Polyakov path integral. The variational approach of Ohrndorf is confirmed in this context.

1. introduction

Witten's open string field theory in the Siegel gauge provides a Feynman rule[1] such that the propagators are represented by strips, and three-string vertex glues three strips. The virtue of this formalism is that one can uniquely define the *time parameters* on the world sheet by gluing these parts together. Here any space-like section of the world sheet describes an on-shell state, which is represented by a Fock state. Furthermore this Feynman rule reproduces the Polyakov path integral[2] and the moduli space is once and only once covered[3]. Now the open string theory includes closed string propagations, and it is natural to search for closed string states in this formalism. The recent progress along this direction has been reported in Ref. 4, where the open-closed string interactions are constructed. Any closed string field theory should reproduce the Polyakov path integral in a perturbative expansion, because this equivalence is verified in the light-cone gauge[5]. Here the light-cone time is defined on the world sheet in terms of the Mandelstam mapping.

If closed string field theory admits a geometrical interpretation, one can define time parameters on arbitrary world sheet in a characteristic way induced in a fixed gauge. Since the on-shell amplitude of the closed string is provided by the Polyakov path integral, it is natural to ask what kind of off-shell states appear in arbitrary section of the world sheet. In this letter we will obtain an intermediate closed string state in the Polyakov path integral. Namely a definition of a ground state wave functional which represents r strings is given by functional integrals over a bordered Riemann surface with r boundaries.

The first quantization of string is firstly performed on a cylinder and one obtains Fock spaces. As we shall see later, this procedure can be generalized over arbitrary bordered Riemann surface. Consequently the string propagation is also generalized to boundary variations. These investigations may make it possible to try the second quantization of interacting strings. The generalization of the Fock space has been utilized in the KP hierarchy[6-9] as follows. The bosonic N-point g-loop amplitude is factorized as the product of the N-point tree amplitude and the tau function[6,7], where the tau function has two kinds of parameters, one is the KP coordinates and the other is the element of UGM (universal grassmann manifold). The family

of Riemann surfaces of g genus is an orbit in UGM. Each element of the orbit is represented[8] by a ground state $|g\rangle$, which satisfies infinite number of equations such that

$$\oint dz \omega_{(i)}^{1-j} \hat{\phi}(z)|g\rangle = 0. \tag{1}$$

Here $\hat{\phi}(z)$ is a spin j field operator, and $\omega_{(i)}^{1-j}$'s are the complete set of infinite number of holomorphic $(1-j)$-differentials on a once punctured Riemann surface (the puncture is located at $z = 0$).

We will verify that the similar equations as (1) are satisfied by the ground state wave functional mentioned before. Actually Ohrndorf[10,11] has already concerned such a wave functional in order to investigate the boundary variations of a bordered Riemann surface. He then found that the boundary variations are generated by the Virasoro operators. His investigation is based on classical solutions, though there exist no solutions for a certain boundary condition. We will investigate this point carefully and confirm his result.

2. Wave Functional

In the path integral formalism, the wave functional is defined by a functional integral over a bordered Riemann surface M. In the bosonic string theory, the functional depends on the boundary values: $X^{\mu}(\sigma)$, $b_{nt}(\sigma)$ and $c^{n}(\sigma)$. Here n and t stand for the normal and tangent directions respectively, and σ parametrizes ∂M with fixed parameter region $[0, 2\pi]$ on each boundary component. This is because the quantum fluctuations of the action S around classical solutions are given by

$$\begin{aligned} \delta S &= \delta\left[\int\limits_{M} \frac{d^2\xi}{2\pi}\left(\frac{1}{4}(\partial_a X)^2 + \frac{1}{2} b_{ab}(Pc)^{ab}\right) - \int\limits_{\partial M} \frac{d\sigma}{2\pi} b_{nt} c^{t}\right] \\ &= \frac{1}{2}\int\limits_{\partial M} \frac{d\sigma}{2\pi} \delta X \partial_n X^{cl} + \int\limits_{\partial M} \frac{d\sigma}{2\pi}(b_{nn}^{cl}\delta c^{n} - \delta b_{nt} c_{cl}^{t}) \end{aligned} \tag{2}$$

where P denotes the covariant derivative with a projection into the second rank anti-symmetric traceless tensor, and b_{ab} is also restricted to the same tensor space.

The last term is induced from the surface term of S, which makes classical solutions genuine stationary points[11]. The wave functional is written down by Ohrndorf[10,11] in terms of the Green functions and functional determinants as

$$\Phi = e^{-S_{cl}} e^{-S_0}. \tag{3}$$

Inserting classical solutions (the explicit forms are given in (13)) into the original action S, one can obtain the classical action S_{cl}. All linear terms of quantum fluctuation vanish and the quadratic terms provide the functional determinants over M, which are represented by e^{-S_0} in (3). Here we assume that there exists a classical solution for any boundary condition, though this is not true in general. For the string coordinates, this is nothing but the Dirichret problem. The problem is solvable for arbitrary boundary conditions and Riemann surfaces[12]. On the other hand, the above assumption is broken for the ghost fields. In fact, let ϕ_{zz} and λ^z be the holomorphic spin 2 and -1 differentials respectively, and satisfy the boundary conditions such that

$$\phi_{nt} = \lambda^n = 0. \tag{4}$$

If there exist such differentials, there are no classical solutions (holomorphic fields on M) unless

$$\begin{aligned} \int_{\partial M} \frac{d\sigma}{2\pi} \phi_{nn} c^n &= 0 \\ \int_{\partial M} \frac{d\sigma}{2\pi} \lambda^t b_{nt} &= 0. \end{aligned} \tag{5}$$

The differentials in (4) are constructed from the holomorphic differentials on the Schottky double M^d (depicted in Fig. 1)[13]. Let g be the genus of M, and r the number of boundaries. Then each space of these differentials has the following dimensions:

$$\begin{aligned} dim\{\phi\} &= 6g + 3r - 6 & \qquad 2g + r \geq 3 \\ &= 1 & g = 0, r = 2 \\ &= 0 & g = 0, r = 1 \end{aligned}$$

and

$$
\begin{array}{lll}
dim\{\lambda\} = 0 & & 2g + r \geq 3 \\
\qquad\quad = 1 & & g = 0, r = 2 \\
\qquad\quad = 3 & & g = 0, r = 1.
\end{array}
\tag{6}
$$

Fortunately, integrating the ghost zero modes, we obtain the factor

$$
C = \prod_i \int_{\partial M} \frac{d\sigma}{2\pi} \phi^{(i)}_{nn} c^n \prod_m \int_{\partial M} \frac{d\sigma}{2\pi} \lambda^t_{(m)} b_{nt}. \tag{7}
$$

Due to this factor, we can suppose the existence of classical solutions in the rest of functional integrations, and obtain the following result

$$
\Phi[X, b_{nt}, c^n] = C e^{-S_{cl}} e^{-S_0}
$$

where

$$
S_{cl} = \int_{\partial M} \frac{d\sigma_1}{2\pi} \frac{d\sigma_2}{2\pi} \left[\frac{1}{2} X(\sigma_1) \partial^1_n \partial^2_n G(\sigma_1, \sigma_2) X(\sigma_2) + b_{nt}(\sigma_1) F^t{}_{nn}(\sigma_1, \sigma_2) c^n(\sigma_2) \right]
$$

$$
e^{-S_0} = (det\Delta)^{-\frac{d}{2}} \left(\frac{det' P^\dagger P}{det(\phi_i|\phi_j) det(\lambda_l|\lambda_m)} \right)^{\frac{1}{2}}. \tag{8}
$$

Owing to the Dirichret boundary condition, there are no coordinate zero modes. On the other hand, the ghost zero mode integration gives not only the prefactor C, but also the denominators which compensate the subtraction of zero modes in the ghost determinant $det' P^\dagger P$. Here G and F are the Dirichret and the ghost Green functions respectively, which are determined by the relations

$$
\begin{aligned}
\Delta G(\xi, \eta) &= 2\pi \delta^{(2)}(\xi - \eta) \\
(PF)^{ab}_{cd} &= (1 - Q)^{ab}_{cd}(\xi, \eta) \\
(P^\dagger F)^a_b &= (1 - R)^a_b\,(\xi, \eta)
\end{aligned}
\tag{9}
$$

where

$$Q^{ab}_{cd} = \sum_{i,j} \phi^{(i)\,ab}(\xi)\,(\phi^{(i)}|\phi^{(j)})^{-1}\phi^{(j)}_{cd}(\eta)$$
$$R^a_b = \sum_{l,m} \lambda^a_{(l)}(\xi)\,(\lambda_{(l)}|\lambda_{(m)})^{-1}\lambda_{(m)\,b}(\eta)$$

are the projections onto b (and c resp.) ghost zero modes. These Green functions satisfy the boundary conditions $G = F^a_{\ nt} = F^n_{\ bc} = 0$.

Let us define the differential operators by

$$\begin{aligned}
\hat{P}_\xi &= -\left(\frac{\delta}{\delta X(\sigma)} + \frac{i}{2}\partial_\sigma X\right) \\
\hat{b}_{\xi\xi} &= \left(\frac{\delta}{\delta c^n(\sigma)} - i b_{nt}(\sigma)\right) \Big/ 2 \\
\hat{c}^\xi &= c^n(\sigma) + i\frac{\delta}{\delta b_{nt}(\sigma)}
\end{aligned} \tag{10}$$

and conjugate operators $\hat{P}_{\bar{\xi}}$ etc. by their complex conjugate ones. The coordinate ξ is defined as in Fig. 2 on each boundary. These operators satisfy the commutation relations on ∂M as

$$\begin{aligned}
[X(\sigma), \hat{P}_{\xi'}] &= 2\pi\delta(\sigma - \sigma') \\
\{\hat{c}^\xi, \hat{b}_{\xi'\xi'}\} &= 2\pi\delta(\sigma - \sigma') \\
\{\hat{c}^{\bar{\xi}}, \hat{b}_{\xi'\xi'}\} &= 0.
\end{aligned} \tag{11}$$

The ordinary Fock operators satisfy same relations on S^1. These boundary operators act on Φ such that

$$\begin{aligned}
\hat{P}_\xi \Phi &= \partial_\xi X^{cl}\Phi \\
\hat{b}_{\xi\xi}\Phi &= b^{cl}_{\xi\xi}\Phi + \sum_i \phi^{(i)}_{\xi\xi}\left(2\int_{\partial M}\frac{d\sigma}{2\pi}\phi^{(i)}_{nn}c^n\right)^{-1}\Phi \\
\hat{c}^\xi\Phi &= c^\xi_{cl}\Phi + i\sum_m \lambda^\xi_{(m)}\left(\int_{\partial M}\frac{d\sigma}{2\pi}\lambda^t_{(m)}b_{nt}\right)^{-1}\Phi.
\end{aligned} \tag{12}$$

The field equations are solved by using the Green functions as follows:

$$X^{cl}(\xi) = \int\limits_{\partial M} \frac{d\sigma'}{2\pi} \partial'_n G(\xi, \sigma') X(\sigma')$$
$$b^{cl}_{\xi\xi} = \int\limits_{\partial M} \frac{d\sigma'}{2\pi} b_{nt}(\sigma') F^t{}_{\xi\xi}(\sigma', \xi)$$
$$c^{\xi}_{cl} = -\int\limits_{\partial M} \frac{d\sigma'}{2\pi} F^{\xi}{}_{nn}(\xi, \sigma') c^n(\sigma'). \tag{13}$$

The ghost classical solutions of (12) are holomorphic because of the prefactor C. Since the boundary operators are holomorphic when they act on Φ, we can expand them in terms of $e^{i\sigma} = z$. Then the coefficients become the normal modes of the Fock operator, and the boundary operators are identified with the Fock operators. Finally we obtain the infinite number of differential equations

$$\int\limits_{\partial M} \frac{d\xi}{2\pi i} f(\xi) \hat{P}_{\xi} \Phi = 0$$
$$\int\limits_{\partial M} \frac{d\xi}{2\pi i} v^{\xi} \hat{b}_{\xi\xi} \Phi = 0$$
$$\int\limits_{\partial M} \frac{d\xi}{2\pi i} h_{\xi\xi} \hat{c}^{\xi} \Phi = 0 \tag{14}$$

and their complex conjugate equations. (Note that Φ is real.) Here f, v^{ξ} and $h_{\xi\xi}$ are the holomorphic fields with no specific boundary conditions. If M has no boundaries but punctures, any holomorphic field on M is meromorphic on M^c (the closure of M). In the once punctured case, due to the Weierstrass gap theorem[8], above equations are easily solved by the Bogoliubov transformation of the Fock vacuum. In the bordered case, the prefactor C corresponds to the $3g-3$ gaps. However the bordered Riemann surface and its puncture limit are quite different. In fact, the holomorphic and anti-holomorphic sectors are decoupled each other at the puncture limit, whereas for the bordered Riemann surfaces these are coupled in Φ of (14).

3. Boundary Variation

Any bordered Riemann surface M is conformally equivalent to a surface which consists of r standard cylinders glued together[3]. Then the boundary variations δ^ξ is realized by gluing a ring domain of width $\delta^n(\sigma)$ on each boundary (as depicted in Fig. 3). δ^n cannot be holomorphically extended on M unless $\int_{\partial M} d\sigma \phi_{nn}\delta^n = 0$ (the same condition as (5)). The number of moduli of M is equal to the number of independent boundary conditions which have no holomorphic extensions. Hence any Teichmüller deformation can be generated by a corresponding δ^n. Let $\hat{\delta}$ be a variational operator with respect to the boundary variation δ^ξ. If one fixes the parameter region of the world sheet, the variation of the metric is given by $\hat{\delta} g^{ab} = -(\nabla^a \delta^b + \nabla^b \delta^a)$. Hence the change of S_0 is calculated[14] as

$$\hat{\delta} S_0 = -\int_{\partial M} \frac{d\sigma}{2\pi} \delta^a \langle T^{qu}_{na} \rangle \tag{15}$$

and the expectation value of the quantum energy-momentum tensor is evaluated by

$$\langle T^{qu}_{\xi\xi} \rangle = -\lim_{\eta \to \xi} \left[2\partial_\xi \partial_\eta G + 2\partial_\eta \tilde{F}^\eta{}_{\xi\xi} + \partial_\xi \tilde{F}^\eta{}_{\xi\xi} \right] \tag{16}$$

where $\tilde{F}$ includes zero-mode contributions which were excluded from F. This is because the ghost zero modes contribute to $\langle T^{qu}_{\xi\xi} \rangle$ from the denominator of the ghost determinant in (8). $\tilde{F}^\eta{}_{\xi\xi}$ becomes holomorphic with respect to both ξ and η, though the tensorial property is lost, *i.e.* the periods around nontrivial cycles do not vanish. The tilde Green function is given by the Poincaré theta series[14,15] of the Schottky group Γ^d, which is defined on the double M^d. In the single boundary case, we can choose a convenient single coordinate z to cover M, where the real axis denotes the boundary. Then Γ^d is defined to be self conjugate as

$$\bar{\gamma} \in \Gamma^d \qquad \text{if} \qquad \gamma \in \Gamma^d \tag{17}$$

where bar denotes the complex conjugation. The singularity at a coincident point is caused by the identity element $\gamma = 1$ of Γ^d, and all other elements provide regular

terms. One can regularize this singularity by using the ζ-function. The result is independent of any geometrical data of M, and (16) becomes

$$\langle T^{qu}_{\xi\xi}\rangle = \alpha + \langle T^{qu}_{\xi\xi}\rangle\Big|_{\gamma\neq 1} \tag{18}$$

where $\langle T^{qu}_{\xi\xi}\rangle|_{\gamma\neq 1}$ denotes that the identity element $\gamma = 1$ is excluded from the Poincaré theta series. The constant α is the intercept, and $\alpha = -1$ for the bosonic closed string with 26 space-time dimensions.

The change of S_{cl} was already calculated by Ohrndorf[11] as

$$\hat{\delta} S_{cl} = -\int_{\partial M} \frac{d\sigma}{2\pi}\delta^a T^{cl}_{na}. \tag{19}$$

Especially the classical ghost action is given by

$$S_{cl} = \int_{\partial M} \frac{d\sigma_1}{2\pi}\frac{d\sigma_2}{2\pi} b_{nt}(\sigma_1)\kappa_1^{-1}\kappa_2^2 F^t{}_{nn}(\sigma_1,\sigma_2)c^n(\sigma_2) \tag{20}$$

where $\kappa_i = \left|\frac{d\xi}{d\sigma_i}\right|$. The boundary parameter σ_i and the boundary values $b_{nt}(\sigma_1)$ and $c^n(\sigma_2)$ are independent of any geometrical data of M. (Remember that the parameter region of σ is fixed to $[0, 2\pi]$.) Hence only the term $\kappa_1^{-1}\kappa_2^2 F^t{}_{nn}$ depends on M. The variation is divided into the following two pieces:

$$\begin{aligned}\hat{\delta}(\kappa_1^{-1}\kappa_2^2 F^t{}_{nn}) = \hat{\delta}_A &\left[\kappa_1^{-1}\kappa_2^2 t^a(\sigma_1)n^b(\sigma_2)n^c(\sigma_2)F^a{}_{bc}(\sigma_1,\sigma_2)\right] \\ &+ \hat{\delta}_R(\kappa_1^{-1}\kappa_2^2 F^t{}_{nn}).\end{aligned} \tag{21}$$

Here $\hat{\delta}_A$ denotes the variation of the argument ξ, whereas $\hat{\delta}_R$ denotes the variation of the domain on which $\kappa_1^{-1}\kappa_2^2 F^t{}_{nn}$ is defined. One can evaluate the change under $\hat{\delta}_A$ by using the following equations

$$\begin{aligned}\hat{\delta}_A(\kappa t^a) &= \frac{d}{d\sigma}\delta^a \\ \hat{\delta}_A(\kappa n^a) &= \frac{d}{d\sigma}(n^a\delta^t - t^a\delta^n)\end{aligned}$$

$$\hat{\delta}_A F^a_{bc} = \left[\delta^d(\sigma_1)\nabla^1_d + \delta^d(\sigma_2)\nabla^2_d\right] F^a_{bc}. \tag{22}$$

In our argument, due to the prefactor C, we can replace F by $\tilde{F}$ in the classical action (20). The above arguments do not change for $\tilde{F}$. Since $\kappa_1^{-1}\kappa_2^2\tilde{F}^t{}_{nn}$ has a holomorphic extension on M, the same equation as (13) is satisfied. In fact, we obtain

$$\begin{aligned}\hat{\delta}_R(\kappa_1^{-1}\kappa_2^2\tilde{F}^t{}_{nn}) &= \int_{\partial M} \frac{d\sigma}{2\pi}\delta_R\left[\kappa_1^{-1}\kappa^2\tilde{F}^t{}_{nt}(\sigma_1,\sigma)\right]\kappa^{-1}\kappa_2^2F^t{}_{nn}(\sigma,\sigma_2) \\ &= -\int_{\partial M} \frac{d\sigma}{2\pi}\delta_A\left[\kappa_1^{-1}\kappa^2\tilde{F}^t{}_{nt}(\sigma_1,\sigma)\right]\kappa^{-1}\kappa_2^2F^t{}_{nn}(\sigma,\sigma_2).\end{aligned} \tag{23}$$

The last equality is induced from the boundary condition $\tilde{F}^a{}_{nt} = 0$. Hence the change of S_{cl} is written as

$$\hat{\delta}S_{cl} = \int_{\partial M} \frac{d\sigma_1}{2\pi}\frac{d\sigma_2}{2\pi} b_{nt}(\sigma_1)\left[\hat{\delta}_A(\kappa_1^{-1}\kappa_2^2\tilde{F}^t{}_{nn})c^n(\sigma_2) + \hat{\delta}_A(\kappa_1^{-1}\kappa_2^2\tilde{F}^t{}_{nt})c^t(\sigma_2)\right]. \tag{24}$$

Using (22) in (24), we obtain the result (19).

The ghost Virasoro operator is defined by

$$\hat{T}_{\xi\xi} = :2\hat{b}_{\xi\xi}(\partial_\xi - \partial_{\bar{\xi}})\hat{c}^\xi + (\partial_\xi - \partial_{\bar{\xi}})\hat{b}_{\xi\xi}\hat{c}^\xi: \tag{25}$$

where : : denotes a normal ordering defined later. Since the operator $\hat{b}_{\xi\xi}$ and $\hat{c}^\xi$ are defined on boundaries, one can use only the tangent derivative $\partial_\xi - \partial_{\bar{\xi}}$. The additional derivative $\partial_{\bar{\xi}}$ vanishes if one replace these boundary operators by the Fock operators mentioned before.

Owing to the prefactor C, the classical energy-momentum tensor can be rewritten as

$$\hat{T}^{cl}_{\xi\xi}e^{-S_{cl}} = \lim_{\eta\to\xi}\Big[2\hat{b}_{\xi\xi}(\partial_\eta - \partial_{\bar{\eta}})\hat{c}^\eta + (\partial_\xi - \partial_{\bar{\xi}})\hat{b}_{\xi\xi}\hat{c}^\eta$$

$$-2\left\{\hat{b}_{\xi\xi}\,,\,(\partial_\eta-\partial_{\bar\eta})c^\eta_{cl}\right\}-\left\{(\partial_\xi-\partial_{\bar\xi})\hat{b}_{\xi\xi}\,,\,c^\eta_{cl}\right\}\Big]e^{-S_{cl}} \tag{26}$$

where we use $\tilde{F}$ in both c^η_{cl} and S_{cl}. Choosing the convenient coordinate z defined before, we can derive the following boundary equations

$$\begin{aligned}
\partial_w\tilde{F}^{\bar w}{}_{zz}&=\partial_{\bar w}\tilde{F}^{w}{}_{zz}=0\\
\partial_{\bar z}\tilde{F}^{\bar w}{}_{zz}&=\partial_z\tilde{F}^{w}{}_{\bar z\bar z}=0\\
\partial_{\bar w}\tilde{F}^{w}{}_{zz}&=\partial_{\bar z}\tilde{F}^{w}{}_{zz}=0\\
\partial_w\tilde{F}^{w}{}_{zz}&=\partial_w\tilde{F}^{\bar w}{}_{zz}\\
\partial_z\tilde{F}^{w}{}_{zz}&=\partial_{\bar z}\tilde{F}^{w}{}_{zz}.
\end{aligned} \tag{27}$$

In the single boundary case, one can verify these equations in terms of the self conjugacy of Γ^d. Using (27), we have

$$\begin{aligned}
(\partial_z+\partial_{\bar z})\left\{\hat{b}_{zz}\,,\,c^w_{cl}\right\}&=-\partial_z\tilde{F}^w{}_{zz}\\
(\partial_w+\partial_{\bar w})\left\{\hat{b}_{zz}\,,\,c^w_{cl}\right\}&=-\partial_w\tilde{F}^w{}_{zz}.
\end{aligned} \tag{28}$$

The singular parts of (28) as $w\to z$ subtract the singularities of operator products in (26), and the normal ordered operator $\hat{T}_{\xi\xi}$ is derived. Hence the classical energy-momentum tensor can be written as

$$T^{cl}_{\xi\xi}e^{-S_{cl}}=\left[\hat{T}_{\xi\xi}-\langle T^{qu}_{\xi\xi}\rangle\Big|_{\gamma\neq 1}\right]e^{-S_{cl}}. \tag{29}$$

We can obtain the same result as (29) for the string coordinates by using the equation: $\left[\hat{P}_z,\partial_w X^{cl}\right]=-2\partial_z\partial_w G$. Collecting these results (eqs. 15, 18, 19 and 29), we conclude that

$$\hat{\delta}\Phi=\left[\hat{\delta}C+C\int_{\partial M}\frac{d\sigma}{2\pi}\delta^a(\hat{T}_{na}+\alpha)\right]e^{-S_{cl}}e^{-S_0}. \tag{30}$$

The arguments deriving (24) are applicable to the calculation of $\hat{\delta}C$, and we have

$$\hat{\delta}\int_{\partial M}\frac{d\sigma}{2\pi}\kappa^2\phi_{nn}c^n=\int_{\partial M}\frac{d\sigma}{2\pi}\left[\hat{\delta}_A(\kappa^2\phi_{nn})c^n+\hat{\delta}_A(\kappa^2\phi_{nt})c^t\right]$$

$$= [\int_{\partial M} \frac{d\sigma}{2\pi} \delta^a \hat{T}_{na} , \int_{\partial M} \frac{d\sigma}{2\pi} \phi_{nn} c^n]. \tag{31}$$

Here we use the holomorphic extension of c^n induced from $\tilde{F}$. Finally the Ohrndorf's result is confirmed as

$$\hat{\delta}\Phi = 2\, Re \left[\int_{\partial M} \frac{d\xi}{2\pi i} \delta^\xi (\hat{T}_{\xi\xi} + \alpha) \right] \Phi. \tag{32}$$

The boundary variations are generated by the Virasoro operator defined on the boundaries. As described in (12), the boundary operators $\hat{P}_\xi$, $\hat{c}^\xi$ and $\hat{b}_{\xi\xi}$ are identified with the Fock operators when they act on Φ. Consequently the Virasoro operator $\hat{T}_{\xi\xi}$ can be also reduced to an ordinary operator defined by the normal ordered product of the Fock operators. The net difference between them only appears in the ground state Φ.

4. Discussion

Let us define a *cap state* by the functional integral over a cap (disk). Convolution of Φ and r cap states should be reduced to the integral over the compact Riemann surface M^c (Fig. 4). This is a special case of the sewing problem for closed string amplitudes discussed in Ref. 16. Inserting the operator

$$exp \left[2Re \int_{\partial M} \frac{d\xi}{2\pi i} \delta^\xi (\hat{T}_{\xi\xi} + \alpha) \right] \tag{33}$$

among Φ and r cap states, one can vary the moduli of M^c. Any δ^n on S^1 has a holomorphic extension δ^ξ_{cap} over the cap, hence $\delta^\xi_{cap}(\hat{T}_{\xi\xi} + \alpha)$ vanishes on the cap state. Namely the boundary deformation δ^n disappears in (33), and there remains boundary reparametrization δ^t. Hence we can find that the Teichmüller deformations are generated by the discontinuities of boundary parameters between M and caps. Since the moduli space can be divided into the deformation orbits, the future problem is to obtain their representatives and weights of the discrete topological sum.

As was shown in Ref. 17, the covariant closed string field theory based on flat propagators and a geometrical vertex is hardly successful. This suggests us that the on-shell description is not satisfactory for the interacting strings. Therefore we have considered an off-shell extension of string states beyond the perturbation expansion. Especially boundary evolutions are generated by a number of *times*. It may be possible to build the covariant closed string field theory inspired by these off-shell states. More conservative purpose of our investigation may be give an inherent measure of UGM in the moduli space of the Polyakov path integral[18].

The boundary rametrizations are used to fix the string field theory gauge in Ref. 19. But the gauge fixing on arbitrary Riemann surface is unknown. As another aspect, the boundary reparametrizations generate a kind of rotation[9,11] among the Fock spaces. This fact possibly may lead us to a direct investigation of the moduli in the operator level.

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FIGURE CAPTIONS

1. The Schottky double of M.

2. ξ coordinate near a boundary which is parametrized by $\sigma \in [0, 2\pi] \cong S^1$.

3. Boundary variation δ^ξ.

4. Compact Riemann surface constructed by sewing M and r caps together.

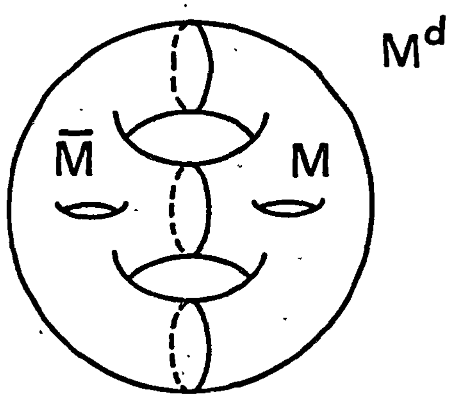

Fig. 1

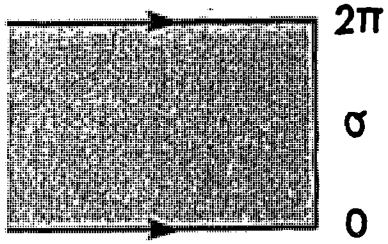

Fig. 2

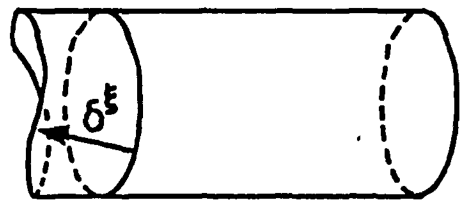

Fig. 3

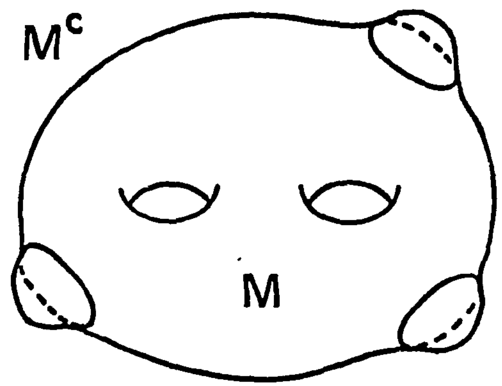

Fig. 4

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BRS FORMULATION OF BOSONIC STRING THEORY AND THE PROPERTIES OF NON-CRITICAL STRINGS†

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ABSTRACT

We review our recent study[1] of a construction of covariant tensorial (BRS, energy-momentum, and ghost number) operators for bosonic string theory in the conformal approach. By incorporating the 2-dimensional world-sheet metric freedom explicitly in the theory, it is shown that these covariances are maintained for an arbitrary space-time dimension D. After this construction we examine the covariant quantization of the metric freedom. We find an interesting solution at $D < 26$, which maintains the above covariance and also the closure of the BRS symmetry. Using a string field theoretical technique, we investigate the physical spectrum of this non-critical string[2]. We find that the 2-dimensional gauge symmetry realizes itself as the Higgs mechanism in D-dimensional space-time. We discuss briefly the implications of our results on the quantization of interacting non-critical strings.

I. MOTIVATION

In the first quantized string theory, the BRS formulation is very useful to clarify the structures of the theory in a covariant manner[3]. It is usually assumed that the condition for the critical dimension is recovered by requiring the nilpotency of the BRS charge. On the other hand, since the string action has the Weyl symmetry, the 2-dimensional metric formally disappears from the action by choosing the conformal gauge. Thus one may use the usual conformal field theoretical technique on the flat world surface at least locally[4]. However, the problem is not so simple. In the

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papers of Witten[5] and also Friedan, Martinec and Shenker[4] , they pointed out that the BRS current requires an anomalous term as

$$J_z = J_z^{(0)} + \frac{3}{4\pi}\partial_z^2 c^z \tag{1.1}$$

to maintain the covariance under the conformal transformation, where $J_z^{(0)}$ is the naive definition of the BRS current and c^z is the ghost field. This extra term is added by hand and does not come from the first principle. Moreover, this current is covariant only at $D = 26$. In the first part of my talk, I review our basic logic to clarify the origin of this extra term in (1.1). Here the BRS anomaly in the covariant path integral approach[3] plays an essential role. As a by-product, we obtain an expression for the BRS current in the conformal approach which is covariant at arbitrary D. We apply this same logic to the energy-momentum tensor and the ghost number current. These tensors include 2-dimensional metric dependences explicitly and they exhibit nice conformal properties. However this conformal symmetry is not yet in the operator level. As is well known, the quantum string theory at $D \neq 26$ contains the Weyl freedom as a dynamical variable[6] . Thus in the second part of my talk, we propose a way to quantize the metric. There by performing the renormalization due to an anomaly which arises from the metric sector to one-loop order, we obtain a partition function for a non-critical string at $D < 26$. On the basis of this formulation it is shown that for any $D < 26$ the above tensors are primary fields and, especially, the BRS charge is nilpotent. Our result thus suggests the possibility of a consistent quantization of non-critical strings. Since the nilpotency of the BRS charge is one of the main bases of string field theory, it is natural to examine our BRS charge for the construction of a field theory at $D < 26$. Thus in the third part of my talk, I show the physical spectrum of first exited states of a free string via the string field theoretical technique. It is shown that the general coordinate symmetry in 2-dimensions realizes itself as the Higgs mechanism in D-dimensional space-time. In the last section, I summarize my talk and comment on further remaining problems. For the details of calculations, the interested readers are referred to Refs.[1, 2].

2. ORIGIN OF THE ANOMALOUS TERM[1]

A. Derivation of the anomaly in the BRS current

In the following, we mainly consider the path integral quantization. In this approach the choice of the integration measure is very important because it determines which symmetries are imposed in the quantization procedure. As is well known, string theory[6] has two gauge freedoms, namely, the general coordinate invariance and the Weyl invariance. From the field theoretical point of view, the bosonic string action is free from the gravitational anomaly, and thus it is natural to use the general coordinate invariant measure. The investigation along this line has been done in Ref.[3]. There it was pointed out that the covariant BRS current j^α has an anomaly

$$\partial_\alpha < j^\alpha >= \Big(\frac{D+2}{48\pi}\Big)\partial_\alpha(c^\alpha\sqrt{g}R) + \Big(\frac{D-26}{24\pi}\Big)\omega\sqrt{g}(R+\mu), \tag{2.1}$$

where ω is an extra ghost associated with the Weyl gauge fixing and it is expressed as $\omega = (-1/2)\partial_\alpha(c^\alpha\sqrt{g})/\sqrt{g}$ by means of the equation of motion, and μ is a regularization dependent c-number. The first term of the right hand side in (2.1) survives even at $D = 26$; however, note that the quantity in the bracket has no normal component at the boundary of the world-sheet and thus it does not break the global conservation of the BRS charge. The second term in (2.1) corresponds to the conformal anomaly and thus it vanishes at $D = 26$. This equation (2.1) is our starting point, that is to say, in the following we show that the first term in (2.1) corresponds to the anomalous term of the BRS current in (1.1) in the conformal approach.

In terms of the conformal notation, the action and the measure in Ref.[3] are expressed as

$$S = \int[-\partial_z X^\mu \partial_{\bar{z}} X^\mu + b_{zz}\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}]d^2x, \tag{2.2}$$

and

$$d\mu = \mathcal{D}(\sqrt{\rho}X^\mu)\mathcal{D}\Big(\frac{1}{\sqrt{\rho}}b_{zz}\Big)\mathcal{D}(\rho c^z)\mathcal{D}\Big(\frac{1}{\sqrt{\rho}}b_{\bar{z}\bar{z}}\Big)\mathcal{D}(\rho c^{\bar{z}}), \tag{2.3}$$

where $\rho = g_{z\bar{z}}$. As noted above, this measure is designed to preserve the general

coordinate invariance. The covariant BRS current j_z in (2.1) is written as

$$j_z = -c^z\partial_z X^\mu \partial_z X^\mu + c^z b_{zz}\partial_z c^z - c^{\bar{z}} b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}} \,. \tag{2.4}$$

The last term in (2.4) which contains $c^{\bar{z}}$ can be shifted to the right-hand side of (2.1) by using an anomalous relation. Finally we have[1]

$$\langle J_z\rangle = \langle j_z - (10/48\pi)c_z\sqrt{g}R\rangle \,, \tag{2.5}$$

with

$$J_z \equiv -c^z\partial_z X^\mu \partial_z X^\mu + c^z b_{zz}\partial_z c^z \,. \tag{2.6}$$

Note that the second term of the right hand side of (2.5) is a covariant vector and thus J_z is also a covariant vector. In the course of this calculation, we set the reguralization dependent term, i.e., μ in (2.1), to be zero. By combining (2.1) and (2.5), we have the fundamental BRS identity as

$$\partial_{\bar{z}} < J_z > = \partial_{\bar{z}}\Big\langle\Big\{-\frac{3}{4\pi}\partial_z(c^z\partial_z \ln\rho) + \Big(\frac{26-D}{24\pi}\Big)c^z[-\frac{1}{2}(\partial_z\ln\rho)^2 + \partial_z^2\ln\rho]\Big\}\Big\rangle, \tag{2.7}$$

where we have used the relation $\sqrt{g}R = -2\partial_z\partial_{\bar{z}}\ln\rho$ in the conformal gauge and the safe equation of motion $\partial_{\bar{z}}c^z = 0$. This identity indicates the breakdown of the holomorphic property of J_z which is expected naively from the action (2.2); this anomaly in (2.7) can be regarded as an essential consequence of our general coordinate invariant quantization.

B. Property of J_z

Now we return to the usual conformal approach[4] . From the path-integral point of view, the integration measure for the usual conformal approach is written as

$$d\mu = \mathcal{D}X^\mu \mathcal{D}b_{zz}\mathcal{D}c^z\mathcal{D}b_{\bar{z}\bar{z}}\mathcal{D}c^{\bar{z}}, \tag{2.8}$$

without any ρ dependence, and the action is the same as in (2.2). These correspond to the use of free field correlation functions in the usual conformal approach; in this

approach the naive BRS current is defined as

$$J_z^{(0)} \equiv -c^z \partial_z X^\mu \partial_z X^\mu + c^z b_{zz} \partial_z c^z . \tag{2.9}$$

This expression (2.9) has the same appearance as the current in (2.6) and it is classically a covariant vector. However their meanings are different because the integration variables are not the same. Note that the current (2.9) is holomorphic

$$\partial_{\bar{z}} < J_z^{(0)} >= 0 \tag{2.10}$$

which can be derived by setting $\rho = 1$ in (2.7). However this current (2.9) is not a covariant tensor in a quantum sense because the measure (2.8) is <u>not</u> general coordinate invariant. To obtain a covariant BRS current we can make use of the BRS anomaly identity (2.7). Recall that the identity (2.7) is a result of the (background) covariant quantization[3].

Suppose that in (2.7) we attempt to go from the covariant measure (2.3) to the naive one (2.8). Since the right hand side has the same value in both measures, we must have the same identity as (2.7) for a covariant BRS current in the conformal approach. Thus we obtain the covariant BRS current in the conformal approach as

$$\begin{aligned} J_z = & - c^z \partial_z X^\mu \partial_z X^\mu + c^z b_{zz} \partial_z c^z - \frac{3}{4\pi} \partial_z (c^z \partial_z \ln \rho) \\ & + \frac{26 - D}{24\pi} c^z [-\frac{1}{2} (\partial_z \ln \rho)^2 + \partial_z^2 \ln \rho] . \end{aligned} \tag{2.11}$$

This current (2.11) satisfies the same identity as in (2.7) by using the measure (2.8), as a result of (2.10).

Now we identify the anomalous team in (1.1) to the third term in (2.11). To see this we consider the conformal transformation $z \longrightarrow w(z)$ (and under this $\rho(w) = |\frac{dz}{dw}|^2 \rho(z)$ classically). It is expected that under these transformations J_z behaves covariantly as

$$J_w = \frac{dz}{dw} J_z . \tag{2.12}$$

This can be readily checked. Actually, the last two extra terms in (2.11) give rise

to two extra terms

$$-\frac{3}{4\pi}\Big(\frac{dz}{dw}\Big)\partial_z\big[c^z\partial_z\ln\Big(\frac{dz}{dw}\Big)\big]+\Big(\frac{26-D}{24\pi}\Big)\frac{dw}{dz}c^z\big[-\frac{1}{2}\Big(\partial_w\ln\Big(\frac{dz}{dw}\Big)\Big)^2+\partial_w^2\ln\Big(\frac{dz}{dw}\Big)\big], \tag{2.13}$$

in addition to the covariant part; the second term in (2.13) cancels the Schwarzian derivative[7] arising from the first two holomorphic terms in (2.11). On the other hand, the anomalous term in (1.1) transforms as

$$\frac{3}{4\pi}\partial_w^2 c^w=\frac{dz}{dw}\big[\frac{3}{4\pi}\partial_z^2 c^z\big]-\frac{3}{4\pi}\Big(\frac{dz}{dw}\Big)\partial_z\big[c^z\partial_z\ln\Big(\frac{dz}{dw}\Big)\big]\,. \tag{2.14}$$

From these equations (2.13) and (2.14) we can see that the anomalous term in (1.1) was introduced to simulate the effects of the background metric at $D=26$, and clearly the covariance of (1.1) is spoiled at $D\neq 26$. By the same logic we have a covariant ghost number current in the conformal approach as[1]

$$j_z^{gh}=b_{zz}c^z+\frac{3}{4\pi}\partial_z\ln\rho\,. \tag{2.15}$$

This current reproduces the ghost number anomaly[3]

$$\partial_{\bar z}<b_{zz}c^z>=\frac{3}{4\pi}\partial_z\partial_{\bar z}\ln\rho\,. \tag{2.16}$$

As a covariant energy momentum tensor, we can use the covariant expression $\delta b_{zz}=T_{zz}$, where δ means the BRS transformation. The result is

$$T_{zz}=\partial_z X^\mu\partial_z X^\mu-b_{zz}\partial_z c^z-\partial_z(b_{zz}c^z)-\Big(\frac{26-D}{24\pi}\Big)\Big[-\frac{1}{2}(\partial_z\ln\rho)^2+\partial_z^2\ln\rho\Big]\,. \tag{2.17}$$

The covariance of these tensors, (2.15) and (2.17), for any D can be checked easily. However this covariance is not in an operator level yet because in the above discussions we varied the metric ρ by hand. This fact motivates us to quantize the metric.

3. QUANTIZATION OF THE METRIC IN THE CONFORMAL GAUGE[1]

A. Choice of path integral measure

In the bosonic string theory at $D \neq 26$, the Weyl mode of the metric does not decouple from the theory[6]. For the quantization of metric in the path integral approach, we have to specify an integration measure. Our principle is the general coordinate invariance. However a way of ensuring this type of measure (or regularizations) is not clear at this moment. Here we instead use the background invariant measure for the metric. In this 'background' formulation, after splitting the metric

$$g_{z\bar{z}} \equiv \rho + \varphi, \tag{3.1}$$

we regard ρ as a background metric and φ as a field living in the metric ρ. In this notation, the measure which is invariant under the general coordinate transformation with respect to the background metric ρ is

$$d\mu \equiv \mathcal{D}\Big[\frac{1}{\sqrt{\rho}}\varphi\Big]\mathcal{D}\big[\sqrt{\rho+\varphi}X^{\mu}\big]\mathcal{D}\Big[\frac{b_{zz}}{\sqrt{\rho+\varphi}}\Big]\mathcal{D}\big[(\rho+\varphi)c^{z}\big]\mathcal{D}\Big[\frac{b_{\bar{z}\bar{z}}}{\sqrt{\rho+\varphi}}\Big]\mathcal{D}\big[(\rho+\varphi)c^{\bar{z}}\big], \tag{3.2}$$

where we have used the full covariant measure for matter and ghost fields. This is our starting point for the quantization of metric.

B. Renormalization of the coupling constant in the Liouville action

One of the advantages of the conformal approach[4] is that one can use free field correlation functions. Thus we extract the metric dependence from the general coordinate invariant measure (2.3) (or (3.2)) to obtain the naive measure (2.8). As is well known[6], after this procedure, we have the Liouville action as a phase factor in the path integral

$$S_L = -\Big(\frac{26-D}{48\pi}\Big)\int \partial_z \ln(\rho+\varphi)\partial_{\bar{z}} \ln(\rho+\varphi)d^2x, \tag{3.3}$$

which arises from the matter and ghost sectors in (3.2). However this action is highly non-linear in terms of the integration variable $\varphi/\sqrt{\rho}$ in (3.2). This situation

makes the quantization of the Liouville action non-trivial (even in the case of a vanishing mass term as in (3.3)). In general, if anomalies are absent, one can choose the integration variable freely and thus the action (3.3) can be reduced to a trivial one. With this situation in mind, we extract a one-loop anomaly which arises from the metric sector in (3.2) as a renormalization of "coupling constant" in the Liouville action (3.3). In the spirit of the background field technique, the relevant term for the 1-loop anomaly is given by

$$\int \mathcal{D}\widetilde{\varphi} \exp\Big[-\frac{26-D}{48\pi}\int \partial_z\Big(\frac{\widetilde{\varphi}}{\sqrt{\rho}}\Big)\partial_{\bar{z}}\Big(\frac{\widetilde{\varphi}}{\sqrt{\rho}}\Big)d^2x\Big], \tag{3.4}$$

where we have set $\widetilde{\varphi} \equiv \varphi/\sqrt{\rho}$. This integration gives precisely the same contribution to the Liouville action for ρ as one of X^μ. Thus the bare coupling constant $(26-D)/4\pi$ is renormalized to $(25-D)/4\pi$ up to a one-loop order. From these considerations, we propose a partition function for non-critical strings

$$\int d\mu \exp\Big[\int (-\partial_z X^\mu \partial_{\bar{z}} X^\mu + b_{zz}\partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}} - \kappa\partial_z\sigma\partial_{\bar{z}}\sigma)d^2x\Big], \tag{3.5}$$

where

$$\kappa \equiv \frac{25-D}{48\pi}, \tag{3.6}$$

and

$$d\mu \equiv \mathcal{D}\sigma\mathcal{D}X^\mu\mathcal{D}b_{zz}\mathcal{D}c^z\mathcal{D}b_{\bar{z}\bar{z}}\mathcal{D}c^{\bar{z}}, \tag{3.7}$$

with $\sigma = \ln\rho$. This generates a renormalized effective correlation function for the metric. Our argument for the renormalized κ is somewhat subtle and it is valid only to one-loop order, but as will be seen, this value works; from the recently developed[9,10] 2-dimensional quantum gravity point of view, this value of κ may be exact, if the non-renormalization theorem of the one-loop anomaly is valid.

C. The modified T_{zz}, J_z and j_z^{gh}

After the renormalization $D \to D+1$, the covariant tensors are replaced by

$$\begin{aligned}
J_z &\equiv -c^z\partial_z X^\mu \partial_z X^\mu + c^z b_{zz}\partial_z c^z - \frac{3}{4\pi}\partial_z(c^z\partial_z\sigma) - \kappa c^z[(\partial_z\sigma)^2 - 2\partial_z^2\sigma], \\
T_{zz} &\equiv \partial_z X^\mu \partial_z X^\mu - b_{zz}\partial_z c^z - \partial_z(b_{zz}c^z) + \kappa[(\partial_z\sigma)^2 - 2\partial_z^2\sigma], \\
j_z^{gh} &\equiv b_{zz}c^z + \frac{3}{4\pi}\partial_z\sigma .
\end{aligned} \tag{3.8}$$

The correlation functions derived from (3.5) are

$$\begin{aligned}
&< X^\mu(z)X^\nu(w) >= -\frac{\delta^{\mu\nu}}{4\pi}\ln(z-w)(\bar{z}-\bar{w}), \\
&< \sigma(z)\sigma(w) >= -\frac{1}{4\pi}\left(\frac{1}{\kappa}\right)\ln(z-w)(\bar{z}-\bar{w}), \\
&< b_{zz}c^w >=< c^z b_{ww} >= -\frac{1}{2\pi}\frac{1}{z-w}.
\end{aligned} \tag{3.9}$$

4. CHECK OF CONFORMAL AND BRS PROPERTIES[1].

By using correlation functions (3.9), one can confirm that J_z in (3.8) generates the BRS transformations[3] (the BRS charge is given by $Q = (-i)\oint J_z dz$),

$$\begin{aligned}
\delta X^\mu &= c^z\partial_z X^\mu, \\
\delta c^z &= c^z\partial_z c^z, \\
\delta b_{zz} &= T_{zz}, \\
\delta\sigma &= c^z\partial_z\sigma + (\partial_z c^z),
\end{aligned} \tag{4.1}$$

as is classically expected. One can also confirm the conformal properties

$$\begin{aligned}
T_{zz}T_{ww} &\sim \left(\frac{-1}{2\pi}\right)\left[\frac{2}{(z-w)^2}T_{ww} + \frac{1}{z-w}\partial_w T_{ww}\right], \\
T_{zz}J_w &\sim \left(\frac{-1}{2\pi}\right)\left[\frac{1}{(z-w)^2}J_w + \frac{1}{z-w}\partial_w J_w\right], \\
T_{zz}j_w^{gh} &\sim \left(\frac{-1}{2\pi}\right)\left[\frac{1}{(z-w)^2}j_w^{gh} + \frac{1}{z-w}\partial_w j_w^{gh}\right],
\end{aligned} \tag{4.2}$$

for any $D < 26$. It is remarkable that the possible anomalous terms precisely cancel in eqs.(4.2) independently of the value of D provided that κ is chosen as in (3.6).

One can also calculate

$$J_z J_w \sim \frac{1}{4\pi^2}\Big(\frac{9}{16\pi\kappa}-1\Big)\partial_w\Big[\frac{1}{(z-w)^2}\partial_w c^w c^w\Big],$$
$$J_z j^{gh}_w \sim \Big(\frac{-1}{4\pi^2}\Big)\Big(\frac{9}{16\pi\kappa}-1\Big)\partial_w\Big[\frac{1}{(z-w)^2}c^w\Big]+\Big(\frac{-1}{2\pi}\Big)\frac{1}{z-w}J_w, \tag{4.3}$$

for any $D<25$; (4.3) shows that the BRS current J_z anti-commutes with the BRS charge Q at any $D<25$ (and thus Q is of course nilpotent).

When one rescales σ as $\sqrt{\kappa}\sigma \equiv \widetilde{\sigma}$ and lets $\kappa \to 0$, one essentially recovers the conventional theory at $D=26$ with $\widetilde{\sigma}$ replacing one of 26 variables X^μ. However, the crucial term $(-3/4\pi)\partial_z(c^z\partial_z\widetilde{\sigma})/\sqrt{\kappa}$ in J_z in (3.8) diverges in this limit.

5. FREE STRING FIELD THEORY AT $D<26$[2]

As was mentioned above, our BRS charge is nilpotent at any $D<26$. Thus it is natural to consider a construction of sting field theory[6] as a simple application. In this talk we consider the free open string case only. Nevertheless, we can see an interesting role of the Weyl freedom.

The free string classical action is

$$S=\frac{1}{2}\int A*QA, \tag{5.1}$$

or in terms of the state vectors of the first quantization

$$S=\frac{1}{2}\langle V_2|\,\widetilde{|A\rangle}\otimes Q\,|A\rangle, \tag{5.2}$$

with a vertex state

$$\langle V_2| \equiv \Big\langle -\frac{1}{2}\Big| \otimes \widetilde{\Big\langle -\frac{1}{2}\Big|}(c_0+\tilde{c}_0)\exp\{-\sum_{n=1}^{\infty}(\frac{d_n\tilde{d}_n}{n}+\frac{a^\mu_n\tilde{a}^\mu_n}{n})(-1)^n + \sum_{n=1}^{\infty}(b_n\tilde{c}_n+\tilde{b}_n c_n)(-1)^n\}, \tag{5.3}$$

and the string field is represented as

$$|A\rangle = \phi(x)\left|-\frac{1}{2}\right\rangle + A_\mu(x)a^\mu_{-1}\left|-\frac{1}{2}\right\rangle + \theta(x)d_{-1}\left|-\frac{1}{2}\right\rangle + B(x)b_{-1}c_0\left|-\frac{1}{2}\right\rangle + \cdots . \tag{5.4}$$

In this notation, x means the position of the center of mass of the string and a^μ_n, d_n, c_n and b_n are the mode expansions of X^μ, σ, c^z and b_{zz}, respectively. One interesting feature in our scheme is that the ghost number is linked to the Weyl mode excitation as

$$\begin{aligned} G &\equiv (-i)\oint dz j^{gh}_z \\ &= -\sum_{n\geq 2} b_{-n}c_n + \sum_{n\leq 1} c_n b_{-n} + \frac{3}{2}\frac{i}{\sqrt{4\pi\kappa}}\hat{d}_0, \end{aligned} \tag{5.5}$$

where $\hat{d}_0$ is the zero mode of σ defined on the plane (see Ref.[2]). The vacuum $\left|-\frac{1}{2}\right\rangle$ corresponds the vacuum on the strip (or cylinder), which can be seen by noting the existence of one conformal killing vector $\left\langle-\frac{1}{2}\right| c_0 \left|-\frac{1}{2}\right\rangle = 1$.

By using our BRS charge, we obtain the action $S = -\int \mathcal{L} d^D x$,

$$\begin{aligned} \mathcal{L} =& \frac{1}{2}\phi(\Box + 1 - 2\pi\kappa)\phi + \frac{1}{2}A_\mu \Box A^\mu - \frac{2\pi\kappa}{2}(A_\mu + \frac{1}{\sqrt{2\pi\kappa}}\partial_\mu\theta)^2 \\ & - \sqrt{2}i(B - \sqrt{\pi\kappa}i\theta)\partial_\mu A^\mu + (B - \sqrt{\pi\kappa}i\theta)^2 + \cdots . \end{aligned} \tag{5.6}$$

The first exited states become massive and this mass shift is consistent with the Casimir energy of the open string at D.

The gauge symmetry, which is a consequence of the nilpotency of the BRS charge in this system, is

$$|A\rangle \to |A\rangle + Q\,|\Lambda\rangle\,, \tag{5.7}$$

or in terms of component fields

$$\begin{aligned} \phi(x) &\to \phi(x), \\ A_\mu(x) &\to A_\mu(x) + \partial_\mu\lambda(x), \\ \theta(x) &\to \theta(x) - m\lambda(x), \\ B(x) &\to B(x) + (i/\sqrt{2})(\Box - m^2)\lambda(x), \end{aligned} \tag{5.8}$$

where we have set $|\Lambda\rangle \equiv (i/\sqrt{2})\lambda(x)b_{-1}\left|-\frac{1}{2}\right\rangle$. One can see that the world-sheet

general coordinate invariance, which remains in this system, realizes itself as the Higgs mechanism and the Weyl freedom excitation $\theta(x)$ becomes an unphysical Higgs scalar. Thus in the free string level our analysis suggests that the non-critical string at $D < 26$ is free from the negative norm states[3] .

6. CONCLUSION

In the course of clarifying the origin of the anomalous term of the BRS current in the conformal approach[4] , we found the general construction of covariant tensors (BRS, energy-momentum and ghost number) in such an approach. At the classical level, these covariances are maintained for an arbitrary space-time dimension D. We then examined the quantization of the metric variable. Our scheme allowed a simplified treatment of the Liouville action. By using the string field theoretical machinery, we found the interesting properties of non-critical bosonic strings at $D < 26$.

One interesting remaining problem is to clarify the connection of our treatment of the Liouville action with the recently developed[9,10] (presumably exact) 2-dimensional quantum gravity where the authors assume the existence of the full general coordinate invariant measure (or regularization) contrary to ours. Our treatment may be regarded as a flat background limit of these treatments. Another interesting problem is an extension of our scheme to fermionic string theory.

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ＢＣ系の bosonization について

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§Ｉ．導入

ＢＣ系とは超弦理論のＢＲＳＴ量子化において現れる超共形ゴーストを範とするfirst order lagrangian に従う系である。それは超共形ウェイト λ - 1/2 の Grassmann even な超場 Ｂと 超共形ウェイト 1 - λの Grassmann odd な超場 Ｃにより構成される。[1]

$$B(z,\theta) = \beta + \theta b , \quad C(z,\theta) = c + \theta\gamma \qquad (1.1)$$

成分場で見るとＢＣ系はフェルミオン・ゴースト対（b，c）とボゾン・ゴースト対（β，γ）から成り、これらが超対称性により統合されている。

超共形理論における超共形ゴーストは λ ＝ 2 の場合である。以下においてはλは整数、或は半整数とする。

（b，c）系のボゾン化に対しては既にかなり深い理解が得られている。その Bose-Fermi同等性は相関関数、Chiral determinant等の解析を通して、一般リーマン面上で確立されており、その結果はリーマン面の代数幾何学と深く関わっている。[2,3]

（β，γ）系のボゾン化（正確に云えばスカラー化と云うべきであろう）は Friedan, Martinec, Shenker (FMS)により超弦理論へ応用において導入された。[4] その中でボゾン化は時空間フェルミオンの散乱を記述する頂点演算子の構成、及びその際、必要となるピクチャーの変更と呼ばれる操作において重要な役割を果たす。（β，γ）系はその統計性のために（同等でない Fock 表現が無限個存在する等）（b，c）系ほど扱い易くなく、そのボゾン化は（b，c）系と比較して十分調べられているとは云えない。[5]

さて、以上のように従来のＢＣ系のボゾン化では、二つのカイラル対で別々に考えられてきた。然るに本来ＢＣ系は（1,1）のように超対称な形にまとめられるものである。超弦理論の定式化には超リーマン面（SRS）の族を考える事が必要とされる。従って一般には、局所座標を張り合わせる超共形変換に odd な超モジュライ変数を含む non-split な超リーマン面を考える事になる。[6] このような non-split な超リーマン面上の系を考える際には明白な超対称性という事が重要となる。

従来のボゾン化ではスカラー場は超共形変換のもとで非線形に変換し、超対称性に関して非常に不透明なものとなっている。この意味に於て超対称性が明白に保たれる

ような形式を考える事は自然であろう。

以下の話でこのようなボゾン化 — 簡単に超ボゾン化と呼ぶ事にする — が存在する事を述べる。[7] 超ボゾン化では背景チャージと結合した二つの共役なスカラー超場により、頂点演算子構成を通してＢＣ系が再構成される。超ボゾン化はフェルミオン・ゴーストのボゾン化の超対称化と云うよりも、むしろボゾン・ゴーストのスカラー化の超対称化と云うべきものとなっている。いずれにしても異なる成分場を持つ二つの超共形理論の間に或る種の同等性が在る事を示唆しており興味深い。

§Ⅱ．ＢＣ系の諸性質

ＢＣ系の超共形変換に対する諸性質をまとめておく。

作用は次式で与えられる。

$$S = \frac{1}{\pi}\int dz\,d\theta\,d\bar{\theta}\ \ B\bar{D}C \tag{2.1}$$

Ｂ，Ｃは既に述べたようにウェイト λ-1/2 及びウェイト 1 -λ の超場である。

量子化は次の基本的な演算子積によって表現できる。

$$B(z_1,\theta_1)C(z_2,\theta_2) \sim 1 \,/\, z_{12} \tag{2.2}$$

但し、 $\theta_{12} = \theta_1 - \theta_2$,　 $z_{12} = z_1 - z_2 - \theta_1\theta_2$

系の stress energy tensor は

$$T(z,\theta) = (1-\lambda)C(\partial B) + 1/2\,(DC)(DB) + (1/2-\lambda)(\partial C)B \tag{2.3}$$

で与えられる。この系の持つ超ヴィラソロ代数は次の演算子積から読み取ることができる。

$$T(z_1,\theta_1)T(z_2,\theta_2) \sim \frac{6-8\lambda}{4\,z_{12}^{3}} + \left(\frac{3/2\,\theta_{12}}{z_{12}^{2}} + \frac{1}{2z_{12}}D_2 + \frac{\theta_{12}}{z_{12}}\partial_2\right)T(2) \tag{2.4}$$

central charge は　$c = 3/2 \times (6-8\lambda) = 9 - 12\lambda$ である。

次にゴースト数カレントを定義する。これはウェイト 1/2 の超場である。

$$I(z,\theta) = -BC = -\beta c + \theta(-bc - \beta\gamma) = \psi + \theta j$$

$$j = j_{bc} + j_{\beta\gamma}\ ,\quad j_{bc} = -bc\ ,\quad j_{\beta\gamma} = -\beta\gamma\ ,\quad \psi = -\beta c \tag{2.5}$$

B，C のゴースト数はそれぞれ -1，1 と定めている。即ち

$$I(1)B(2) \sim -1 / z_{12}\ ,\quad I(1)C(2) \sim 1 / z_{12} \tag{2.6}$$

B，C のゼロモード起因するアノマリーにより、カレントＩは共変的に変換しない。つまり純粋な primary field ではない。 stress energy tensor T との演算子積は

$$T(z_1,\theta_1)I(z_2,\theta_2) \sim \frac{-1}{2\ z_{12}{}^2} + \left(\frac{1/2\,\theta_{12}}{z_{12}{}^2} + \frac{1}{2z_{12}}D_2 + \frac{\theta_{12}}{z_{12}}\partial_2\right) I(2) \tag{2.7}$$

となり、右辺第一項が異常項である。この項の分子に現れる因子を背景チャージと呼ぶ。今の場合カレントＩの背景チャージは -1 である。演算子積（2.7）の意味は後で述べる。

カレントＩの重要な性質は互いに相関を持たない事である。

$$I(z_1,\theta_1)I^{\cdot}(z_2,\theta_2) \sim O\ (z_{12}) \tag{2.8}$$

この為にＢＣ系の超ボゾン化は通常のボゾン化とは様相の異なるものとなる。

さて、Ref.[4]で指摘されているようにＢＣ系は N ＝ 2 の超共形対称性を持っている。よく知られているように N ＝ 2 の超共形代数は stress energy tensor T（z,θ）とウェイト 1 の超場 J（z,θ）により生成される。ＢＣ系に於て J は

$$J(z,\theta) = (1 - 2\lambda)B(DC) + (2\lambda - 2)(DB)C \tag{2.9}$$

により与えられる。 J は真の primary field である事に注意しておく。

Ⅲ．異常カレントの変換性と背景チャージ

背景チャージ Q の異常カレント $S(z,\theta) = \psi + \theta j$ を考えよう。

$$T(z_1,\theta_1)S(z_2,\theta_2) \sim \frac{Q}{2\ z_{12}{}^2} + \cdots \tag{3.1}$$

演算子積(3.1)は超共形変換 $(z,\theta)\ \longrightarrow (\tilde{z},\tilde{\theta})$ に対して S が

$$S(z,\theta) = (D\tilde{\theta})\tilde{S}(\tilde{z},\tilde{\theta}) + Q\ \partial\tilde{\theta}/D\tilde{\theta} \tag{3.2}$$

と変換する事を意味する。スプリットした超共形変換

$$(z,\theta) \rightarrow (\tilde{z},\tilde{\theta}) =: (\tilde{z}(z), (\partial\tilde{z})^{1/2}\theta) \tag{3.3}$$

においては超場の成分で考える事が意味を持つ。この時、変換(3.2)は次式に帰着する。

$$\begin{aligned} j(z) &= (\partial\tilde{z})\tilde{j}(\tilde{z}) + Q/2\ \partial^2\tilde{z}/\partial\tilde{z} \\ \psi(z) &= (\partial\tilde{z})^{1/2}\psi(\tilde{z}) \end{aligned} \tag{3.4}$$

カレント S（或は j）の変換性は相関関数のチャージの保存に関して次のような重要な帰結をもたらす。A_i $i=1\sim N$ をリーマン面 Σ（種数 g とする）上で定義されたチャージ q_i の局所演算子とするとき $\langle A_1(x_1)\cdots A_N(x_N)\rangle \neq 0$ と成るためには

$$\Sigma\ q_i = -Q\chi/2 = Q(g-1) \tag{3.5}$$

但し、χ はオイラー数で $\chi = 2-2g$.

を満たさなければならない。

ゴースト数カレント j と j の背景チャージは

$$Q_{bc} = 1-2\lambda\ ,\quad Q_{\beta\gamma} = 2\lambda-2 \tag{3.6}$$

となる。ＢＣ系のカレントについての以上に述べた現象は軸性U(1)アノマリーの場合と全く同様にカイラル演算子 — 今の場合 $\bar{\partial}$ — に関する指数定理に関連している。Riemann - Roch の定理によれば

$$\begin{aligned} &\dim\ (\text{holo}\ (\lambda,0)\ \text{form}) - \dim\ (\text{holo}\ (1-\lambda,0)\ \text{form}) \\ &\quad = -(2\lambda-1)\chi/2 = (2\lambda-1)(g-1) \end{aligned} \tag{3.7}$$

ｂｃ系、或は$\beta\gamma$系の相関関数においては(3.7)によるゼロモードの出現のため汎関数積分の中で適当な演算子を挿入してこれらのゼロモードを吸収する必要がある。(3.5)はこの操作の結果を述べたものに他ならない。

§Ⅳ．ＢＣ系の超ボゾン化

ＢＣ系の超ボゾン化は通常のボゾン化と同じく strss energy tensor をカレントから菅原構成する事により遂行される。ＢＣ系の場合、カレントＩの(2.8)の性質のためにＩだけからは何もできない。従って次のように二つの共役なスカラー超場 $\Phi(z,\theta)=\phi+\theta\psi$ ，$\Phi^{*}(z,\theta)=\phi^{*}+\theta\psi^{*}$

$$\Phi(z_1,\theta_1)\Phi^{*}(z_2,\theta_2)\sim\log z_{12} \tag{4.1}$$

を用意して

$$I=D\Phi \qquad I^{*}=D\Phi^{*}$$
$$I(z_1,\theta_1)I^{*}(z_2,\theta_2)\sim 1/z_{12} \tag{4.2}$$

と定義するのが最も自然であろう。I^{*}の意味は後でみる。次に central charge がＢＣ系と同じく $9-12\lambda$で、かつＴＩの演算子積が(2.7)になるように stress energy tensor を菅原構成する。これは唯一に定まり次式で与えられる。

$$T(z,\theta)=1/2\{I^{*}DI+I\,DI^{*}-Q\,\partial I+\partial I^{*}\} \tag{4.3}$$
但し、　$Q=1-2\lambda$

(4.3)よりI^{*} はやはり異常カレントでその背景チャージは Q となる。$N=2$ の超共形代数の存在はこの系ではむしろ明かでありＪ(z,θ)は

$$J(z,\theta)=I\,I^{*}-Q\,DI^{*}-DI \tag{4.4}$$

で与えられる。

頂点演算子として基本的なものは

$$U_{(q,q^{*})}=e^{q\Phi^{*}}e^{q^{*}\Phi} \tag{4.5}$$

$$\text{weight} \cdots\cdots \; qq^* + 1/2\,(\,qQ - q^*\,)$$
$$\text{I charge} \cdots\cdots \; q \qquad \text{I}^*\text{charge} \cdots\cdots \; q^*$$

である。次に簡単なものとしては

$$V_{(q,q^*)} = \{\,(\,q-1\,)D\Phi^* - (\,q^*+Q\,)D\Phi\,\}\; e^{q\Phi^*}\; e^{q^*\Phi} \tag{4.6}$$
$$\text{weight} \cdots\cdots \; qq^* + 1/2\,(\,qQ - q^* + 1\,)$$

である。$V_{(q,q^*)}$が超場として定まったゴースト数（Iチャージ）を持つのは $q = 1$ の場合に限られる。

ＢとＣは上記二つの型の頂点演算子により

$$B = e^{-\Phi^*} \qquad\qquad C = -\,D\Phi\;\; e^{\Phi^*} \tag{4.7}$$

と同一視する事ができる。(4.3)、(4.4)及び(4.7)によりＢＣ系のオペレーター代数が再現される。

(4.7)より最初に導入したＩ*は謂所るピクチャー・カレントである事が分かる。

$$\frac{1}{2\pi i}\oint dz d\theta \;\; I^*(z,\theta)$$

はＢ、Ｃと可換であり、そのチャージはＢＣ系の同等でない Fock空間を特徴ずける。

§Ｖ．超ボソン化の諸性質

ＢＣ系の重要な演算子と超ボソン化における頂点演算子の対応は次の如くである。

$$\delta\,(B) = D\Phi\;\; e^{\Phi^*-\Phi} \qquad \text{weight} \cdots\; 1/2 - \lambda \tag{5.1}$$

$$\delta\,(DC) = D\Phi^*e^{\Phi-\Phi^*} \qquad\qquad (\,\lambda - 3/2\,) \tag{5.2}$$

但し、超共形場となるのは(5.2)ではなく

$$(2 - \lambda)\delta\,(DC) + (1 - \lambda)D\{C\,\delta'\,(DC)\}$$

$$= \{ D\Phi^{*} + (1 - \lambda) D\Phi \} e^{\Phi - \Phi^{*}}$$

である。さらに

$$DB\,\delta(B) = - e^{-\Phi} \qquad \text{weight} \cdots 1/2 \qquad (5.3)$$

$$\Theta(DC) = - D\Phi^{*} e^{\Phi} \qquad (0) \qquad (5.4)$$

やはり超共形場となるのは(5.4)ではなく

$$\Theta(DC) + (2 - \lambda) D\{C\,\delta(DC)\}$$

$$= - \{ D\Phi^{*} + (2 - \lambda) D\Phi \} e^{\Phi}$$

これらの表式より超ボゾン化は（β,γ）系のスカラー化において、従来のＦＭＳによる表現

$$\beta = \partial\xi\, e^{-\varphi}, \qquad \gamma = \eta\, e^{\varphi}$$

$$\xi = \partial\Theta(\beta), \qquad \eta = \partial\Theta(\gamma) \qquad (5.5)$$

と異なり（β,γ）系としてむしろ

$$\beta = \eta\, e^{-\varphi}, \qquad \gamma = -\partial\xi\, e^{\varphi}$$

$$\xi = -\partial\Theta(\gamma), \qquad \eta = \partial\Theta(\beta) \qquad (5.6)$$

を採る事に注意する必要がある。ξのゼロモードによる（η,ξ）系 のオペレーター代数の拡大は(5.5)と(5.6)で異なる。最近 Horowitz 達は(5.5)により拡大された表現空間を相（I）、(5.6)によるものを相（II）と呼び、$\lambda = 2$ の場合に BRST 不変性及び BRST コホモロジーに関する二つの相違を論じている。[8]

次に相関関数における頂点演算子の挿入規則について考えよう。ＩとＩ*の背景チャージはそれぞれ -1 と $1 - 2\lambda$ であるから素朴には§Ⅲの議論より

$$\# e^{-\Phi^{*}} = g - 1, \qquad \# e^{-\Phi} = (2\lambda - 1)(g - 1)$$

と考えられる。しかし Ref.[9]に指摘されているようにカレント j — 言い替えればピクチャー・カレント $j_{pic} = -j_{\beta\gamma} \pm j_{\xi\eta}$ （＋は相(I) −は相(II)） — は相関関数の中でオペレーター代数には現れぬ g 個の「非物理的」な極を持つ。従って正しい頂点演算子の挿入は

$$\# e^{-\Phi^*} = g - 1 \quad , \qquad \# e^{-\Phi} = (2\lambda - 1)(g - 1) - g \qquad (5.7)$$

である。

§VI. まとめ

以上の話で我々はＢＣ系の明白に超対称性を保つボゾン化が存在することを示し、基本的なＢＣ系の演算子と頂点演算子との対応を与えた。

今後の課題としては次のようなものが考えられる。

(i) ＢＣ系との同等性を超リーマン面上において確立するためには超場 Φ, Φ^* に基ずく計算ができなければならない。ラグランジアンは形式的には

$$L = \frac{1}{2\pi}\{ D\Phi^* D\Phi + D\Phi D\Phi^* + 1/4\ E\ R_s(Q\Phi - \Phi^*) \}$$

但し，Rs は superscalar curvature

と書けるが、これをきちんと超リーマン面上で定義して、ゼロモード即ちソリトン・セクターについての足し上げに関する計算規則を定めねばならない。通常のボゾン化においてソリトン・セクターの足し上げは Poisson の和公式を通してテータ函数の絶対値の自乗のスピン構造についての和となる事を想起すると、これは興味深い。ここにはテータ函数の拡張の問題と、超リーマン面においてスピン構造を変える操作が意味を持つように定義できるのかという問題が関わっている。

(ii) 通常のボゾン化が KP hierarchy において果たす役割を考えるならば、この超対称なボゾン化と super KP hierarchy との関連を期待することは全く自然である。[10] 超ボゾン化は（b，c）系の性質より（β，γ）系の性質に大きく依存しており、通常のボゾン化に於けるような数学的広がりと深さを持つかどうか今後の発展に期待される。

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One-Particle-Irreducible Effective Lagrangian of String Modes ⋆

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ABSTRACT

We propose a new method to obtain the effective Lagrangian from the vanishing of β-functions. Although the β-functions are not one-particle-irreducible at more than five-string interaction level, we can rewrite them in a one-particle-irreducible form by redefining the background fields. The vanishing of the β-functions is then considered as the equations of motion of the modified background fields.

⋆ This talk is based on ref. [0].

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§1 INTRODUCTION

String propagation in the presence of background fields is described by a non-linear sigma model on the world sheet.[1-5] The consistency requires that the β-functions should vanish. These conditions are supposed to give the correct equations of motion of the background fields. At present, there are two different methods to calculate the β-functions: one is the normal coordinate expansion (α' expansion)[1-3] and the other is the weak field expansion.[4-8] In the normal coordinate expansion it is hard to incorporate massive background fields, on the other hand, it is easy in the weak field expansion.

In the latter case some nontrivial problems arise. One of the problems is whether the effective Lagrangian of the background fields is one-particle-irreducible (1PI) or not. Up to four-tachyon interaction at the string tree level it was verified that the effective Lagrangian is 1PI.[4] In the case of five or more tachyons or in the presence of massless or other massive background fields, it is an open question whether the effective Lagrangian is 1PI or not.

This talk is based on ref. [0] which studies this problem in the context of bosonic closed string theory by using the weak background field expansion. We propose a new method to obtain the effective Lagrangian of background fields from the vanishing of the β-functions. In this method β-functions $= 0$ are rewritten by redefining the background fields so as to subtract the physical poles from the off-shell amplitudes. The physical pole means the pole at the physical mass in this paper. The resulting equations are considered as the equations of motion of the modified background fields. Thus, one obtains the effective Lagrangian. If the off-shell amplitudes appeared in the effective Lagrangian have the physical poles, the effective Lagrangian becomes 1PI.

§2 DEFINITION OF β-FUNCTION

The action of closed bosonic string in the orthonormal gauge is given by

$$
\begin{aligned}
I &= I_{\text{free}} + I_{\text{BG}}, \\
I_{\text{free}} &= \frac{1}{2\pi\alpha'}\int d^2\sigma\ \partial_+ X^\mu \partial_- X^\nu \delta_{\mu\nu}, \\
I_{\text{BG}} &= \frac{1}{2\pi\alpha'}\int d^2\sigma\ [\ \frac{\alpha'}{\varepsilon^2}\Phi(X)\ +\ \partial_+ X^\mu \partial_- X^\nu A_{\mu\nu}(X)\ +\ \cdots],
\end{aligned}
\tag{1}
$$

where $\sigma^\pm = (\sigma^1 \pm i\sigma^2)/\sqrt{2}$ are complex coordinates on the world sheet and X^μ ($\mu = 1, \cdots, D$) are string coordinates in a D-dimensional external space. D-dimensional vector indices are contracted by $\delta_{\mu\nu}$. $\Phi(X), A_{\mu\nu}(X)$ and dots denote tachyon, massless and higher massive fields, respectively. In order to regularize the ultraviolet divergences we introduce the point splitting constant parameter ε on the world sheet. The $1/\varepsilon$ is the only parameter which has the mass dimension in the two-dimensional field theory.

The definition of β-functions is given by

$$
\begin{aligned}
&-\varepsilon\frac{d}{d\varepsilon} < \exp[-I_{\text{BG}}(X)] > = < \beta(X)\exp[-I_{\text{BG}}(X)] >, \\
&\beta(X)\ =\ \frac{1}{2\pi\alpha'}\int d^2\sigma\ [\ \frac{\alpha'}{\varepsilon^2}\beta_\Phi(X)\ +\ \partial_+ X^\mu \partial_- X^\nu \beta_{\mu\nu}(X)\ +\ \cdots],
\end{aligned}
\tag{2}
$$

where X^μ are expanded around a background X_0^μ: $X^\mu = X_0^\mu + \xi^\mu$, and $<\mathcal{O}(X)>$ is defined by

$$
< \mathcal{O}(X) > \equiv \int \mathcal{D}\xi\ \mathcal{O}(X_0 + \xi)\exp[-I_{\text{free}}(\xi)]. \tag{3}
$$

The invariance under a constant conformal transformation of $<\exp[-I_{\text{BG}}(X)]>$, i.e., $\varepsilon\frac{d}{d\varepsilon} < \exp[-I_{\text{BG}}(X)] > = 0$, requires that the β-functions should vanish. One may rewrite I_{BG} as

$$
I_{\text{BG}}\ =\ \frac{1}{2\pi}\int\frac{d^D k}{(2\pi)^D}[\ \frac{1}{\varepsilon^2}\widetilde{\Phi}(k)V_\Phi(k) + \widetilde{A}_{\mu\nu}(k)V_A^{\mu\nu}(k)\ +\ \cdots], \tag{4}
$$

where

$$\Phi(X) = \int \frac{d^D k}{(2\pi)^D} \tilde{\Phi}(k) \exp(ikX), \qquad V_\Phi(k) = \int d^2\sigma \exp(ikX) \tag{5}$$

and so on. Therefore, in order to calculate the β-functions in (2), we have to study the ε-dependence of each vertex function and also the products of vertex functions.

In the case of a single vertex function there exist the divergences arising from the tadpole diagrams. The point splitting regularization for the tadpole is

$$< \xi^\mu(\sigma)\xi^\nu(\sigma) > = -\frac{\alpha'}{2}\delta^{\mu\nu} \log \varepsilon^2. \tag{6}$$

By using the relation $\exp(ik\xi) = \exp(-\frac{k^2}{2} < \xi\xi >) : \exp(ik\xi) :$ and (6), one obtains

$$\varepsilon^{\frac{\alpha'}{2}m^2} V(k) = \varepsilon^{\Delta_k} :V(k):, \qquad \Delta_k \equiv \frac{\alpha'}{2}(k^2 + m^2), \tag{7}$$

where no more contractions are taken inside the colons.

Next, let us concentrate on the products of several vertex functions. In this case the singularities in the product of N ($N \geq 2$) vertex functions are

$$\begin{aligned} &:V(k_1): :V(k_2): \cdots :V(k_N): \\ &\sim - \sum_{\text{particles}} \frac{\varepsilon^{\omega_N}}{\omega_N} S_{N+1}(k_1, \cdots, k_N, -k) :V(k): \end{aligned} \tag{8}$$

with

$$\begin{aligned} &\omega_N = \Delta_k - \sum_{i=1}^N \Delta_{k_i}, \quad k = \sum_{i=1}^N k_i, \\ &S_N(k_1, \cdots, k_N) = 2\pi < :V(k_1): \cdots :V(k_N): >, \end{aligned} \tag{9}$$

where $1/\omega_N$ is a pole arising from factorizing out the N vertices together from the world sheet and these poles correspond to tachyon, massless and higher massive particles. S_N is an N-point off-shell amplitude which becomes a correct on-shell amplitude if all the external particles are on-shell ($\Delta_{k_i} = 0$).

§3 β-FUNCTION UP TO FOUR STRING INTERACTION

Now, let us calculate the β-functions by using (7) and (8). One need not use the explicit forms of off-shell amplitudes S_N. In the case of tachyon field only, β_Φ becomes

$$\beta_\Phi = \sum_{N=1}^{\infty} \frac{1}{N!}\beta_\Phi^{(N)}, \tag{10}$$

$$\beta_\Phi^{(1)} = -(\frac{\alpha'}{2}\partial^2 + 2)\Phi,$$

$$\beta_\Phi^{(2)} = \frac{1}{2\pi} S_{\Phi^3}(\partial_1, \partial_2, -\partial_1 - \partial_2)\Phi_1\Phi_2,$$

$$\beta_\Phi^{(3)} = -(\frac{1}{2\pi})^2\Big\{S_{\Phi^4}(\partial_1, \partial_2, \partial_3, -\partial_1 - \partial_2 - \partial_3)$$

$$- \frac{3}{\alpha'\partial_3(\partial_1 + \partial_2) - 2} S_{\Phi^3}(\partial_1, \partial_2, -\partial_1 - \partial_2) S_{\Phi^3}(\partial_1 + \partial_2, \partial_3, -\partial_1 - \partial_2 - \partial_3)\Big\}\Phi_1\Phi_2\Phi_3,$$

up to four-tachyon interaction level, where ∂_i denotes a derivative on Φ_i, e.g., $\partial_2\partial_3 A_1 B_2 C_3 = A\partial B\partial C$, $(\partial_1 + \partial_2 + \partial_3)A_1B_2C_3 = \partial(ABC)$, and S_{Φ^N} denotes an N-point amplitude of tachyon fields. Up to this order, we can use $\beta_\Phi^{(1)} + O(\Phi^2) = 0$ in $\beta_\Phi^{(3)}$ if $\beta_\Phi = 0$ is required. So, $\beta_\Phi = 0$ leads to $\mathcal{E}_\Phi = 0$ with

$$\mathcal{E}_\Phi = \sum_{N=1}^{\infty} \frac{1}{N!}\mathcal{E}_\Phi^{(N)}, \tag{11}$$

$$\mathcal{E}_\Phi^{(1)} = \beta_\Phi^{(1)},$$

$$\mathcal{E}_\Phi^{(2)} = \beta_\Phi^{(2)},$$

$$\mathcal{E}_\Phi^{(3)} = -(\frac{1}{2\pi})^2\Big\{S_{\Phi^4}(\partial_1, \partial_2, \partial_3, -\partial_1 - \partial_2 - \partial_3)$$

$$+ \frac{3}{\frac{\alpha'}{2}(\partial_1 + \partial_2)^2 + 2} S_{\Phi^3}(\partial_1, \partial_2, -\partial_1 - \partial_2) S_{\Phi^3}(\partial_1 + \partial_2, \partial_3, -\partial_1 - \partial_2 - \partial_3)\Big\}\Phi_1\Phi_2\Phi_3.$$

The effective Lagrangian which reproduces $\mathcal{E}_\Phi = 0$ is

$$\mathcal{L}_{\text{eff}} = \sum_{N=2}^{\infty} \frac{1}{N!}\mathcal{L}^{(N)}, \tag{12}$$

$$\mathcal{L}^{(2)} = -\Phi(\frac{\alpha'}{2}\partial^2 + 2)\Phi,$$

$$\mathcal{L}^{(3)} = \frac{1}{2\pi} S_{\Phi^3}(\partial_1, \partial_2, \partial_3)\Phi_1\Phi_2\Phi_3,$$

$$\mathcal{L}^{(4)} = -(\frac{1}{2\pi})^2 \Big\{ S_{\Phi^4}(\partial_1, \partial_2, \partial_3, \partial_4)$$

$$+ \frac{3}{\frac{\alpha'}{2}(\partial_1+\partial_2)^2 + 2} S_{\Phi^3}(\partial_1, \partial_2, \partial_3+\partial_4) S_{\Phi^3}(\partial_1+\partial_2, \partial_3, \partial_4)\Big\} \Phi_1\Phi_2\Phi_3\Phi_4.$$

In $\mathcal{L}^{(4)}$ the tachyon poles are subtracted from the four-tachyon amplitude. Therefore, the effective Lagrangian (12) is 1PI to this order.[4] Similarly, one can obtain the 1PI effective Lagrangian in the presence of other background fields.

§4 β-FUNCTION UP TO FIVE STRING INTERACTION

Next, we study $\beta^{(N)}$ for $N \geq 4$. Following ref. [8], we calculate $\beta_\Phi^{(4)}$ and obtain

$$\begin{aligned}\beta_\Phi^{(4)} = & \left(\frac{1}{2\pi}\right)^3 \Big\{ S_{\Phi^5}(\partial_1,\partial_2,\partial_3,\partial_4,-\partial_1-\partial_2-\partial_3-\partial_4) \\ & - \frac{4}{\alpha'\partial_4(\partial_1+\partial_2+\partial_3)-2} S_{\Phi^4}(\partial_1,\partial_2,\partial_3,-\partial_1-\partial_2-\partial_3) \\ & \qquad \times S_{\Phi^3}(\partial_1+\partial_2+\partial_3,\partial_4,-\partial_1-\partial_2-\partial_3-\partial_4) \\ & - \frac{6}{\alpha'\partial_3\partial_4+\alpha'(\partial_3+\partial_4)(\partial_1+\partial_2)-4} S_{\Phi^3}(\partial_1,\partial_2,-\partial_1-\partial_2) \\ & \qquad \times S_{\Phi^4}(\partial_1+\partial_2,\partial_3,\partial_4,-\partial_1-\partial_2-\partial_3-\partial_4) \\ & + \frac{12}{[\alpha'\partial_4(\partial_1+\partial_2+\partial_3)-2][\alpha'\partial_3(\partial_1+\partial_2)-2]} S_{\Phi^3}(\partial_1,\partial_2,-\partial_1-\partial_2) \\ & \times S_{\Phi^3}(\partial_1+\partial_2,\partial_3,-\partial_1-\partial_2-\partial_3) S_{\Phi^3}(\partial_1+\partial_2+\partial_3,\partial_4,-\partial_1-\partial_2-\partial_3-\partial_4) \Big\} \Phi_1\Phi_2\Phi_3\Phi_4. \end{aligned} \tag{13}$$

At first sight, β_Φ seems not to be 1PI, because double poles are not cancelled in $\beta_\Phi^{(4)}$. We will show that the double poles are cancelled if the contribution from $\beta_\Phi^{(1)}$, $\beta_\Phi^{(2)}$ and $\beta_\Phi^{(3)}$ is considered. In order to study whether $\beta_\Phi = 0$ is 1PI or not up to Φ^4, we must take into account $\beta_\Phi^{(3)}$ first. We rewrite the pole of the second term in $\beta_\Phi^{(3)}$ as

$$\begin{aligned} & -\frac{1}{\alpha'\partial_3(\partial_1+\partial_2)-2} \\ & = \frac{1}{\frac{\alpha'}{2}(\partial_1+\partial_2)^2+2} - \frac{\frac{\alpha'}{2}(\partial_1+\partial_2+\partial_3)^2+2-(\frac{\alpha'}{2}\partial_3^2+2)}{[\frac{\alpha'}{2}(\partial_1+\partial_2)^2+2][\alpha'\partial_3(\partial_1+\partial_2)-2]}. \end{aligned} \tag{14}$$

Since we want to find more simple form of the field equation which is equivalent to $\beta_\Phi = 0$, we use $\beta_\Phi^{(1)} + \frac{1}{2}\beta_\Phi^{(2)} + O(\Phi^3) = 0$ in (14). When one uses it and redefines the field Φ as

$$\begin{aligned} \Phi \longrightarrow \Phi + \frac{1}{2}\left(\frac{1}{2\pi}\right)^2 & \frac{1}{[\frac{\alpha'}{2}(\partial_1+\partial_2)^2+2][\frac{\alpha'}{2}\partial_3(\partial_1+\partial_2)-2]} \\ & \times S_{\Phi^3}(\partial_1,\partial_2,-\partial_1-\partial_2) S_{\Phi^3}(\partial_1+\partial_2,\partial_3,-\partial_1-\partial_2-\partial_3)\Phi_1\Phi_2\Phi_3, \end{aligned} \tag{15}$$

the second term in the right-hand side of (14) becomes a term with double pole and contributes to $\beta_\Phi^{(4)}$. The contribution to $\beta_\Phi^{(4)}$ from $\beta_\Phi^{(1)}$, $\beta_\Phi^{(2)}$ and $\beta_\Phi^{(3)}$ is

$$\begin{aligned}
&\Big(\frac{1}{2\pi}\Big)^3\Big\{\frac{12}{[\frac{\alpha'}{2}(\partial_1+\partial_2)^2+2][\alpha'\partial_3(\partial_1+\partial_2)-2]}S_{\Phi^3}(\partial_1,\partial_2,-\partial_1-\partial_2)\\
&\quad\times S_{\Phi^3}(\partial_1+\partial_2,\partial_3,-\partial_1-\partial_2-\partial_3)S_{\Phi^3}(\partial_1+\partial_2+\partial_3,\partial_4,-\partial_1-\partial_2-\partial_3-\partial_4)\\
&\qquad-\frac{6}{[\frac{\alpha'}{2}(\partial_1+\partial_2)^2+2][\frac{\alpha'}{2}(\partial_1+\partial_2)(\partial_3+\partial_4)-2]}S_{\Phi^3}(\partial_1,\partial_2,-\partial_1-\partial_2)\\
&\quad\times S_{\Phi^3}(\partial_3,\partial_4,-\partial_3-\partial_4)S_{\Phi^3}(\partial_1+\partial_2,\partial_3+\partial_4,-\partial_1-\partial_2-\partial_3-\partial_4)\Big\}\Phi_1\Phi_2\Phi_3\Phi_4.
\end{aligned}\tag{16}$$

Adding (16) to $\beta_\Phi^{(4)}$, we obtain the following form,

$$\begin{aligned}
\mathcal{E}_\Phi^{(4)}=\Big(\frac{1}{2\pi}\Big)^3\Big\{&S_{\Phi^5}(\partial_1,\partial_2,\partial_3,\partial_4,-\partial_1-\partial_2-\partial_3-\partial_4)\\
&+\frac{4}{\frac{\alpha'}{2}(\partial_1+\partial_2+\partial_3)^2+2}S_{\Phi^4}(\partial_1,\partial_2,\partial_3,-\partial_1-\partial_2-\partial_3)\\
&\qquad\times S_{\Phi^3}(\partial_1+\partial_2+\partial_3,\partial_4,-\partial_1-\partial_2-\partial_3-\partial_4)\\
&+\frac{6}{\frac{\alpha'}{2}(\partial_1+\partial_2)^2+2}S_{\Phi^3}(\partial_1,\partial_2,-\partial_1-\partial_2)\\
&\qquad\times S_{\Phi^4}(\partial_1+\partial_2,\partial_3,\partial_4,-\partial_1-\partial_2-\partial_3-\partial_4)\\
&+\frac{12}{[\frac{\alpha'}{2}(\partial_1+\partial_2)^2+2][\frac{\alpha'}{2}(\partial_1+\partial_2+\partial_3)^2+2]}S_{\Phi^3}(\partial_1,\partial_2,-\partial_1-\partial_2)\\
&\qquad\times S_{\Phi^3}(\partial_1+\partial_2,\partial_3,-\partial_1-\partial_2-\partial_3)S_{\Phi^3}(\partial_1+\partial_2+\partial_3,\partial_4,-\partial_1-\partial_2-\partial_3-\partial_4)\\
&+\frac{3}{[\frac{\alpha'}{2}(\partial_1+\partial_2)^2+2][\frac{\alpha'}{2}(\partial_3+\partial_4)^2+2]}S_{\Phi^3}(\partial_1,\partial_2,-\partial_1-\partial_2)S_{\Phi^3}(\partial_3,\partial_4,-\partial_3-\partial_4)\\
&\qquad\times S_{\Phi^3}(\partial_1+\partial_2,\partial_3+\partial_4,-\partial_1-\partial_2-\partial_3-\partial_4)\Big\}\Phi_1\Phi_2\Phi_3\Phi_4.
\end{aligned}\tag{17}$$

The tachyon equation $\mathcal{E}_\Phi=0$ with (11) and (17) is equivalent to $\beta_\Phi=0$ up to Φ^4

order. The effective Lagrangian has the form (12) and

$$
\begin{aligned}
\mathcal{L}^{(5)} = \left(\frac{1}{2\pi}\right)^3 \Big\{ & S_{\Phi^5}(\partial_1, \partial_2, \partial_3, \partial_4, \partial_5) \\
& + \frac{10}{\frac{\alpha'}{2}(\partial_1+\partial_2)^2+2} S_{\Phi^3}(\partial_1, \partial_2, \partial_3+\partial_4+\partial_5) S_{\Phi^4}(\partial_1+\partial_2, \partial_3, \partial_4, \partial_5) \\
& + \frac{15}{[\frac{\alpha'}{2}(\partial_1+\partial_2)^2+2][\frac{\alpha'}{2}(\partial_3+\partial_4)^2+2]} S_{\Phi^3}(\partial_1, \partial_2, \partial_3+\partial_4+\partial_5) \\
& \qquad \times S_{\Phi^3}(\partial_3, \partial_4, \partial_1+\partial_2+\partial_5) S_{\Phi^3}(\partial_1+\partial_2, \partial_3+\partial_4, \partial_5) \Big\} \Phi_1\Phi_2\Phi_3\Phi_4\Phi_5 .
\end{aligned}
\tag{18}
$$

If the amplitudes S_{Φ^N} have physical poles even if the external particles are off-shell, the equation of motion (11) and (17) and the effective Lagrangian (12) and (18) become 1PI up to five-string interaction level. From the point of view of the *string field theory* it is natural that the off-shell amplitudes have physical poles.

§5 COMMENTS

There remain two problems. One is whether the effective Lagrangian is 1PI or not if the off-shell amplitudes S_N do not have physical poles. After the field redefinition of background fields, it may be possible that the effective Lagrangian will become 1PI. The other problem is that whether the field redefinition (15) change the space of solution or not.

Finally, assuming that the effective Lagrangian is 1PI, we try to calculate the effective potential $V_{\text{eff}}(\Phi) \equiv \mathcal{L}_{\text{eff}}(\Phi)\big|_{\Phi=\text{constant}}$ to all order in Φ. Under this assumption, the effective potential becomes

$$V_{\text{eff}} = -\Phi^2 - 2\sum_{N=3}^{\infty} \frac{1}{N!}\left(-\frac{1}{2\pi}\frac{d}{dx}\right)^{N-2}\left(1 - \frac{1}{2}\sum_{n=3}^{\infty}\frac{A_n}{(n-1)!}x^{n-2}\right)^{-N+1}\Phi^N\Bigg|_{x=0}, \tag{19}$$

where $A_n \equiv S_{\Phi^n}\big|_{\partial=0}$.

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Evaluation of one-loop mass shifts in open superstring theory*

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I. Introduction

In previous papers we worked out some two-point functions of massive bosons in superstring theories and discussed whether there exist mass shifts in these models and/or whether mass degeneracies observed at the tree level can be violated by the loop effect[1] - [3]. In ref.[1] vertex operators which describe emissions of bosons on the leading and the next-to-leading Regge trajectories are constructed and their two-point functions are calculated. It is conjectured that mass shifts depend on the mass level number complicately and hence can not be absorbed into a slope parameter redefinition. In these calculations two point functions for both trajectories have the same form at the same mass level, but the results of ref.[2],[3]

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*Work collaborated with Tsuyoshi Nishioka and Hisashi Yamamoto

strongly suggest that these mass degeneracies would also be accidental. All these investigations are far from being satisfactory, however, from some points of view described below. First of all, we have not yet given exact values of mass shifts considering the over-all definite factor. Our previous discussions were entirely based on the forms of integrands of two-point functions. It has not yet been accomplished to estimate integral expressions of two-point amplitudes which are proportional to mass shifts, and, consequently, we have neither rigorous proof of non-zeroness of mass shifts nor of violations of mass degeneracies. The difficulty of estimations is not only due to being lacking of adequate approximation method such as α' expansion in the case of massless amplitudes. What is worse, as is discussed in ref.[2], all integral expressions of two-point functions given in ref.[1],[3] are divergent even after we eliminate dilaton tadpole divergences by the well-known Green-Schwarz' procedure. We have to manipulate these divergences to obtain meaningful answers.

It is not so hard to understand the origin of such divergences. A transition matrix obtained from an one-loop two-point amplitude must have an imaginary part required from unitarity, but the standard method to calculate on-shell string amplitudes does not give such an imaginary part. (Recall that two-point functions have the form $\sim \mathrm{Tr}(V^{\dagger}V)$. See also ref.[6].) We expect that the on-shell two-point function thus calculated is on a cut as a function of external momentum, therefore proper analytic continuation would be needed to extract

correct answer from it. In this process we would gain the imaginary part of the mass shift, which is also expected to give the damping factor of lifetime of the massive mode and express its instability.

Our main purpose is to estimate correct values of mass shifts of the bosons of first and second mass levels in Type I superstring theory at one-loop level. Our analysis at the second mass level is not completed so that in this report we concentrate on evaluating the value of mass shift at the first excited level. A two point function at a given mass level is a complex value, not a function of external momentum. So it can give a touchstone to see whether ordinary string perturbation expansion is a sensible one, by comparing the order of the values of loop effect to those of tree part. Our calculation is a concrete example of how to obtain finite results from the form of string amplitudes which are divergent in general.

II Estimation of mass shift at the first exited level

In open superstring theories there exist two different states at the first exited level, corresponding to the following vertices in F_1 formalism :

$$\zeta_{\mu\nu}\psi^{\mu}\partial X^{\nu}e^{ikX}$$

$$\frac{1}{3!}\zeta_{\mu\nu\rho}\psi^{\mu}\psi^{\nu}\psi^{\rho}e^{ikX} \tag{2.1}$$

It is straightforward to evaluate two-point functions from vertices (2.1), and their forms with the definite coefficients

determined by unitarity are found to be the same (For planar part):

$$A = \frac{g^2 2(2\pi)^2}{2(2\pi)^{10}\alpha}\ \frac{N}{2}\int_0^1 \frac{dw}{w}\int_0^{\tau} dv\ \frac{1}{\tau^5}\ [\ e^{\pi i v^2/\tau}\ \frac{\theta_1(v|\tau)}{\theta_1(0|\tau)}\ ]^{-k^2} \qquad (2.2)$$

Where $\tau \equiv \frac{\ln w}{2\pi i}$ and integration over v is along its imaginary axis. As is shown in ref.[1], amplitudes with external massive bosons are finite in the region $w \sim 1$ (ultraviolet limit), if we incorporate contributions from non-orientable part and do standard procedure discovered by Green and Schwarz. In the region $w \sim 0$ (infrared limit), however, naive power counting does not work and we should pay special attention to the behaviour of the amplitude so long as we use parameter representations of propagators to calculate string on-shell amplitudes.[2] As noted in the previous section, this integral diverges reflecting the fact that there should exist a cut in the amplitude as a function of external momentum. For simplicity we introduce ultraviolet cut-off Λ and do standard procedure of Green and Schwarz to eliminate dilaton tadpole divergences in the ultraviolet region. The whole of the amplitude does not depend on the value of Λ. We divide the amplitude in the infrared region as

$$A = A_1 + A_2$$

$$A_1 = \frac{g^2\ N/2}{2(2\pi)^8\alpha}\int_0^{\Lambda} \frac{dw}{w}\int_0^{\tau} dv\ \frac{1}{\tau^5}\ \{[\ e^{\pi i v^2/\tau}\ \frac{\theta_1(v|\tau)}{\theta_1(0|\tau)}\ ]^{-k^2} - e^{\pi i v\ k^2(1-\frac{v}{\tau})}\}$$

$$A_2 = \frac{g^2\ N/2}{2(2\pi)^8\alpha}\int_0^{\Lambda}\frac{dw}{w}\int_0^{\tau}dv\ \frac{1}{\tau^5}\ e^{\pi i v\ k^2(1-\frac{v}{\tau})} \tag{2.3}$$

One can prove that A_2 is divergent at the on-shell value of external momentum, and A_1 is convergent in the momentum region which includes the on-shell value. Note that the integrand of A_2 comes from the leading term (that is, a constant)in the expansion of Theta-functions. One can think that A_2 is the contribution from massless particles as intermediate states in a rough sense, which becomes dominant in the 'infrared' region. It is possible to prove that both of A_1 and A_2 are convergent in the region $0< k^2< 2$, so that we integrate the amplitude in this momentum region first. Since it can be shown that A_1 is analytic as a function of k^2 at least in the region $-2\leqq k^2 <2$, we can integrate A_1 at the on-shell momentum value $k^2 = -2$ from the beginning numerically. Part of the integration of A_2 can be done by hand, and function $\ln k^2$ emerges in the process of it. So there exists cut in the k^2-plain which arises at the origin and runs along the real axis to $-\infty$. We return the value of k^2 back to that of on-shell, and get the imaginary part of the amplitude. We have already emphasized the importance of this imaginary part in the previous section. A_2 also contains terms which approach zero exponentially when $w \longrightarrow 0$ ($\tau \longrightarrow -\infty$). These terms of course correspond to boundary terms of the

Teichmuller space which are divergent if we integrate out the amplitude with the values of external momentum fixed to be on-shell. All processes described above is equivalent to simply drop such boundary terms, which are precisely the roles of contact terms.

It would be instructive to compare these circumstances to those of non-planar part of the four-point amplitude. This amplitude contains the function $\frac{\theta_4(v|\tau)}{\dot{\theta}_1(0|\tau)} = q^{-1/4}[\sum_n A_n q^{2n}]$ which gives intermediate closed string poles by the well-known analytic continuation procedure. θ_1 in the amplitude (2.1) contains the factor $\sin v$ instead of $q^{-1/4}$, which is the very origin of the divergence (because v is imaginary) and gives a cut instead of poles. Of course, closed string poles appear in the 'ultraviolet' region (For non-planar amplitudes these analytic continuations are usually done after Jacobi transformation), so our correspondence is not complete.

Obviously even in four-point amplitudes of closed string models which have modular invariance we need analytic continuation in the 'infrared' region because these amplitudes have cuts as functions of external momentum in any one of three channels. There is no 'completely finite' on shell amplitude in string theories. It is believed that these divergences can be removed by adding suitable contact terms, but complete analysis is not yet reported. (For a detailed discussion, see ref.[4].) Our analysis discussed above for a massive two-point amplitude is

the simplest example of manipulating divergences applicable to any other string amplitudes.

We determined the over-all factor of mass shift as follows by factorizing tree- and one-loop four-point massless amplitudes

$$A_{\text{planar}} + A_{\substack{\text{non-}\\\text{orientable}}} + A_{\substack{\text{non-}\\\text{planar}}} = i\delta m^2 \langle \zeta_2 k | \zeta_1 k \rangle$$

After numerically evaluating, we find the result of our estimation to be

$$\delta m^2 = \frac{g^2}{\alpha(2\pi)^8}\left((-8.03061\times 10^{-2} \pm 8.25\times 10^{-5}) + \frac{58}{60}\frac{1}{14}\pi^3 i\right)$$

It turns out that the value of the integral expression is rather small compared to the factor $(2\pi)^D$ coming from phase factor of momentum space. Some phenomenological arguments give the plausible order of the value of the loop expansion parameter g to be $\sim \frac{4\pi}{100}$. Therefore our result supports the validity of perturbation ordinary used in string calculations.

III Mass shifts at the second excited state

As noted in the introduction, discussions on mass shifts at the second mass level, including details of constructing all vertex operators at this level and calculations of two-point functions, see ref.[9].

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Note added

Very recently we received a preprint [8] in which similar results are obtained.

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Two Dimensional Conformal Gauge Theories

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Abstract:

The 2-dim quantum gauge theories are investigated in the approach of the conformal field theory. Some important points such as gauge fixing, the residual symmetry, physical quantities and the energy-momentum tensor are examined in detail. The renormalized gauge coupling constant is shown to be determined absolutely. Some obscure aspects of Polyakov et al's 2-dim quantum gravity are clarified from the standpoint of general gauge theories.

§1. Introduction

Since the advent of 2-dim conformal quantum field theory by Belavin et al [1], much understanding has been obtained about the role of the conformal symmetry in the 2-dim theoretical physics (including string theories). Some models such as the Ising and Potts models are exactly solved in the conformal limit by the formalism. It is known, however, that the conformal invariant approach has, in general, some special difficulties in systems with local (gauge) symmetries [2]. In the present paper we investigate some important points (gauge fixing, residual symmetry, physical quantities, energy-momentum tensor, etc.) in the conformal gauge theories. The recent series of works by Polyakov et al [3], on 2-dim quantum gravity have stimulated the present investigation. We clarify some obscure aspects of them from the general standpoint of the conformal gauge theories.

§2. 2-Dim Local Gauge Theories ($(QCD)_2$)

We consider the 2-dim non-Abelian (SU(2)) local gauge theory ($(QCD)_2$),

$$\mathcal{L}_{inv}[\ \psi,\ \bar{\psi},\ A_\mu\ ] = \bar{\psi}\ \gamma^\mu\ (\ \partial_\mu + e\ A_\mu\ )\ \psi\ , \qquad (1)$$

$$A_\mu = A^i_\mu\ \tau^i\ ,$$

where ψ is the 2-dim Dirac spinor, τ^i is the SU(2) generators

and e is the dimensionless (bare) gauge coupling constant. The 'induced gauge theory' $S_{inv}[\ A_\mu\ ]$ is defined as the effective theory based on the 'microscopic' theory of $(QCD)_2$ (1).

$$\exp \left(S_{\mathrm{inv}}[A_\mu] \right) = \int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \exp \int d^2x \;\; \mathcal{L}_{\mathrm{inv}}[\psi, \bar{\psi}, A_\mu] \; . \qquad (2)$$

Before directly treating the theory (1) (see sect.5), we consider first the induced theory $S_{\mathrm{inv}}[A]$. $S_{\mathrm{inv}}[A]$ can be quantized relatively easily in the conformal invariant way compared with $(QCD)_2$ (1).

The explicit form of $S_{\mathrm{inv}}[A]$ was obtained by Polyakov and Wiegman [4] for the axial gauge,

$$A_- \equiv A_0 - A_1 = 0 \; . \qquad (3)$$

In the derivation they used the anomaly equations satisfied by the vector and axial vector currents. The final results are

$$\begin{aligned} & S[A_+] \equiv S_{\mathrm{inv}}[A_+ , A_- = 0] , \\ & A_+ \equiv A_0 + A_1 = \frac{1}{e} g^{-1} \partial_+ g , \qquad \partial_+ = \partial_0 + \partial_1 , \\ & J_- \equiv \frac{\delta S}{\delta A_+} = - g^{-1} \partial_- g , \end{aligned} \qquad (4a)$$

$$\begin{aligned} & \delta S [A_+] = \int d^2x \;\; \mathrm{Tr} \left(\partial_- A_+ \, \delta g \, g^{-1} \right) , \\ & S [A_+] = \frac{1}{e} \Bigg(\frac{1}{2} \int d^2x \; \mathrm{Tr} \, \partial_\mu g^{-1} \, \partial^\mu g \\ & + \frac{i}{8 \pi^2} \int_Q d^3\xi \; \epsilon^{ABC} \, \mathrm{Tr} \left(g^{-1} \partial_A g \, g^{-1} \partial_B g \, g^{-1} \partial_C g \right) \Bigg) \\ & \qquad + 2 \pi N i \quad , \end{aligned} \qquad (4b)$$

where $\partial Q = S^2$ and N is an arbitrary integer. Note that the lagrangian (4b) is the $SU(2) \times \overline{SU(2)}$ Wess-Zumino-Witten model [5] with the level = $\frac{1}{e}$. We treat, however, $S [A_+]$ not as the chiral model but as the gauge-fixed lagrangian of the gauge theory $S_{\mathrm{inv}}[A_\mu]$.

§3. Conformal Ward-Identities

Now we consider the conformal structure of the quantum gauge

system $S_{inv}[\, A_+, A_- \,]$ under the axial gauge (3). The dynamical variable is $A_+ = \frac{1}{e}\, g^{-1}\, \partial_+ g$. Due to this gauge condition we must restrict our consideration to the chirality plus(+) part (left-moving part) of the full coordinate transformation.

$$\delta x^+ = \epsilon^+ (\, x^+, x^- \,) , \qquad \delta x^- = 0 \tag{5}$$

We regard g as the primary field with the conformal dimension 0 under (5).

$$\delta g = \epsilon^+\, \partial_+ g \,, \tag{6a}$$

$$\delta A_+ = \partial_+ \epsilon^+\, A_+ + \epsilon^+\, \partial_+ A_+ \,. \tag{6b}$$

For later use we introduce an arbitrary primary field with the conformal dimension λ .

$$\delta \phi = \epsilon^+\, \partial_+ \phi + \lambda\, \partial_+ \epsilon^+\, \phi \,,$$
$$\phi = \phi^i\, \tau^i \,. \tag{6c}$$

Let us derive the conformal Ward-identities. Consider the n-point Green function of ϕ's.

$$\langle\, \phi^{i_1}(\, x_1\,)\, \phi^{i_2}(\, x_2\,) \ldots \phi^{i_n}(\, x_n\,)\, \rangle$$
$$\equiv \int \mathcal{D}A_+\, \phi^{i_1}(\, x_1\,)\, \phi^{i_2}(\, x_2\,) \ldots \phi^{i_n}(\, x_n\,) \;\; \exp S[\, A_+\,] \,. \tag{7}$$

We require the invariance of (7) under the transformation (6).

$$0 = \delta\, \langle\, \phi^{i_1}(\, x_1\,)\, \phi^{i_2}(\, x_2\,) \ldots \phi^{i_n}(\, x_n\,)\, \rangle$$
$$= \langle\, J^i_-(\, z\,)\, \delta A^i_+(\, z\,)\, \phi^{i_1}(\, x_1\,)\, \phi^{i_2}(\, x_2\,) \ldots \phi^{i_n}(\, x_n\,)\, \rangle$$
$$+ \sum_{k=1}^{n} \delta(\, z - x_k\,)\, \langle\, \phi^{i_1}(\, x_1\,) \ldots \delta\phi^{i_k}(\, x_k\,) \ldots \phi^{i_n}(\, x_n\,)\, \rangle \,. \tag{8}$$

Inserting eqs.(6b,c) into (8) and making use of the following relations,

$$\mathrm{Tr}\, A_+\, \partial_+ J_- = -\,\frac{e}{2}\, \partial_-(\, \mathrm{Tr}\, A_+^{\,2}\,) \,,$$

$$\delta (z - x) = \frac{1}{\pi} \frac{\partial}{\partial z^-} \frac{1}{z^+ - x^+} , \tag{9}$$
$$\frac{\partial}{\partial z^+} \delta (z - x) = \frac{1}{\pi} \frac{\partial}{\partial z^-} \frac{\partial}{\partial z^+} \frac{1}{z^+ - x^+} ,$$

we can integrate the equation (8) as

$$- \frac{e}{2} < A_+^i(z)\, A_+^i(z)\, \phi^{i_1}(x_1)\, \phi^{i_2}(x_2) \ldots \phi^{i_n}(x_n) >$$
$$= \frac{1}{\pi} \sum_{k=1}^{n} \left(\frac{1}{z^+ - x_k^+} \frac{\partial}{\partial x_k^+} + \frac{\lambda}{(z^+ - x_k^+)^2} \right) < \phi^{i_1}(x_1) \ldots \phi^{i_n}(x_n) > , \tag{10}$$

These identities say $- \frac{e}{2}\, A_+^i(z)\, A_+^i(z)$ is the energy-momentum tensor of the system [6]. Therefore we may define its normal-ordered quantity as

$$- \frac{1}{2} : A_+^i(z)\, A_+^i(z) : = \frac{1}{\varepsilon} T(z) , \tag{11}$$

where ε is the renormalized coupling constant which will be related with e (bare one) later.

§4. Gauge Ward-Identities

The gauge-fixed action $S[A_+]$ (4) has the residual gauge symmetry,

$$\delta A_+^i = \epsilon^{ijk} \sigma^j(x^+)\, A_+^k + \frac{i}{e} \partial_+ \sigma^i(x^+) ,$$
$$\delta \phi^i = \epsilon^{ijk} \sigma^j(x^+)\, \phi^k . \tag{12}$$

Gauge Ward-identities are obtained by the requirement of the invariance under the residual symmetry.*

$$0 = \delta < \phi^{i_1}(x_1)\, \phi^{i_2}(x_2) \ldots \phi^{i_n}(x_n) >$$
$$= < J_-^i(z)\, \delta A_+^i(z)\, \phi^{i_1}(x_1) \ldots \phi^{i_n}(x_n) > \tag{13}$$

* See the last part of this section.

$$+ \sum_{k=1}^{n} \delta(z - x_k) \langle \phi^{i_1}(x_1) \ldots \delta\phi^{i_k}(x_k)\ldots\phi^{i_n}(x_n) \rangle \quad .$$

By use of the relations (9) and the following one

$$\epsilon^{ijk} A_+^i J_-^j \tau^k - \frac{i}{e} \partial_+ J_- = i \partial_- A_+ \quad , \tag{14}$$

we can integrate the eq.(13).

$$- i \langle A_+^i(z) \phi^{i_1}(x_1) \ldots \phi^{i_n}(x_n) \rangle$$

$$= \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{z^+ - x_k^+} (t^i{}_k)^{i_1 i_2 \ldots i_n}{}_{j_1 j_2 \ldots j_n} \langle \phi^{j_1}(x_1)\ldots\phi^{j_n}(x_n) \rangle ,$$

$$(t^i{}_k)^{i_1 i_2 \ldots i_n}{}_{j_1 j_2 \ldots j_n} = \epsilon^{i\, j_k\, i_k} \delta_{i_1 j_1} \delta_{i_2 j_2} \ldots \cancel{\delta_{i_k j_k}} \ldots \delta_{i_n j_n} . \tag{15}$$

where the symbol $\nearrow$ means omission of that part. We see $t^i{}_k$ (i=1,2,3 ; k=fixed) are a representation of SU(2) generators and $- i A_+^i$ are corresponding currents [6].

The energy-momentum tensor obtained in sect.3 , (11), is just the Sugawara form constructed from the current $- i A_+^i$. (Note that we have not assumed the form, but have derived the form of T) Therefore the chirality plus(+) sector has the conformal structure of SU(2) Virasoro-Kac Moody symmetry. From the result of ref.[6] we obtain the relation between the renormalized coupling constant ε and the bare one e , the anomalous dimension * Δ of g and the central charge c of the Virasolo algebra as

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* We can introduce the anomalous dimension Δ of g in (6a) as $\delta g = \epsilon^+ \partial_+ g + \Delta \partial_+ \epsilon^+ g$. In this case the relation between A_+ and g , (4a), must be modified as $e A_+ = g^{-1} \partial_+ g - \Delta A_+^{-1} \partial_+ A_+$ in order to keep the conformal transformatio of A_+ (6b).

$$\frac{1}{\varepsilon} = -\frac{1}{2}\left(2 + \frac{1}{e} \right) ,$$
$$\Delta = \frac{2}{2 + \frac{1}{e}} , \qquad (16)$$
$$c = \frac{3}{2e + 1} .$$

This result (16) can also be obtained by treating the action $S[A_+]$ as $SU(2) \times \overline{SU(2)}$ Wess-Zumino-Witten model. In this case the SU(2) Virasolo-Kac Moody symmetry appears both in the chirality plus(+) sector and in the minus(-) one. Seemingly the situation is different from the present case. We must note, however, that as far as the 'physical quantities' (renormalized charge, anomalous dimension, central charge, etc.) are concerned, both treatment give the same amount of information.

The present way to derive the gauge Ward-identities is distinct from the case for usual (non-conformal) gauge theories. Usually all residual local symmetries are fixed by additional gauge-fixing conditions in order to make a lagragian invertible (or in order to obtain proper kinetic terms in the perturbative approach). The gauge Ward-identities are obtained by the requirement of invariance under the differnt choices of the gauge-fixing condition. The situation is different in the presaent case of the conformal invariant approach. Essentially we need no gauge-fixing because the requirement of the conformal symmetry characterizes the theory so strongly that the dynamics are determined unambiguously. The gauge choice $A_- = 0$, in the present case, is required only to obtain an explicit form of $S_{inv}[A_+, A_-]$ and the Ward-identities easily. The fixing is never for making the lagragian invertible.

§5. Conformal Structure of $(QCD)_2$

Now we consider the conformal structure of the system at the 'microscopic' level. We must directly treat the $(QCD)_2$ (1). It can be rewritten, in terms of chiral components, as

$$\mathcal{L}_{inv} = - \psi_-^\dagger (\partial_- + e A_-) \psi_- - \psi_+^\dagger (\partial_+ + e A_+) \psi_+ ,$$

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} . \tag{17}$$

After the axial gauge-fixing, $\mathcal{L}_{inv}$ reduces to

$$\mathcal{L} = - \psi_-^\dagger \partial_- \psi_- - \psi_+^\dagger (\partial_+ + e A_+) \psi_+ . \tag{18}$$

The field ψ_- decouples from the system.

Because we cannot find easily primary fields in this interacting system of ψ_+ and A_+, it is difficult to derive the energy-momentum tensor. We can, however, guess the form based on the dimensional analysis and the assumption of locality.

$$T = - \psi_+^\dagger \partial_+ \psi_+ - \mathrm{Tr}\, A_+ A_+ . \tag{19}$$

Then we can obtain the absolute value of the coupling constant from the vanishing condition of the total central charge. All equations so far in this paper are generalized easily to the case of k-flavour, SU(n) gauge symmetry. In this general case, the charge is given as

$$c^{tot} = k + \frac{n^2 - 1}{n e + 1} = 0 ,$$

$$e = - \frac{n^2 + k - 1}{k n} . \tag{20}$$

This is the 2-dim realization of the old idea that the electric charge can be fixed by the conformal invariance of QED [7].

§6. 2-Dim Quantum Gravity

Finally we comment on Polyakov et al's results on 2-dim quantum gravity [3]. The situation is more complicated than the case of gauge theories.

1. We can not keep the light-cone gauge

$$g_{+-} = \frac{1}{2} \quad , \quad g_{--} = 0 \quad , \tag{21a}$$

in quantization because the metric obtains the Weyl anomaly as the quantum effect.

$$g_{+-} = f \quad , \quad g_{--} = 0 \quad , \tag{21b}$$

$$f = \frac{1}{2} + O(\, h \,) \, ,$$

f is a fixed function and must be determined consistently. Then we must take into account of the effects of Faddeev-Popov ghosts.

2. The conformal transformation of fields must be consistent with the residual symmetry of the gauge (21b).

3. The present results of conformal gauge theories suggest strongly that the origin of SL(2,R) symmetry might be that symmetry (SO(2,2)) which appear in deriving the 2-dim conformal gravity from the gauge theory.

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Pre-geometrical Field Theory of Open String*

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ABSTRACT

We propose a gauge invariant, background independent string action, which contains open and closed string fields and no kinetic terms. The kinetic term is generated through the condensation of the string fields, which is the solution of equations of motion. We solve the equations and show that the action is classically equivalent to the open string action proposed by Hata *et al.*

There are several problems related to closed strings in open string field theory. We now point out two of them: a) In the light-cone gauge formulation, a closed string field must be added as an elementary field since the pure open string field theory is inconsistent.[2] On the other hand, in the Lorentz covariant formulation, the closed string field arises as loop effects of the open string field.[5,6] b) Though a space-time-metric-independent formulation of closed string field theory was proposed,[7] that of open string field theory of Hata *et al.*[8] is not known yet.† — The above problems have close connection with each other. The first problem a) suggests the existence of an open string action which has a larger gauge symmetry which includes general covariance and contains also a closed string field as an elementary field. We will obtain the light-cone gauge theory and the covariant theory by fixing the gauge symmetry of the action in different ways.

⋆ This talk is based on the work with M.M. Nojiri.[1]

† The background independent formulatiom of Witten's open string field theory[9] is already given in Ref.10.

On the other hand, the second problem b) arises probably because the covariant theory does not contain, as an elementary field, a closed string field which determines the geometry of space-time and because the theory has no stringy general coordinate invariance.[11] Therefore both of the above two problems will be solved by considering the system of open strings coupled with closed strings. In this talk, we give an answer to the second problem and discuss the first one.*

Hata and Nojiri proposed a new transformation on the open string field with a closed string functional parameter[12] in the formulation of string field theory proposed in Ref.8):

$$\delta_c \Phi = [r] - [r\Phi] \,. \tag{1}$$

Here r is a closed string functional parameter. We denote an open string field which is given by the transition from an closed string field A by $[A]$ and the product of a closed string field A and open string field Φ by $[A\Phi]$.[13]† The structure of the transformation and its algebra are those of stringy general coordinate transformation known in the closed string field theory.[11] The gauge invariant open string action:

$$S_o = \Phi \cdot Q_B \Phi + \frac{2}{3} g \Phi \cdot \Phi * \Phi + \frac{2}{4} g^2 \Phi \cdot \Phi \circ \Phi \circ \Phi \tag{2}$$

is, however, not invariant under the transformation (1) and the variation of the

* In this talk, we follow the notation of Refs.8), 11), 12) and 13) and the discussion given here is based on the theory of Hata *et al.*[5,7,8,11] The 1-loop scattering amplitude in covariant closed string field theory of Hata *et al.* obtained by applying the conventional canonical quantization violates unitarity since the integration over the moduli parameter covers the fundamental region infinitely many times. Recently, however, Hata showed that the conventional one is inapplicable since the interaction vertex is non-local in time coordinate.[3] By applying Hayashi's theory[4] of Hamiltonian formulation for field theories with non-local interactions, he found that the resulting 1-loop amplitude coincides with that in light-cone gauge string field theory. The same result was also obtained by modifying the string field theory action and the BRS transformation order by order in $\hbar$ to recover the BRS symmetry violated by path-integral measure.

† We denote open string fields by Φ, Ψ, Λ, $\cdots$ and closed string fields by A, B, $\cdots$, J.

action (2) is given by,

$$\delta_c S_o = -Q_B r \cdot \Big\{ -(\Phi) + \frac{g}{2}(\Phi\Phi) \Big\} \equiv -Q_B r \cdot T \ . \tag{3}$$

Here (Φ) is a closed string field which is given by the transition from an open string field Φ and we denote the product of two open string fields Φ and Ψ, which is a closed string field, by $(\Phi\Psi)$.[13] Equation (3) tells that the action (2) is invariant if $Q_B r$ is proportional to π_c^0. This indicates that the action is invariant under only the global part of the transformation.

Recently the authors proposed a string action, which contains both open and closed string fields and has not only the gauge invariance of the open string field theory[8] but also stringy general coordinate invariance. We now explain this action briefly.

We start with the following pre-geometrical action,

$$S^{\mathrm{pre}} = g^2 \{ \frac{2}{4} \Phi \cdot \Phi \circ \Phi \circ \Phi + \frac{1}{2} J \cdot (\Phi\Phi) + J * J \cdot A \} \ . \tag{4}$$

This action is invariant under the gauge transformations:

$$\begin{aligned} \delta_o^{\mathrm{pre}} : \ & \delta_o^{\mathrm{pre}} \Phi = g^2 \{ -\Phi \circ \Phi \circ \Lambda + \Phi \circ \Lambda \circ \Phi - \Lambda \circ \Phi \circ \Phi + \frac{1}{2} [J\Lambda] \} \ , \\ & \delta_o^{\mathrm{pre}} J = 0 \ , \qquad \delta_o A = -\pi g^2 (\Lambda \Phi) \ , \end{aligned} \tag{5}$$

$$\delta_c^{\mathrm{pre}} : \ \delta_c^{\mathrm{pre}} \Phi = -g[r\Phi] \ , \quad \delta_c^{\mathrm{pre}} J = 4\pi g J * r \ , \quad \delta_c^{\mathrm{pre}} A = 4\pi g A * r \ , \tag{6}$$

$$\delta_A^{\mathrm{pre}} : \qquad \delta_A^{\mathrm{pre}} \Phi = \delta_A^{\mathrm{pre}} J = 0 \ , \qquad \delta_A^{\mathrm{pre}} A = 2\pi g^2 a * J \ . \tag{7}$$

Note that we only need o-product for open string fields, $*$-product for closed string fields and $(\Phi\Psi)$-product ($[J\Phi]$-product) in the action (4) and the transformations (5) $\sim$ (7).

The action (4) is background independent like in case of pure closed string.[7] We obtain the action with kinetic terms by considering the condensation.

The equations of motion of (4) are given by,

$$2\Phi \circ \Phi \circ \Phi + [J\Phi] = 0 \ , \tag{8}$$

$$J * J = 0 \ , \tag{9}$$

$$(\Phi\Phi) + 2J * A = 0 \ . \tag{10}$$

Let $\{\Phi_0,\ J_0,\ A_0\}$ be a solution of Eqs.(8) $\sim$ (10) and we express the fields Φ, J, A as the solution plus fluctuation:

$$\Phi \to \Phi_0 + \Phi \ , \quad J \to J_0 + J \ , \quad A \to A_0 + A \ . \tag{11}$$

Then we define $Q^{\rm open}$, $Q^{\rm closed}$, $\Phi * \Psi$, (Φ) and $[J]$ as follows,

$$Q^{\rm open}\Phi \equiv g^2\{(-)^{|\Phi|+1}\Phi \circ \Phi_0 \circ \Phi_0 + \Phi_0 \circ \Phi_0 \circ \Phi + \Phi_0 \circ \Phi \circ \Phi_0 + \frac{1}{2}[J_0\Phi]\} \ , \tag{12}$$

$$\Phi * \Psi \equiv g\{\Phi \circ \Psi \circ \Phi_0 + (-)^{|\Psi|+1}\Phi \circ \Phi_0 \circ \Psi + (-)^{|\Phi|+|\Psi|}\Phi_0 \circ \Phi \circ \Psi\} \ , \tag{13}$$

$$(\Phi) \equiv -g(\Phi\Phi_0) \ , \tag{14}$$

$$[J] \equiv -g[J\Phi_0] \ , \tag{15}$$

$$Q^{\rm closed}J \equiv 2\pi g^2 J_0 * J \ . \tag{16}$$

Here $|\Phi|$ is 0(1) if Φ is Grassmann even(odd). These definitions (12) $\sim$ (16) reproduce the same properties as those proved in Ref.8), *i.e.* (5.73a), (5.73b), (5.73c) and also Eqs. (5), (7) $\sim$ (9) in Ref.13), using Eq. (5.73d) in Ref.8) and

Eqs. (6), (10), (30) in Ref.13). We can also show the nilpotency of Q^{open} and Q^{closed} :

$$(Q^{\text{open}})^2 = (Q^{\text{closed}})^2 = 0 \, . \tag{17}$$

Equation (17) allows us to regard these operators Q^{open} and Q^{closed} as BRS charges. Now after the redefinition (11), the pre-action (4) and the gauge transformations (5) $\sim$ (7) are rewritten as follows,

$$S = S_{\text{o}} + gJ \cdot T + \frac{1}{\pi} J \cdot Q^{\text{closed}} A + g^2 J * J \cdot A_0 + g J * J \cdot A \, , \tag{18}$$

$$\begin{aligned}
\delta_{\text{o}} \ : \quad & \delta_{\text{o}} \Phi = Q^{\text{open}} \Lambda + g \Phi * \Lambda - \Lambda * \Phi - g^2 \{ \Phi \circ \Phi \circ \Lambda - \Phi \circ \Lambda \circ \Phi + \Lambda \circ \Phi \circ \Phi \} \\
& \qquad + \frac{g^2}{2} [J\Lambda] \, , \\
& \delta_{\text{o}} J = 0 \, , \quad \delta_{\text{o}} A = \pi g (\Lambda) + g^2 (\Phi \Lambda) \, ,
\end{aligned} \tag{19}$$

$$\begin{aligned}
\delta_{\text{c}} \ : \quad & \delta_{\text{c}} \Phi = [r] - g[r\Phi] \, , \quad \delta_{\text{c}} J = \frac{2}{g} Q^{\text{closed}} r + 2\pi g J * r \, , \\
& \delta_{\text{c}} A = 4\pi g \{ A_0 * r + A * r \} \, ,
\end{aligned} \tag{20}$$

$$\delta_A \ : \quad \delta_A \Phi = \delta_A J = 0 \, , \quad \delta_A A = Q^{\text{closed}} a + 2\pi g^2 J * a \, , \tag{21}$$

The transformation δ_{o} corresponds to the gauge transformation of open string field theory[8] and δ_{c} to the stringy general coordinate transformation,[11] *i.e.* Eq.(1). δ_A is an unfamilier transformation which is important in the discussion of gauge fixing.

In this talk, we now solve the equations of motion (8) $\sim$ (10) and, after that, we show that the action in Eq.(18) is classically equivalent to that in Eq.(2).

Equation (10) is already solved in case of flat background[7] and in case of curved background.[14] Hereafter we consider a solution J_0 which gives Eq.(16) in

the flat background. Using (12) $\sim$ (16), we rewrite Eqs. (8) and (10) as follows,

$$Q_{\mathrm{B}}\Phi_0 - \frac{2}{3}g\Phi_0 * \Phi_0 = 0 \ , \tag{22}$$

$$g(\Phi_0) + \frac{1}{\pi}Q_{\mathrm{B}}A_0 = 0 \ . \tag{23}$$

A solution $\{\Phi_0,\ A_0\}$ should have vanishing string "length" parameter $\alpha = 0$, or else, the condensation of $\{\Phi_0,\ A_0\}$ breaks the conservation of the string "length" parameters.

There is a subtlety in the limiting procedure when the string "length" parameter α goes to zero.[7] We note, however, that an adequate procedure gives the correct properties of products, transition etc. For example, the product of two string fields with vanishing string "length" parameter should be defined so that it gives the commutator of corresponding vertex operators.[15]

Let $\widetilde{\Phi}_0$ be an arbitrary string functional with ghost number -1 and infinitesimal string "length" parameter $\alpha = \epsilon$. Analysis of Neumann coefficients give following equations,[15]

$$\begin{aligned}
\widetilde{\Phi}_0 \circ \Phi \circ \Psi &\propto \Phi * \Psi + O(\epsilon^{\frac{1}{2}}) \ , \\
\widetilde{\Phi}_0 * \Phi &\propto (\frac{1}{\epsilon} + O(1))\pi_{\bar{c}}(0)\Phi + O(\epsilon) \ , \\
\Phi * \widetilde{\Phi}_0 &\propto (\frac{1}{\epsilon} + O(1))\pi_{\bar{c}}(\pi)\Phi + O(\epsilon) \ , \\
(\widetilde{\Phi}_0\Phi) &\propto (\Phi) + O(\epsilon^{\frac{1}{2}}) \ , \\
[\widetilde{\Phi}_0 J] &\propto [J] + O(\epsilon^{\frac{1}{2}}) \ ,
\end{aligned} \tag{24}$$

Here $\pi_{\bar{c}}$ is the FP ghost on the string.[8] In particular, if $\widetilde{\Phi}_0$ has a form as follows,

$$\widetilde{\Phi}_0 = Q_{\mathrm{B}}\bar{c}_0\Psi_0 \ . \tag{25}$$

we obtain,

$$\widetilde{\Phi}_0 * \Phi = O(\epsilon) \ , \qquad \Phi * \widetilde{\Phi}_0 = O(\epsilon) \ , \tag{26}$$

Here Ψ_0 is another arbitrary string functional with ghost number -1 and $\bar{c}_0$ is

the zero mode of the FP anti-ghost.[8]

Equations(25), (26) and the nilpotency of the BRS charge Eq.(17) tell that a solution of Eq.(22) is given by,

$$\Phi_0 = \lim_{\epsilon \to 0} Q_{\rm B} \bar{c}_0 \Psi_0 \ . \tag{27}$$

We can easily confirm that this solution Φ_0 gives the non-vanishing $*$-product and open–closed transition through Eqs.(13) $\sim$ (15), by using one simple example,

$$\Psi_0 = \frac{N}{g} \bar{c}_{-2} |0> \delta(p) \delta(\alpha - \epsilon) \ . \tag{28}$$

Here N is a finite normalization constant.

Equations (25), (26) and (27) tell $\Phi_0 * \Phi = 0$ for an arbitrary field Φ in Fock space. Due to this property, there remain less ambiguities in the limiting procedure when string "length" parameters vanish.[15]

Using the solution in Eq.(27), Equation(23) is rewritten as,

$$Q_{\rm B} \{ g \lim_{\epsilon \to 0} (\bar{c}_0 \Psi_0) + \frac{1}{\pi} A_0 \} = 0 \ . \tag{29}$$

Here we used the following equation,[13]

$$(Q_{\rm B} \Phi) = Q_{\rm B} (\Phi) \ . \tag{30}$$

By solving Eq.(29), we obtain,

$$A_0 = -\pi g \lim_{\epsilon \to 0} (\bar{c}_0 \Psi_0) + B \ . \tag{31}$$

Here B is a closed string functional which satisfies the "on-shell" condition,

$$Q_{\rm B} B = 0 \ . \tag{32}$$

We now show that the action(18) or (4) is classically equivalent to the action (2) proposed by Hata *et al.*[8]

First we expand the closed string field J and A with respect to the anti-ghost zero mode $\bar{c}_0$ into,

$$J = -\bar{c}_0\phi_J + \psi_J \,, \qquad A = -\bar{c}_0\phi_A + \psi_A \,. \tag{33}$$

Using the gauge transformation (20) and (21), we choose the following gauge condition,

$$\psi_J = \psi_A = 0 \,. \tag{34}$$

Second we note that $J * J \cdot A_0 = A_0 * J \cdot J$ in the action(18) diverges due to the tachyon.* By regularizing this divergence by letting the string "length" parameter α finite ($\alpha = \epsilon \neq 0$), we obtain,

$$A_0 * J = \frac{1}{\epsilon^2}\frac{\partial}{\partial \bar{c}_0} J + O(1) \,. \tag{35}$$

Using Equations (34) and (35), the action (18) is rewritten as follows,

$$S = S_o - g\phi_J T_\psi + \frac{1}{\pi}\phi_J L \phi_A + g^2\epsilon^{-2}\phi_J\phi_J + g^2\phi_J\phi_J\phi_A + O(\epsilon) \,. \tag{36}$$

Here we expand T in Eq.(3) with respect to the anti-ghost zero mode:

$$T = -\bar{c}_0 T_\phi + T_\psi \,. \tag{37}$$

After the redefinition

$$\epsilon^{-1}\phi_J \to \phi_J \,, \tag{38}$$

and letting ϵ go to zero, we obtain,

$$S = S_o + g^2\phi_J\phi_J \,. \tag{39}$$

The action (39) is obviously equivalent to that in Eq.(2).

---

* See Equation(37) in Ref.13.

We proposed a gauge invariant and background independent string action, which contains both open and closed string field. We showed that this action is classically equivalent to that proposed by Hata *et al.*, by choosing the gauge condition (34). Another gauge condition might give the light-cone string field theory but we need further investigation.[15]

The subject similar to that given here was discussed by Strominger and others[16~20] on the basis of Witten's action.[9] Strominger constructed closed string states in terms of open string oscillators[18] and proposed an action describing closed string field theory.[19] His dicussion is based on the cubic action[10] and associativity anomalies.[16,18] Although the discussion given in this talk is sometimes analogous to theirs, the relation is not clear at present.

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Off-shell Amplitude in Witten's Bosonic String Field Theory

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The off-shell amplitude are calculated in the Witten's bosonic string field theory. We use ordinary methods in the first quantization by extending the Giddings' approach to this theory.

1. Introduction

In the first quantization of the string we treat string coordinates $X^{\mu}(z)$ as 2-dimensional scalar fields in a curved surface, which is equivalent to the Riemann surface in the conformal gauge. There are two formalisms in the first quantization; one of these is the operator formalism, which is equivalent to the conformal field theory[1] , and the other is the path-integral formalism[2] . N-point amplitudes are expressed as correlations of N vertex operators on a Riemann surface. The vertex operator means the emission of a particle state characterized by the momentum and the polarization tensor. The on-shell condition of this operator is necessary and sufficient for the conformal invariance of amplitudes. The conformal transformation relates several Riemann surfaces so that conformally equivalent surfaces give the same on-shell amplitude. If we can make a systematic choice of a specific surface from conformally equivalent surfaces, we'll obtain the consistent off-shell amplitudes. However in the first quantization it is difficult to make such a choice.

On the other hand, if we have a second quantization of the string, i.e., the quantization of string fields, we can calculate off-shell amplitudes in the sense of usual field theory. We have Feynman rules for this theory. Following these rules we can construct Feynman diagrams and regard them as specific Riemann surfaces. We expect that the amplitudes of string fields are reduced to those in the first quantization on their Feynman diagrams. In fact Giddings[3] evaluated the four-tachyon on-shell amplitude of the Witten's bosonic open string field theory[4] in

the path-integral formalism by considering its Feynman diagram. We here extend his method to off-shell amplitudes.

In such extension it is difficult to find the expression of the string field in the path-integral formalism. On the other hand in the operator formalism that expression is given fairly easily by using vertex operators. Therefore we reduce a three-string amplitude in the second quantization to an expectation value of three vertex operators corresponding to string fields in the operator formalism. Next let's reformulate this amplitude in the path-integral by starting from the construction of its Feynman diagram. Comparing two amplitudes we will find the relation of string fields in two formalisms. Finally we also give a four-string off-shell amplitude in the path-integral formalism. For the detailed discussions, see Ref.[5].

2. Off-shell three-string amplitude

The Witten's string field theory has a gauge symmetry. In its quantization this symmetry is fixed usually by the Siegel-Feynman gauge[6] ,

$$b_0|\Psi\rangle = 0. \tag{1}$$

If the field is off-shell then it has no other condition; an on-shell condition is $Q|\Psi\rangle = 0$. Since in this theory the field has a ghost number -1/2, $|\Psi\rangle$ is constructed from the vacuum $|\hat{0}\rangle = c_1|0\rangle$ by acting the polynomials of mode operator a_m's and monomials containing only an equal number of c_n's and b_m's $(n, m < 0)$,

$$\begin{aligned}|\Psi\rangle = \Big[&(\phi(k) + \zeta(k)\cdot a_1^\dagger + \cdots)e^{ik\cdot x} \\ &+ (\phi_{(1)}(k) + \zeta_{(1)}(k)\cdot a_1^\dagger + \cdots)e^{ik\cdot x}c_{-1}b_{-1} + \cdots\Big]|\hat{0}\rangle.\end{aligned} \tag{2}$$

These states can be expressed by using ordinary vertex operators in the first quantization so that we assume the string field to be written in the form

$$|\Psi\rangle = \lim_{z\to 0}\Psi(z)|0\rangle, \tag{3}$$

where $\Psi(z)$ will consist of the ordinary vertex operators with normal-ordered polynomials of $c(z)$ and $b(z)$. If $|\Psi\rangle$ is off-shell $\Psi(z)$ is not a primary conformal field in general.

The three-string amplitude in Witten's string field theory is expressed as

$$A_3 = \langle V^W | \prod_{r=1}^{3} |\Psi_{(r)}\rangle. \tag{4}$$

We rewrite this expression into the expectation value in the single Hilbert space, i.e., the correlation function of standard conformal field theory.

Using the relation[7] between the Witten vertex $\langle V^W|$ and the CSV vertex $\langle V^{CSV}|$ and also using the overlap condition[8] of the CSV vertex we can reduce (4) as follows[5] ,

$$A_3 = \langle \Psi_{1G_1(0)} \Psi_{2G_2(0)} \Psi_{3G_3(0)} \rangle, \tag{5}$$

where $\Psi_{G(z)}$ is the conformal transformed $\Psi(z)$ by $G(z)$. Each $G_r(z)$ $(r = 1,2,3)$ is defined by

$$G_1(z) = g(z), \;\; G_2(z) = g(-1/z), \;\; G_3(z) = g(z), \tag{6}$$

and

$$g^{-1}(w) = \frac{-1}{3\sqrt{3}} \frac{(2-w)(1-2w)(1+w) - 2(1-w-w^2)^{3/2}}{w(1-w)}. \tag{7}$$

$g(z)$ has branch cuts so that the variables z of $G_1(z)$ and $G_3(z)$ stay on different branches near $z \sim 0$, i.e., $G_1(0) = \infty$ and $G_3(0) = 0$.

The simplest non-trivial example is the case where the off-shell massless vector operator $V_\zeta(z) =: \zeta \cdot \partial X e^{(ik\cdot X)}(.:)$: is contained because it exhibits the inhomogeneity under conformal transformation. Choosing $V_{\zeta_r}(z)c(z)$ as $\Psi_r(z)$ we can calculate the amplitude for three massless vectors. The behavior of $V_\zeta(z)$[5] under any finite conformal transformation $g(z)$ is

$$V_{\zeta\, g(z)} = (g'(z))^{1+k^2/2}\Big[V_\zeta(g(z)) - \frac{i}{2}\zeta \cdot k : e^{ik\cdot X} : \frac{g''(z)}{(g'(z))^2}\Big]. \tag{8}$$

Using this we can calculate the correlation function (5) for the massless vertices

and that result is

$$A_3(\zeta_1,\zeta_2,\zeta_3) = -\frac{1}{2}\zeta_1^\mu\zeta_2^\nu\zeta_3^\rho\{\delta_{\mu\nu}(k_1-k_2)_\rho + \delta_{\nu\rho}(k_2-k_3)_\mu + \delta_{\rho\mu}(k_3-k_1)_\nu \\ \frac{1}{4}(k_1-k_2)_\mu(k_2-k_3)_\nu(k_3-k_1)_\rho\}\left(\frac{4}{3\sqrt{3}}\right)^{\Sigma k_r^2/2}. \tag{9}$$

This result coincides with the previous one[9] by different method, which were evaluated directly from (4).

3. Reformulation of three-string amplitude

The three-string amplitude is written in the path-integral[4] as follows;

$$\begin{aligned} A_3 = \int \prod_{r=1}^{3}\prod_{\sigma=0}^{\pi} & DX^{(r)}(\sigma)D\phi^{(r)}(\sigma)e^{3i\phi(\pi/2)/2} \\ & \times \prod_{r=1}^{3}\prod_{\sigma=0}^{\pi/2}\delta(X^{(r)}(\sigma)-X^{(r+1)}(\pi-\sigma))\delta(\phi^{(r)}(\sigma)-\phi^{(r+1)}(\pi-\sigma)) \\ & \times \prod_{r=1}^{3}\Phi_r[X^{(r)}(\sigma),\phi^{(r)}(\sigma)]. \end{aligned} \tag{10}$$

Here $\phi(\sigma)$ is a bosonized ghost and $\Phi[X(\sigma),\phi(\sigma)]$ is the string field which is a functional of the string $X(\sigma)$ and the ghost $\phi(\sigma)$ in general. The factor $e^{3i\phi(\pi/2)/2}$ comes from the ghost number anomaly $-(i/\pi)\int d\tau d\sigma R(\tau,\sigma)\phi(\tau,\sigma)$ and a curvature $3\pi/2$ at the mid point $(\tau=0,\sigma=\pi/2)$.

We would like to alter this equation to the form in §2. For this purpose we rewrite this equation as the path-integral on the upper half complex plane U because the operator formalism is equivalent to the path-integral on U. Our procedures are divided into two parts; the first is rewriting the path-integral in (10) to the one on an infinite surface corresponding to a Feynman diagram F_3, and the second is conformal transforming it from F_3 to U.

To construct the Feynman diagram we redefine the string field by

$$\Phi[X(\sigma),\phi(\sigma)] = \int \prod_{u\in D} DX(u)D\phi(u)e^{-S_D}\widetilde{\Phi}(u_\infty) \\ \times \delta(X(u_0)-X(\sigma))\delta(\phi(u_0)-\phi(\sigma)), \tag{11}$$

where D is a surface like a ribbon with a width π of which one boundary stretches to infinity. We parametrize the surface D as Fig.1 by $u = \tau + i\sigma$ which we call the proper coordinate of string. At the boundary of D, $u_\infty = -\infty + i\sigma$, we put the string field $\widetilde{\Phi}(u_\infty) = \widetilde{\Phi}[X(u_\infty),\phi(u_\infty)]$ and propagate it along D to the other boundary $u_0 = i\sigma$ where they interact together.

Because of the delta-functional in (10) and (11) the three regions represented by the D's are connected together at u_0 so that we can construct a surface as Fig.2(a). Cutting it along C in Fig.2(a) and spreading it, we obtain a flat surface F_3 parametrized by w as Fig.2(b) which has double sheet structure in Re $w < 0$. The coordinate w is relate to each proper coordinate $u_r (r = 1, 2, 3)$ in such a way that

$$\begin{aligned} &\text{in Re } w = \tau < 0 \\ &\quad w = \begin{cases} u_1 \equiv w_1(u_1) & \text{lower sheet} \\ u_3 \equiv w_3(u_3) & \text{upper sheet,} \end{cases} \\ &\text{in Re } w = \tau > 0 \\ &\quad w = \pi - iu_2 \equiv w_2(u_2). \end{aligned} \tag{12}$$

If strings $X^{(r)}(u_r)$ and ghosts $\phi^{(r)}(u_r)$ are conformal transformed by w_r, these fields can be identified with those on F_3. In (10) and (11) only the string fields $\widetilde{\Phi}_r$ are not conformal invariant. Therefore A_3 can be rewritten in the following path-integral on F_3.

$$\begin{aligned} A_3 = &\int \prod_{w\in F_3} DX^{(r)}(w)D\phi^{(r)}(w)e^{-S_{F_3}} \prod_{r=1}^{3} \widetilde{\Phi}_{rw_r(u_{r\infty})} \\ &\times \prod_{w_C\in C, w_{C'}\in C'} \delta(X(w_C)-X(w_{C'}))\delta(\phi(w_C)-\phi(w_{C'})) \\ \equiv &\langle \widetilde{\Phi}_{1w_1(u_{1\infty})} \widetilde{\Phi}_{2w_2(u_{2\infty})} \widetilde{\Phi}_{3w_3(u_{3\infty})} \rangle_{(\text{on } F_3)}. \end{aligned} \tag{13}$$

The delta-functionals in the first equation of (13) mean that the two lines C and

C' in F_3 are identified. S_{F_3} is an action on F_3 of the strings and the ghost.

Next consider a conformal transformation $f: F_3 \to U$. Since $SL(2,R)$ transforms U to U we fix this freedom by picking up three representative points on F_3 and U and relating these points by f as follows,

$$f: \begin{cases} w_1(u_{1\infty}) \to \infty \\ w_2(u_{2\infty}) \to 1 \\ w_3(u_{3\infty}) \to 0. \end{cases} \tag{14}$$

This transformation is related to the conformal transformation $G(z)$ defined in §2 by

$$f \circ w_r(u) = G_r \circ E(u) \quad (\text{for } r = 1,2,3), \quad E(w) \equiv e^w. \tag{15}$$

Transforming (13) by f and using the relation (15) we can obtain the final expression for A_3

$$\begin{aligned} A_3 &= \int \prod_{z \in U} DX(z) D\phi(z) e^{-S_U} \prod_{r=1}^{3} \widetilde{\Phi}_{r f \circ w_r(u_{r\infty})} \\ &= \langle \widetilde{\Phi}_{1 G_1 \circ E(u_{1\infty})} \widetilde{\Phi}_{2 G_2 \circ E(u_{2\infty})} \widetilde{\Phi}_{3 G_3 \circ E(u_{3\infty})} \rangle_{(\text{on } U)}. \end{aligned} \tag{16}$$

The path-integral formalism on U is equivalent to the operator formalism or correlation functions of the strings and the ghosts in the former formalism are coincide with those in the later one. To obtain the same amplitude from two formalisms we find the relation

$$\widetilde{\Phi}_{E(u)} = \Psi(E(u)) \qquad \text{at } u = u_\infty, \tag{17}$$

by comparing (16) with (5). Since this relation is independent of the number of external string we can use it as the definition of $\widetilde{\Phi}$ for multi-string amplitudes.

4. Four-string amplitude

Using procedures in the last section we calculate the off-shell 4-string amplitude. A Feynman diagram F_{4s} as in Fig.3 corresponding to the S-channel 4-string

amplitude is parametrized by w related to each proper coordinate u_r $(r = 1 \sim 4)$ as follows;

$$\begin{cases} \text{in Re } w < 0 \quad w = u_r \equiv w_r(u_r) & \text{for r} = 1, 2 \\ \text{in Re } w > t \quad w = t + i\pi - u_r \equiv w_r(u_r) & \text{for r} = 3, 4. \end{cases} \tag{18}$$

This amplitude is

$$A_{4s} = \int_0^\infty dt \langle \prod_{r=1}^4 \widetilde{\Phi}_{r w_r(u_{r\infty})} \Big[\cdot \int_C dw \ b(w) + \text{h.c.} \Big] \rangle_{(\text{on } F_{4s})}. \tag{19}$$

The line integral comes from b_0 in the propagator $(b_0 \int dt e^{-tL_0})$.

We also map the surface F_{4s} to the upper-half complex plane U by a conformal transformation[3] $\hat{f}$. For the overall transformation of each string field we define that

$$\hat{f} w_r(z) \equiv \hat{G}_r \circ E(z) \qquad \text{for } r = 1 \sim 4. \tag{20}$$

Then A_{4s} is reduced to

$$A_{4s} = \int_0^\infty dt \langle \prod_{r=1}^4 \Psi_{r\hat{G}_r(\xi_r)} \Big[\int_{\hat{f}(C)} dz \ (\partial_z \hat{f}^{-1})^{-1} b(z) + \text{h.c.} \Big] \rangle_{(\text{on } U)}. \tag{21}$$

Here we used the relation (17) and $\xi \equiv u_\infty$.

For example the four-tachyon amplitude can be calculated by using the techniques given in Ref.[3]. Here we give only the result;

$$\begin{aligned} A_{4-tachyon} = & -\frac{1}{4} \int_{1/2}^{1} dx \ x^{(k_1 \cdot k_4 + k_2 \cdot k_3)/2} \ (1-x)^{(k_1+k_2)^2/2-2} \ \Big[\frac{exp w_\alpha}{2} \Big]^{\Sigma(k_r^2-2)/2} \\ & + (k_1 \to k_4 \to k_3 \to k_2 \to k_1). \end{aligned} \tag{22}$$

This result also coincides with the previous one[10] by different method.

5. Conclusion

Conformal transformations play important roles for calculations of off-shell amplitudes in the first quantization. The off-shell amplitudes have the informations of Riemann surfaces through the conformal transformations. String field theories can specify the conformal transformations. In other words their Feynman diagrams determine such transformation which map those diagrams to a reference Riemann surface. From this analyses we can enumerate some rules for the calculation of off-shell amplitudes for the string field theory;

1) Relate the coordinate w of the Feynman diagram F to each proper coordinate u_r by conformal transformations w_r.
2) Use $\widetilde{\Phi}_{E(u)} = \Psi(\xi)$, $\xi = e^u$, as each string field at $u = u_\infty$.

Since these rules are concerned with external strings, we can apply them to the multi-string and the multi-loop amplitudes. In other string field theories w_r and $\Psi(z)$ are different from those of Witten's theory but the above rules will be applicable.

Furthermore for the tree amplitudes, we can give a more concrete rule

3) Map from F to the upper-half plane U, as a reference Riemann surface, by a conformal transformation f because the correlation functions of fundamental fields become simple. Then the N-string tree off-shell amplitude can be expressed formally as follows;

$$A_N = \int \prod_{l=1}^{N-3} dt_l \langle \prod_{r=1}^{N} \Psi_{rG_r(\xi_r)} \prod_{l=1}^{N-3} \int_{f(C_l)} dz_l \left(\frac{df^{-1}}{dz_l} \right)^{-1} b(z_l) \rangle_{(\text{on } U)}, \tag{23}$$

where G_r are defined by $f \circ w_r(u) = G_r \circ E(u)$. $f(z)$ must satisfies the following differential equation [11]

$$\frac{df^{-1}}{dz_l}(z) = \frac{\prod_{l=1}^{N-3} \sqrt{(z - Z_l)(z - \bar{Z}_l)}}{\prod_{r=1}^{N}(z - X_r)}, \tag{24}$$

where Z_l are the images of the interaction points on U and X_r are the images of the external states on U ($X_r = G_r(\xi_r)$).

For multi-loop amplitudes it is difficult to give the conformal transformation[12] that map their Feynman diagrams to the reference Riemann surfaces.

In actual calculations we must use the off-shell vertex operators but their behaviors under the finite conformal transformation is different for each vertex operator. However, as explained in Ref.[5], those behavior are determined by the normal ordering and the behaviors of lower mass-level vertices so that the systematic evaluations may be so difficult.

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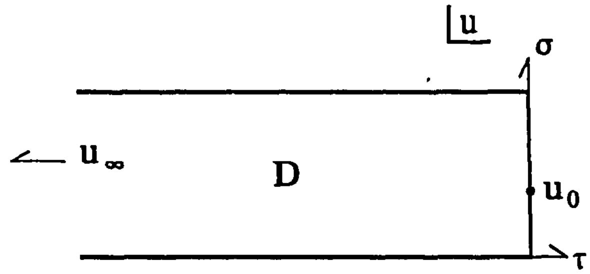

Fig.1

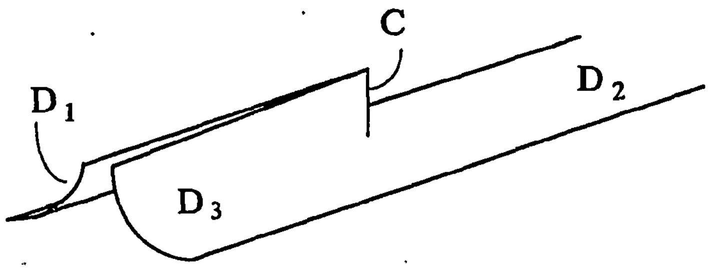

Fig.2(a)

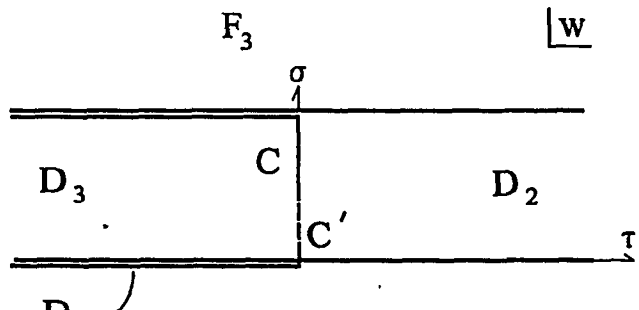

Fig.2(b)

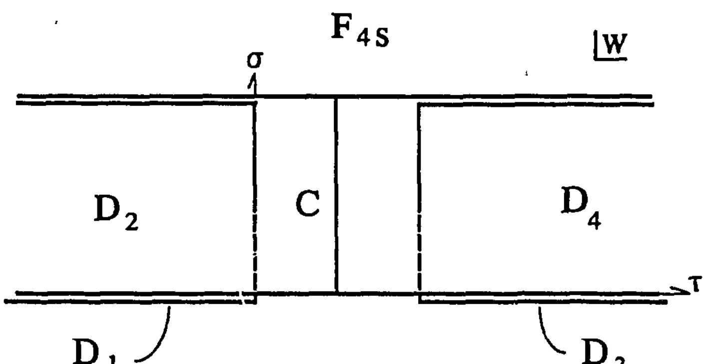

Fig.3

Lorentz Symmetry in the Light-Cone Field Theory of Open and Closed Strings

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1. Introduction

A main subject in the construction of the light-cone string field theories (SFT) [1-3] is the Lorentz covariance. In the light-cone approach the unitarity is manifest but the Lorentz covariance is non-trivial and needs a proof. The covariance of the on-shell S-matrix was proved in [1] many years ago. Recently the transformation law of the string fields was given and the invariance of the action and the closure of the algebra were proved for the bosonic pure closed SFT [4,5] and the bosonic pure open SFT [4]. (The covariance in the first order of a coupling constant was proved also in ref.[6]. For superstrings see ref.[7].) The open SFT is indeed Lorentz invariant without closed string fields at the classical level. At the quantum level, however, the closed string fields may be needed.

In this report,we will examine the Lorentz covariance of the light-cone SFT for the open-closed mixed system. We will consider the 26-dimensional orientable bosonic string theory. We propose a transformation law of the string fields with undetermined coefficients using the vertices appearing in the action [2]. These coefficients are fixed up to one unfixed constant requiring the invariance of the action and the closure of the algebra as much as possible. However, with any choice of this last constant, the invariance of the action and the closure of the algebra are not completely achieved at the classical level due to three diagrams which contain closed string intermediate states. We expect that the quantum effects will determine this last constant and cancel these diagrams and the exact Lorentz symmetry is realized only at the quantum level.

The result of this report was published in ref.[8]. For further details of the discussions, see ref.[8].

2. Action and Lorentz Transformation Law

In the light-cone SFT for the open-closed mixed system we need two independent string fields ϕ and ψ which represent open and closed strings respectively[2]. These fields are functions of the light-cone time $x^+ \equiv \tau$, the light-cone momentum $p^+ \equiv \frac{1}{2}\alpha$ (open) or $p^+ \equiv \alpha$ (closed) and the transverse string coordinates $X^i(\sigma)(i = 1 \cdots 24)$. The open string field ϕ is matrix-valued. For the orientable string, which we will consider here, a possible gauge group is U(N) and ϕ is anN × N matrix in the fundamental representation of its Lie algebra[9]. We use the Fock spase representation of the string fields in the rest of this report : $|\phi(1,\tau)\rangle$, $|\psi(1,\tau)\rangle$. For the notations, see ref.[8].

The action of this light-cone SFT is given by [2-5]

$$
\begin{aligned}
I = \int d\tau \{ \int d1 \; tr \, \langle\phi(1)| \, (\frac{1}{2} i\alpha_1 \partial_\tau - L_0^\perp) \, |\phi(1)\rangle \\
+ \frac{2}{3} g \int d1d2d3 \; tr[\langle\phi(1)| \, \langle\phi(2)| \, \langle\phi(3)|] \left| V^{(3)}(1,2,3) \right\rangle \\
+ \frac{2}{4} g^2 \int d1d2d3d4 \; tr[\langle\phi(1)| \, \langle\phi(2)| \, \langle\phi(3)| \, \langle\phi(4)|] \left| V^{(4)}(1,2,3,4) \right\rangle \\
+ \int d1_c \; \langle\psi(1_c)| \, (i\alpha_1 \partial_\tau - L_0^\perp) \, |\psi(1_c)\rangle \\
+ \frac{2}{3} c_3 g^2 \int d1_c d2_c d3_c \; \langle\psi(1_c)| \, \langle\psi(2_c)| \, \langle\psi(3_c)| \left| V_c^{(3)}(1_c, 2_c, 3_c) \right\rangle \\
+ c_6 g \int d1_c d2 \; \langle\psi(1_c)| \, tr \, \langle\phi(2)| \left| U^{(2)}(1_c, 2) \right\rangle \\
+ c_7 g^2 \int d1_c d2d3 \; \langle\psi(1_c)| \, tr[\langle\phi(2)| \, \langle\phi(3)|] \left| U^{(3)}(1_c, 2, 3) \right\rangle_S \},
\end{aligned}
\tag{2.1}
$$

where $L_0^\perp$ is a Virasoro generator with the only transverse modes and $|V\rangle$ and $|U\rangle$ are the vertices of open and closed strings. The products of string fields in the integrands in eq.(2.7) are at equal τ. In the following, we omit τ in the fields for simplicity. The coefficients c_i will be fixed in sects.3 and 4 by requiring the Lorentz covariance. We have used a relation between the open string coupling constant $g_{op} = g$ and the closed string one g_{cl} :

$$
g_{cl} = c_3 g_{op}^2. \tag{2.2}
$$

The Lorentz transformations, except for a rotation in the [j-]-plane, act on the string fields lineary. The invariance of the action and the closure of the algebra for such transformations can be easily shown. Therefore we only need to study the non-trivial one. We propose a following Lorentz transformation law of the string fields:

$$
\begin{aligned}
\delta \langle\phi(1)| = i\epsilon_{j-}[-\langle\phi(1)|\, M^{j-\dagger}
&+ g\int d2d3d4\ \langle\phi(2)|\,\langle\phi(3)|\left\langle \tilde{R}(4,1)\right| \frac{\sqrt{\pi}}{\alpha_1} X_I^j \left| V^{(3)}(4,2,3)\right\rangle \\
&+ g^2\int d2\cdots d5\ \langle\phi(2)|\,\langle\phi(3)|\,\langle\phi(4)|\left\langle \tilde{R}(5,1)\right| \frac{\sqrt{\pi}}{\alpha_1} X_I^j \left| V^{(4)}(5,2,3,4)\right\rangle \\
&+ c_1 g\int d2_c d3\ \langle\psi(2_c)|\left\langle \tilde{R}(3,1)\right| \frac{\sqrt{\pi}}{\alpha_1} X_I^j \left| U^{(2)}(2_c,3)\right\rangle \\
&+ c_2 g^2\int d2_c d3d4\ \langle\psi(2_c)|\,\langle\phi(3)|\left\langle \tilde{R}(4,1)\right| \frac{\sqrt{\pi}}{\alpha_1} X_I^j \left| U^{(3)}(2_c,3,4)\right\rangle_S], \\
\delta \langle\psi(1_c)| = i\epsilon_{j-}[-\langle\psi(1_c)|\, M^{j-\dagger}
&+ \frac{c_3}{2} g^2\int d2_c d3_c d4_c\ \langle\psi(2_c)|\,\langle\psi(3_c)|\left\langle \tilde{R}(4_c,1_c)\right| \frac{\sqrt{\pi}}{\alpha_1} X_I^j \left| V_c^{(3)}(4_c,2_c,3_c)\right\rangle \\
&+ c_4 g\int d2d3_c\, tr\,\langle\phi(2)|\left\langle \tilde{R}(3_c,1_c)\right| \frac{\sqrt{\pi}}{\alpha_1} X_I^j \left| U^{(2)}(3_c,2)\right\rangle \\
&+ c_5 g^2\int d2d3d4_c\, tr[\langle\phi(2)|\,\langle\phi(3)|]\left\langle \tilde{R}(4_c,1)\right| \frac{\sqrt{\pi}}{\alpha_1} X_I^j \left| U^{(3)}(4_c,2,3)\right\rangle_S],
\end{aligned}
\tag{2.3}
$$

where ϵ_{j-} is a parameter of the transformation and $X_I^j = X^j(\sigma_{int})$ with σ_{int} being the interaction point on the strings determined by the vertices. M^{j-} are the Lorentz generator of the first quantized theory.The transformation laws were already given for the pure open [4] and the pure closed [4,5] SFT. The terms which consist of only open string fields or only closed string fields are present in those cases. The normalizations of those terms in eq.(2.3) have been chosen according to refs.[4,5] using the relation (2.2). We have introduced new terms with the open-closed mixed vertices $|U\rangle$, which appear in the action (2.1).The coefficients c_i will be fixed in sects.3 and 4.

3. Lorentz Transformation of the Action

Having given the action and the transformation laws, we now study the Lorentz invariance of the action (2.1). First let us consider the variation of a part of the action I_τ which has τ-derivatives. Since there is no τ-derivative in the Lorentz transformation, the variation of these terms must vanish by themselves. It is easy to show $\delta I_\tau = 0$ if we choose

$$c_1 = 2c_4, \;\; c_2 = 4c_5. \tag{3.1}$$

We turn to the variation of the remaining part I' of the action (2.7). Order g^0 terms vanish due to

$$L_0^\perp M^{j-} = M^{j-\dagger} L_0^\perp. \tag{3.2}$$

There are 16 kinds of terms of order g^n $(n \geq 1)$ in the variation. We can show that 13 of these terms vanish by using the diagrammatic methods of refs.[10,11], if we choose the coefficients as

$$c_1 = c_2 = \frac{1}{4\pi} c_3 = 2c_4 = 4c_5 = \frac{1}{2} c_6 = c_7 \equiv c, \tag{3.3}$$

where we have used eq. (3.1). Eq.(3.3) fixes all the coefficients up to an overall constant c.

There remain three terms in the variation of the action. These terms do not vanish and therefore the action (2.1) is not invariant under the Lorentz transformation (2.3) at the classical level. The explicit forms of these non-vanishing terms are

$$\delta I = \epsilon_{j-}(g^2 A_2 + g^3 A_3 + g^4 A_4),$$

$$
\begin{aligned}
A_2 =& \frac{1}{2} i c^2 \int d1 \cdots d4_c tr[\langle\phi(1)|] tr[\langle\phi(2)|] \left\langle \tilde{R}(3_c, 4_c) \right| \\
& \times \frac{\sqrt{\pi}}{\alpha_4} (X^j_{31} - X^j_{42}) \left| U^{(2)}(3_c, 1) \right\rangle \left| U^{(2)}(4_c, 2) \right\rangle ;
\end{aligned}
$$

$$
\begin{aligned}
A_3 =& i c^2 \int d1 \cdots d5_c tr[\langle\phi(1)|] tr[\langle\phi(2)| \langle\phi(3)|] \left\langle \tilde{R}(4_c, 5_c) \right| \\
& \times \frac{\sqrt{\pi}}{\alpha_5} (X^j_{234} - X^j_{15}) \left| U^{(3)}(4_c, 2, 3) \right\rangle_S \left| U^{(2)}(5_c, 1) \right\rangle ,
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
A_4 =& i c^2 \int d1 \cdots d6_c tr[\langle\phi(1)| \langle\phi(2)|] tr[\langle\phi(3)| \langle\phi(4)|] \left\langle \tilde{R}(5_c, 6_c) \right| \\
& \times \frac{\sqrt{\pi}}{\alpha_6} (X^j_{125} - X^j_{346}) \left| U^{(3)}(5_c, 1, 2) \right\rangle_S \left| U^{(3)}(6_c, 3, 4) \right\rangle_S .
\end{aligned}
$$

A common feature of these terms is that there is an intermediate closed string state with an infinitesimal propagation time. The above non-vanishing terms are proportional to the difference between X^j's at the interaction points before and after the intermediate state. These two interaction points do not coincide in general due to a twist of the closed string, over which there is an integration. A possible cancellation of these terms by loop effects in the quantum theory will be discussed later.

4. Lorentz Algebra

The commutator of two Lorentz transformations of the form (2.3) can be computed in a similar way as in the previous section. According to the correct Lorentz algebra this commutator should vanish.

It is easy to verify that the order g^0 term in the commutator is vanish. Most of the remaining terms in the commutator vanish quite similarly to the case of the previous section if we choose the coefficients as in eq.(3.3). However there remain three non-vanishing terms and the Lorentz algebra does not close at the classical level.

The final result for the commutator is

$$
\begin{aligned}
&[\delta(\epsilon_{i-}),\delta(\epsilon_{j-})]\,\langle\phi(1)| = -\epsilon_{i-}\epsilon_{j-}(g^2G_2+g^3G_3+g^4G_4),\\
&[\delta(\epsilon_{i-}),\delta(\epsilon_{j-})]\,\langle\psi(1_c)| = 0,
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
G_2 =&\frac{1}{2}c^2\int d2\cdots d5_c tr[\langle\phi(2)|]\left\langle\tilde{R}(3,1)\right|\left\langle\tilde{R}(4_c,5_c)\right|\\
&\times\frac{\sqrt{\pi}}{\alpha_5}X^i_{42}\frac{\sqrt{\pi}}{\alpha_1}X^j_{53}\left|U^{(2)}(4_c,2)\right\rangle\left|U^{(2)}(5_c,3)\right\rangle-(i\leftrightarrow j),\\
G_3 =&\frac{1}{2}c^2\int d2\cdots d6_c tr[\langle\phi(2)|]\,\langle\phi(3)|\left\langle\tilde{R}(4,1)\right|\left\langle\tilde{R}(5_c,6_c)\right|\\
&\times[\frac{\sqrt{\pi}}{\alpha_6}X^i_{52}\frac{\sqrt{\pi}}{\alpha_1}X^j_{634}\left|U^{(2)}(5_c,2)\right\rangle\left|U^{(3)}(6_c,3,4)\right\rangle_S\\
&+\frac{1}{2}\frac{\sqrt{\pi}}{\alpha_5}X^i_{623}\frac{\sqrt{\pi}}{\alpha_1}X^j_{54}\left|U^{(2)}(5_c,4)\right\rangle\left|U^{(3)}(6_c,2,3)\right\rangle_S]-(i\leftrightarrow j),\\
G_4 =&\frac{1}{4}c^2\int d2\cdots d7_c tr[\langle\phi(2)|\,\langle\phi(3)|]\,\langle\phi(4)|\left\langle\tilde{R}(5,1)\right|\left\langle\tilde{R}(6_c,7_c)\right|\\
&\times\frac{\sqrt{\pi}}{\alpha_7}X^i_{623}\frac{\sqrt{\pi}}{\alpha_1}X^j_{745}\left|U^{(3)}(6_c,2,3)\right\rangle_S\left|U^{(3)}(7_c,4,5)\right\rangle_S-(i\leftrightarrow j).
\end{aligned}
\tag{4.2}
$$

These non-vanishing terms have a similar structure to A_2, A_3, A_4 in eq.(3.4).It is likely that if A's in eq.(3.4) are cancelled by quantum effects, G's in eq.(4.2) are also cancelled.

5. Discussions

In this report we have studied the Lorentz symmetry in the light-cone SFT of the open-closed mixed system. Although the classical light-cone SFT for the pure open and the pure closed systems have the exact Lorentz symmetry, SFT for the open-closed mixed system is not Lorentz covariant at the classical level : the action is not invariant and the algebra does not close. This is due to the open-closed mixed terms in the action and the transformation law. Furthermore there remains a unfixed parameter c in the action and the transformation law.

Since the on-shell amplitudes are known to be Lorentz covariant [1], it is expected that the complete Lorentz symmetry is recovered at the full quantum level. The non-vanishing terms in eqs.(3.4) and (4.2) may be cancelled by "anomalies" in the quantum theory. In this respect the open-closed mixed terms in the action are very similar to the "local counter terms" of the Green-Schwarz anomaly cancellation mechanism [12].

The Lorentz anomaly of the 1-loop two-open string amplitude was computed by Hata[13].This anomaly has the same structure as A_2 in eq.(3.4) and may cancell it. Since A_2 has a factor c^2 while the above anomaly is independent of c, the requrirement of the cancellation between these two terms will fix the value of c. The other non-vanishing terms in the classical theory may be cancelled by loop effects in a similar way.

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One-Loop Dynamics of Four-Dimensional String Theory

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1. Introduction.

Since 1984, string theories have been considered as candidates of the theory in the real world.[1] We know that the considerable progress has been made in the past four years. However, we do not have satisfactory results at present. One of the severest difficulties in string theory, which is also the most interesting point, is the fact that the string theory is formulated in the critical dimensions, namely 10 and 26 for superstring and bosonic string respectively. In order to formulate string theory in our world, which is assumed to be four-dimensional, several approaches have tried. The most traditional one is to employ the Kaluza-Klein approach. In this approach, we assume that some of the transverse dimensions are compactified on the small scale compact space and investigate physics in the remaining dimensions. The key point in applying this method to the string theory is how to determine the compactified space and its dimensions. Unfortunately. we can not find the appropriate way to answer to this question. Therefore we can not investigate the compactification in the systematical way. However, we have several criteria in choosing the compact space, which are the conformal invariance of the theory and the modular invariance of the loop amplitudes. Among the candidates of the compact spaces which we have at present, orbifold models have been attracting the interests of many people.[2] Under the restriction of the modular invariance, model searches have been performed for more than two years and we have several phenomenologically interesting models.[3–6] As we have mentioned above, we have no method to determine which model is the most

appropriate to investigate the phenomenology of the string theory. One of the questions concerning these model buildings is that how the model looks like in our energy scale ($\sim 10^2$GeV), whereas the model building is performed in Planck scale ($\sim 10^{19}$GeV). Since we can not deny the presence of different phases along the development of the energy scale, there is no insurance for a model at the Planck scale goes down to the weak scale without modification.

Because of the reasons mentioned in the previous paragraph, we think the investigation in the dynamics of four-dimensional string theory is very important and we can expect that these studies will supply us with several criteria in the model building. In this report, we will discuss the dynamical features of four-dimensional string theory comparing with the field theory of particle.

2. Features of one-loop amplitude in four-dimensional string theory.

Superstring theories have many features which field theories of particles do not have. Their dynamical aspects are worthy of special notice, for example, anomaly cancellations and ultraviolet finiteness of the amplitudes. In closed superstring theories, the origin of these features is the modular invariance of the amplitudes. On the other hand, when we consider the superstring theories as a candidates of GUT and compare them with the field theory, we will find a question. In the field theory of particles, amplitudes are not finite in the ultraviolet (U.V.) region in general. Therefore, we have to perform renormalizations and we obtain the renormalization group equations (R.G.E.) of physical quantities. We know that R.G.E.s play important roles in unifying the strong, weak and electromagnetic interactions. Since there is no U.V. divergences in superstring theories, we need not perform renormalizations. How about R.G.E. in that case ? If there were no R.G.E., what will become of the unification of interactions in superstring theories ? Recently, J. Minahan showed that we can obtain R.G.E. even if superstrings are finite in the U.V. region.[7] As we have mentioned previously, 1-loop amplitudes of closed strings are modular invariant. Hence, the integration region on the moduli parameter is restricted to the fundamental region. In one-loop

case, a moduli parameter is represented by a complex parameter τ. It is easy to see that the imaginary part of τ corresponds to the proper time in the field theory and $Im\tau \ll 1$ region and $Im\tau \gg 1$ region correspond to ultraviolet (U.V.) and infrared (I.R.) regions, respectively. Since the integration region over τ is restricted to the fundamental region, $-1/2 \leq Re\tau \leq 1/2$, $Im\tau \geq 0$, $|\tau| \geq 1$, the U. V. region is removed from the integration. Therefore, we can understand the closed string theories as those which incorporate U. V. cut off naturally. Let us introduce I. R. cut off $1/\alpha'\mu^2$ for a while, where α' is Regge-slope parameter and μ is a small mass which will be taken a limit zero at the end of the calculation. Then the integration region of $Im\tau$ is given approximately by $\alpha' \leq \alpha' Im\tau \leq 1/\mu^2$. From these considerations, we find that α' plays a role of an inverse square of U. V. cut off, $\alpha' \sim 1/\Lambda^2$, where Λ is a U. V. cut off. It might seem strange to relate Regge slope parameter to the U. V. cut off, since Regge slope parameter has a rigid physical meaning in string theories. But we can interpret the R.G.E.s obtained by identifying $\alpha' \sim 1/\Lambda^2$ as those representing the responses of effective physical quantities against the change of energy scale relative to $1/\sqrt{\alpha'}$(Planck energy scale $\sim 10^{19}$GeV). More convincing formulation of these R.G.E.s are possible, namely renormalization group can be formulated as scale transformations of four-dimensional external momenta (situation is very similar to that of critical phenomena.).

Another interesting point concerning the dynamics of four-dimensional string theory is the possibilities of dynamical symmetry breakings. By investigating the amplitudes including massless scalars as external lines, we can study possibilities of supersymmetry breakings[8,9] and gauge symmetry breakings. Because such scalar states have indices of compactified dimensions, their one-loop propagators carry the informations on the structure of the compactified space and the amplitudes including them depend on the structure of internal space. Consequently, whether supersymmetry and gauge symmetry breakings occur or not depends on the structure of the compactified space. Thus it is important to examine the dependence of one-loop amplitudes on the structure of the internal space.

There is an important thing which we should notice here. In the course of analyzing amplitudes in a four-dimensional string theory, we often come across the logarithmic divergences coming from I.R. region, $\sim ln\mu$, where μ is a I.R. cut off. In this region ($Im\tau \gg 1$), only the massless states of the theory contribute to the amplitudes and the behaviors of such logarithmic divergences are the same as those of the field theory consisting of massless states in the string theory.[10] In field theory, there are theorems of Bloch-Nordick and Kinoshita-Lee-Naurenberg, which state that such I.R. divergences are canceled out at the stage of calculating transition probabilities. Therefore, the appearance of these divergences in amplitudes does not contradict the finiteness of superstring theories. However, it is important to notice that we can obtain $ln\alpha'$ dependence of the amplitude from these logarithmic divergences, since the string theory has only one dimensionful parameter, α' . By the use of such $ln\alpha'$ dependences of the amplitudes, the R.G.E. are derived by the previously mentioned identification.

To illustrate the relation between dynamics in four-dimensional superstring theory and those in field theory, let us investigate the logarithmic divergences in four-dim string theory roughly. One-loop amplitude of string in d-dimensional Minkowski space is given as follows,

$$A^{(N)} \sim \int \prod_i d\nu_i \underbrace{(Im\tau)^{-d/2}}_{\Uparrow \atop \int d^d p \, e^{-Im\tau\, p^2}} Z(\tau) \langle V_1 V_2 \cdots V_N \rangle_\tau ,$$

where ν are the positions of external lines and Z is a partition function. In four dimensions ($d = 4$), the measure in the above expression becomes,

$$\prod_i d^2\nu_i (Im\tau)^{-2} \sim \prod_i d\, Im\nu_i \frac{1}{(Im\tau)^2} .$$

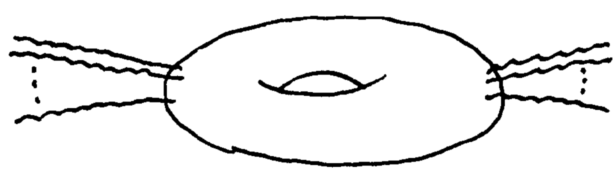

Fig. 1.

Therefore we find the logarithmic divergences appear from the configurations shown in Fig. 1, which will be called a two-punctured tours. If we performed detailed calculations in four-dimensional superstring theory, we obtain the following logarithmic divergences,

$$A \sim \ln(\alpha' k_i \cdot k_j)$$

where k_i denotes momentum of i-th external leg. If we recall the previous discussions, we know that these divergences correspond to those absorbed through the wave function renormalization in the effective field theory of superstring. It is well-known that in supersymmetric field theory there is a theorem called non-renormalization theorem which states that only wave function renormalization have to be performed in such field theory. Hence, the phenomena we have seen here in four-dimensional superstring is consistent with field theory.

3. Questions and results.

In this section, we will present problems in four-dimensional superstring theory and the answers we obtained. For the detailed discussions, please see the paper Ref. [11].

1). Can renormalization group equations derived in the string theory ?

We can obtain R.G.E.s. For the R.G.E.s of supersymmetric ($N = 1$) theory, Beta functions are shown to be the same as those of corresponding field theory.

2). How the structures of the compactified dimensions appear in amplitudes and effective actions ?

As for the model which we have investigated (N=1 supersymmetric orbifold compactification of heterotic string), the structures of internal spaces appear in two ways. The structure of six-dimensional orbifold are reflected in amplitudes through the zero modes of one-loop propagators of the twisted fields and that of sixteen-dimensional internal space through the indices of the representations to which massless states of the theory belong.

3). Is it possible to investigate symmetry breakings ?
Yes, we can encounter Fayet-Illiopoulis model like situations in supersymmetric models. We can obtain one-loop effective potential of massless scalar bosons and discuss the possibilities of symmetry breakings (both gauge symmetry and supersymmetry).
4). Is it possible to obtain Coleman-Weinberg like effective potential and derive different energy scale which is far below the Planck scale ?
In the simplest part of the amplitudes, we can perform the summation to all orders in external lines, in which we can see several interesting features characteristic to string theory. However, the full evaluation of the amplitude to all orders is very hard to perform (see Appendix).

4. Remarks.

In this report, we have discussed the dynamical aspects of four-dimensional string theory. These investigations are important to connect the string theory with phenomena in our energy scale. We also expect that they will supply us with new criteria to search for phenomenologically satisfying models. However, we think that the perturbative method used in this report is not enough to obtain full understandings and we have to study non-perturbative effects in string theory in order to make string theory more realistic.

APPENDIX

In this appendix we will illustrate the derivation of the Coleman-Weinberg type effective potential briefly. We have to explain the reason to have interests in this investigation. Since superstring model buildings are performed in the energy scale around the Planck mass, we must invent the way to explain why small scale like that of electro-weak appear naturally in the theory. In general, this work is hard. Conventional ways to do this is to consider the non-perturbative effects, instantons, or investigate Coleman-Weinberg type effective potential.[12] As we

do not have methods to probe the non-perturbative effects in string theory, the most favorable one at present is to choose the latter.

Let us consider the amplitude with 2M complex scalar bosons as external lines. The effective potential of these scalar bosons are obtained from the one-particle-irreducible non-derivative parts of the amplitude. Non-derivative terms mean that they are 0-th order in external momenta and one-particle-irreducible terms are obtained by extracting pole terms from the amplitude. It is non-trivial but we can see that non-derivative terms appear from the configurations shown in Fig.2, where all external lines make pairs with its conjugate. In each pair, one-particle-reducible poles appear which we have to subtract to obtain correct answer.

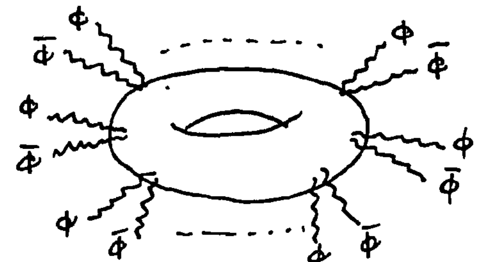

Fig. 2.

When two-scalar boson emission vertices become close, we obtain the following asymptotic forms,

$$
\begin{aligned}
&\int_{|\eta|\ll 1} d^2\eta \, V_1^I(\eta+v)\, \bar{V}_2^J(v) \,\Big|_{(k.h)(\alpha.\beta)\,\tilde{p}} \\
&\sim -\frac{1}{4}\pi\, \delta_{\tilde{k}_1+\tilde{k}_2 .0}\, \delta^{IJ} \\
&\quad -\frac{1}{4}\pi\, \delta_{\tilde{k}_1\cdot\tilde{k}_2 ,-1}\, \delta^{IJ}\, C^{(0)I}_{(k.h)(\alpha.\beta)}(\tau) \\
&\quad -\frac{1}{4}\pi\, \delta_{\tilde{k}_1+\tilde{k}_2 .0}\, \delta^{IJ}\, (-2\pi i\, \tilde{p}\cdot\tilde{k}_1)\, C^{(0)I}_{(k.h)(\alpha.\beta)}(\tau) \\
&\quad +\frac{1}{4}\pi\, \delta_{\tilde{k}_1\cdot\tilde{k}_2 .0}\, \delta^{IJ}\, D^{(0)I}_{(k.h)}(\tau) \;+\; \cdots\cdots
\end{aligned}
\qquad (a-1)\,,
$$

where (k,h), (α,β) denote twist sector and spin structures and $C^{(0)I}$, $D^{(0)I}$ are zero-modes of fermionic and bosonic one-loop propagators, respectively. The full evaluation of the contributions from these terms are hard to carry out at the present technique. Therefore we will evaluate the contributions of first a few

terms. Let us investigate the contribution of the first term in the above equation as an example. After summing to all orders in external lines,

$$-\mathcal{L}_{eff} = 2^{-7}\,\Delta_{susy}$$
$$\times\Big\{-\frac{1}{2\pi}\frac{g^2}{V/(2\alpha')^4}\,tr(\phi\bar\Phi)$$
$$-\frac{1}{2}\left(\frac{1}{2}+\gamma+\ln t\right)\frac{1}{\pi^2}\Big[\frac{g^2}{V/(2\alpha')^4}\,tr(\phi\bar\Phi)\Big]^2$$
$$-\Big[1-\frac{1}{\pi^2}\Big(\frac{g^2}{V/(2\alpha')^4}\,tr(\phi\bar\Phi)\Big)^2\Big]\ln\Big(1+\frac{1}{\pi}\frac{g^2}{V/(2\alpha')^4}\,tr(\phi\bar\Phi)\Big)\Big\},$$

Δ_{susy} = # boson − # fermion in massless sector,

where Δ_{susy} is a parameter which indicates the order of supersymmetry breaking, t is a scale parameter relative to the Planck mass, $1/\sqrt{\alpha'}$, and V is a volume of the compactified space. γ is Eular constant. Due to Δ_{susy} this result vanishes if the model has a tree-level supersymmetry. In such case we have to evaluate the contribution of the second term in eq.(a-1). The evaluation is performed similarly to eq.(a-2), which we do not show here. Several notices concerning to the above results are in order. The appearance of $ln(t)$ corresponds to coupling constant renormalization. The third term in eq.(a-2) has the same structure as that obtained in field theory and inverse of the internal volume V plays a role of the renormalization mass in usual field theory approach. Physical consequences of the above result is under investigation.

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REDUCING THE RANK OF GAUGE GROUPS IN ORBIFOLD COMPACTIFICATION

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Heterotic string theories are promising candidate for unified theories of all known interactions.[1] However if they are to make contact with the real world it is necessary to understand how the enormous symmetry of these theories is broken down to the symmetry which we observe at low energies. A powerful method to implement symmetry breaking in string theories is to consider string propagation on orbifold.[2] In the standard Z orbifold model the gauge group in the $E_8 \times E_8$ heterotic string theory breaks down to $E_6 \times SU(3) \times E_8$ and there are 36 chiral generations. An effective method to break further the gauge symmetry and to reduce the number of generations is an application of the Wilson-line mechanism[3] in the framework of orbifold compactification.[4-5]

In ref.(5) we have made a systematic approach to this problem and given a complete classification of possible gauge symmetry breaking by Wilson lines on the standard Z orbifold. There we have embedded the action of the Z_3 discrete group in the internal degrees of freedom in an Abelian way so that the rank of the gauge group could not be reduced. String propagation on orbifold is expressed by imposing twisted boundary conditions on string variables. In this talk we introduce most general twisted boundary conditions on fermionic string variables and show that a non-Abelian embedding is possible when background gauge field is introduced on orbifold. This leads to reduction of the rank of the gauge group.

In what follows we focus on the heterotic string in the fermionic formulation. We will work in the light-cone gauge and use the NSR formulation to describe the superstring part of the heterotic string. We describe the bosonic variables in the six compactified dimensions in the complex notation as z^{α} ($\alpha=1,2,3$). Then the string propagation on the Z orbifold is described by imposing the following boundary conditions on the bosonic variables:

$$z^{\alpha}(\sigma_1+\pi,\sigma_2) = \exp(2\pi ik/3)\, z^{\alpha}(\sigma_1,\sigma_2) \quad , \tag{1}$$

$$z^{\alpha}(\sigma_1,\sigma_2+\pi) = \exp(2\pi ih/3)\, z^{\alpha}(\sigma_1,\sigma_2) \quad , \tag{2}$$

where $h,k=0,\pm 1$. The center of mass coordinate of z^{α} must be at one of the fixed points for the Z_3 transformation. The group action on the right-moving NSR fermions λ^a ($a=1,\cdots,4$) is determined by the requirement of preserving world sheet supersymmetry and the boundary conditions amount to

$$\lambda^a(\sigma_1+\pi,\sigma_2) = -(-1)^n \exp(2\pi ik\xi^a)\, \lambda^a(\sigma_1,\sigma_2) \quad , \tag{3}$$

$$\lambda^a(\sigma_1,\sigma_2+\pi) = -(-1)^m \exp(2\pi ih\xi^a)\, \lambda^a(\sigma_1,\sigma_2) \quad . \tag{4}$$

$$3\,\xi^a = \text{integer}$$

where $n,m=0,1$ specifies the spin structure.

The internal degrees of freedom of the heterotic string is described by two sets of 16 fermionic coordinates ψ^i and $\tilde{\psi}^i$ ($i=1,\cdots,16$) which transform as the vectors of $SO(16)\times SO(16)$. If we introduce background gauge fields A_{μ}^{ij} and $\tilde{A}_{\mu}^{ij}$ which transform as the adjoint representation of $SO(16)\times SO(16)$, the left-moving gauge fermions ψ^i, $\tilde{\psi}^i$ have a coupling with A_{μ}^{ij}, $\tilde{A}_{\mu}^{ij}$. Under the group action a fixed point of Z_3 must be accompanied by a shift on the lattice Γ_6 defining the six torus T^6, since

different fixed points correspond to the same element of the point group. Under this shift fermions ψ^i, $\tilde{\psi}^i$ pick up the following phase factor due to gauge invariance:

$$\Theta_{f,k} = \exp[2\pi i(pa_k + qb_k + rc_k)] \ , \tag{5}$$

where the background-field contribution is written by

$$2\pi\, a_k = (A_\mu^{ij} T_{ij} + \tilde{A}_\mu^{ij} T_{ij}) e_k^1 \ , \tag{6}$$

and b_k, c_k are given by replacing e_k^1 by e_k^2 and e_k^3, respectively. Here we denote the unit basis vectors of the Γ_6 lattice as e_1^α, e_2^α and T_{ij}, $\tilde{T}_{ij}$ are generators of $SO(16)\times SO(16)$. The fixed point on the Z orbifold can be expressed by $(p,q,r)d$ with $d=\sqrt{1/3}\ e^{i\pi/6}$ and $p,q,r=0, \pm 1$ modulo 3. Note that we can introduce at most three independent Wilson lines a_k, b_k, c_k.

Embedding of the Z_3 group action into the internal degrees of freedom is determined by giving the rotation matrix Ω for the fermionic coordinates ψ^i and $\tilde{\psi}^i$. By choosing appropreate basis of $SO(16)\times SO(16)$, we can always diagonalize the Ω so that

$$\Omega = \exp[2\pi i\, \zeta^\ell H_\ell] \ , \quad 3\zeta^\ell = \text{integer} \tag{7}$$

which obeys $\Omega^3 = 1$ and H_ℓ ($\ell=1,\cdots,16$) is the Cartan subalgebra of $SO(16)\times SO(16)$ defined by $H_1 = T_{12}$, $H_2 = T_{34}$, etc. Thus the boundary conditions for the gauge fermions are given by

$$\psi(\sigma_1+\pi, \sigma_2) = -(-1)^n\, \Omega^k\, \Theta_{f,k}\, \psi(\sigma_1,\sigma_2) \ , \tag{8}$$

$$\psi(\sigma_1, \sigma_2+\pi) = -(-1)^m\, \Omega^h\, \Theta_{f,h}\, \psi(\sigma_1,\sigma_2) \ , \tag{9}$$

and matrix notation is understood here.

Under the $2\pi/3$ rotation the basis vectors e_1^α and e_2^α transform as

$e_1^\alpha \to e_2^\alpha - e_1^\alpha$ and $e_2^\alpha \to -e_1^\alpha$. Correspondingly the Wilson lines must transform as

$$\Omega^{-1}\,\Theta_{f,1}\,\Omega = \Theta_{f,2-1} \quad , \tag{10}$$

$$\Omega^{-1}\,\Theta_{f,2}\,\Omega = \Theta_{f,-1} \quad , \tag{11}$$

where $\Theta_{f,2-1}$ stands for the Wilson-line matrix obtained by replacing a_k by a_2-a_1, etc., in $\Theta_{f,k}$. If the Wilson lines obey (10) and (11) it is shown that

$$[\,\Omega^k\,\Theta_{f,k},\,\Omega^h\,\Theta_{f,h}] = 0 \;, \tag{12}$$

which means that the Z_3 group is Abelian. Notice, However, that the Wilson lines associated with different fixed points, in general, are not commutable and we can use this non-Abelian nature of Wilson lines to reduce the rank of the gauge group. In the previous method adopted by several authors[4-6] the Wilson lines are restricted to the Cartan subalgebra. When the background gauge field is present on orbifold, however, this commutability is not always satisfied due to the non-Abelian nature of the gauge fields.

When background gauge fields have only components corresponding to the Cartan subalgebra, the Wilson-line matrix $\Theta_{f,k}$ and the rotation matrix Ω which is chosen to be diagonal are commutable. Then the conditions on Wilson lines (10) and (11) turn out to be

$$a_2 = 2\,a_1 \;,\quad b_2 = 2\,b_1 \;,\quad c_2 = 2\,c_1 \tag{13}$$

$$3\,(a_k,\,b_k,\,c_k) = \text{integers} \quad \text{mod } 3 \tag{14}$$

When some of the Wilson lines are in the Cartan subalgebra and others are not, the above relation holds for those Wilson lines in the Cartan

subalgebra.

The Wilson-line matrices $\Theta_{f,k}$ and the rotation matrix Ω are not commutable in general. Commutability of these matrices depends on the choice of the background gauge fields A_μ^{ij}, $\tilde{A}_\mu^{ij}$ introduced on the Z orbifold. First we consider the case where all components of the background fields are in the Cartan subalgebra.

(i) Abelian embedding: This case is equivarent to the embedding of the space group by shifts in the $E_8 \times E_8$ lattice.[4] All the Wilson-line matrices and the rotation matrix are commutable and are diagonalized simultaneously in the form;

$$\Omega^k \Theta_{f,k} = \exp[2\pi i k v_f^\ell H_\ell] \ , \ k = 0, \pm 1 \tag{15}$$

and

$$v_f^\ell = \zeta^\ell + (pa_1 + qb_1 + rc_1)^\ell \ . \tag{16}$$

The boundary conditions (8) and (9) for the gauge fermions are characterized by the vector v_f^ℓ up to an $E_8 \times E_8$ lattice vector.

The condition of modular invariance in the presence of the background gauge fields is given by the level matching condition.[7] Taking into account (8), (9) with (15), (16) this reads

$$3 \sum_a \xi^a = 0 \mod 2 \tag{17}$$

$$3 \sum_\ell v_f^\ell = 0 \mod 2 \tag{18}$$

$$3 \{ \sum_\ell (v_f^\ell)^2 - \sum_a (\xi^a)^2 \} = 0 \mod 2 \tag{19}$$

Furthermore the string states on the orbifold must be invariant under the Z_3 group. The group invariant condition has been obtained in ref.(5) by constructing the projection operator onto the invariant subspace of the string Hilbert space. The condition for the k-twisted

sector is given by

$$(V^\ell + kv_f^\ell/2)v_f^\ell + (K^a - k\xi^a/2)\xi^a + m_k = 0 \mod 1 \tag{20}$$

where V^ℓ is the vector in the $E_8 \times E_8$ root lattice and K^a is the vector in the SO(8) vector or spinor lattice. The m_k is the eigenvalue of the operator $\hat{m}_k$, in terms of which the twist operator $\hat{g}_k$ for z^α is written by $\hat{g}_k = \exp(2\pi i\hat{m}_k)$. The twist operator $\hat{g}_k$ acts on the string variable z^α in the k-twisted sector as

$$\hat{g}_k \, z^\alpha \, \hat{g}_k^{-1} = e^{2\pi i/3} \, z^\alpha \quad . \tag{21}$$

The gauge symmetry and massless spectra are obtained by taking into account the conditions of modular invariance (17)–(19) and the group invariant condition (20). Detailed discussions and explicit examples are found in ref.(5) so that we just give here a general prescription to examine the possible symmetry breaking. The massless gauge bosons are obtained in the untwisted sector by the combination with the right-moving ground states with helicity ± 1 in 8_v of SO(8), for which $K^a\xi^a = 0$. Then the group invariant condition (20) implies

$$V^\ell \, v_f^\ell = 0 \mod 1 , \tag{22}$$

and the symmetry corresponding to the root vector V^ℓ obeying (22) for all v_f^ℓ remains unbroken. Massless fermions in the untwisted sector are combined with the right-moving ground states with helicity 1/2 in 8_s of SO(8), for which $K^a\xi^a = 2/3 \mod 1$. The group invariant condition (20) reads

$$V^\ell \, v_f^\ell = 1/3 \mod 1 , \tag{23}$$

and the states obeying this condition for all v_f^ℓ survive as massless

fermions. Massless fermions in the k=1 twisted sector must obey the following massless conditions:

$$\frac{1}{2}(V^{\ell}+v_f^{\ell})^2+N_L-\frac{2}{3}=0\ , \tag{24}$$

$$\frac{1}{2}(K^a-\xi^a)^2+N_R-\frac{1}{6}=0\ , \tag{25}$$

where N_L and N_R are the occupation numbers for the left- and right-moving oscillators of z^{α}.

(ii) Non-Abelian embedding: Now we are ready to consider the case where the background gauge fields have the components other than the Cartan subalgebra. In this case some or all of the Wilson lines are not commutable with each other or with the rotation matrix Ω. Those Wilson lines are not diagonalized simultaneously. On the other hand in order to quantize string states on orbifold we need to diagonalize the boundary conditions (8) and (9). Diagonalization of the boundary conditions is performed at each fixed point f=(p,q,r) as follows;

$$\Omega^k\,\Theta_{f,k}=U_f^{-1}\exp[2\pi i k v_f^{\ell}H_{\ell}]\,U_f\ \ , \tag{26}$$

where transformation matrix U_f belongs to $SO(16)\times SO(16)$ and eigenvalues v_f^{ℓ} must obey $3\,v_f^{\ell}=0$ mod 1 due to Z_3 invariance.

Modular invariance of the theory is guaranteed by imposing the level matching condition for the eigenvalues v_f^{ℓ} with the same form as (17)-(19) of the Abelian case. In the case of the non-Abelian embedding, however, the string states associated with each fixed point are expressed in the different basis which diagonalizes the corresponding boundary conditions. The transformation matrix U_f may be different for different fixed point f. The string Hilbert space must be invariant under the Z_3 group action. Embedding of the group action in the internal degrees of freedom is done by giving the matrices (26), which are

commutable at the same fixed point f as shown by (12) but may not be commutable between different fixed points. The group invariant condition is given now by (20) for the eigen vectors v_f^{ℓ} with the additional condition that the string Hilbert space should be invariant under U_f;

$$U_f\, E_V\, U_f^{-1} = E_V \quad , \tag{27}$$

$$U_f\, H_\ell\, U_f^{-1} = H_\ell \quad , \tag{28}$$

where E_V is the generator corresponding to the root V of $E_8 \times E_8$ and H_ℓ is the one in the Cartan subalgebra.

Gauge symmetry is determined by the condition (22) supplemented with the U_f invariance, (27) and (28). The U(1) factor associated with the Cartan subalgebra H_ℓ which does not obey (28) disappears now and the rank of the unbroken subgroup is reduced. The background gauge fields transform as (120,1)+(1,120) of SO(16) × SO(16). Since electroweak symmetry must be unbroken the possible Wilson lines are restricted considerably. We will focus on the first SO(16) where electroweak symmetry is supposed to be contained. Since $\underline{120}$ = (45,1)+(10,6)+(1,15) of SO(10) × SU(4) and SO(10) $\supset$ $SU(3)_c \times SU(2)_L \times SU(2)_R$, the electroweak-symmetric Wilson lines must be chosen from (1,1,3,1)+(1,1,1,15) of $SU(3)_c \times SU(2)_L \times SU(2)_R \times SU(4)$. In particular, the $v_f^{\ell} H_\ell$ in (26) must lie in the Cartan subalgebra of $SU(2)_R \times SU(4)$ and the transformation U_f is also in $SU(2)_R \times SU(4)$.

Massless spectra of chiral fermions in the untwisted sector are determined by the group invariant condition (23) and the U_f invariant conditions (27) and (28). The number of massless fermions will be reduced as compared with the case of the Abelian embedding by the newly imposed U_f invariance. In the twisted sector massless spectra are determined by the massless condition (24). The number of states is the

same as the case of the Abelian embedding but their group representation is determined by the symmetry of the non-Abelian embedding.

Now we summarize the procedure to obtain the lower-rank gauge groups by the use of non-Abelian Wilson lines. The unbroken gauge group is essentially determined by the eigen vector v_f^ℓ which should obey the level-matching conditions (18) and (19) in order to respect modular invariance of the theory. The eigen vector v_f^ℓ must be chosen also in such a way that the $v_f^\ell H_\ell$ is in the Cartan subalgebra of the subgroup of $SU(2)_R \times SU(4)$ to preserve electroweak symmetry. Then the gauge symmetry is determined by the conditions (22), (27) and (28). In particular, the condition (28) plays the role to reduce the rank of the group. In a practical application we do not need to introduce explicit form of the non-Abelian Wilson lines (5). We start from the introduction of desired eigen vectors v_f^ℓ which are supposed to be obtained by diagonalization of (26) with the appropriate transformation matrix U_f.

In order to reduce the rank by one we can use the Wilson lines which transform as $\underline{3}$ of $SU(2)_R$ or $SU(2) \subset SU(4)$. To reduce the rank by two, possible way is to use the Wilson lines from $SU(2)_R \times SU(2)$ or SU(3) in SU(4). If we use the Wilson lines which transform as SU(4) or $SU(2)_R$ SU(3), the rank is reduced by three. Finally the rank is reduced by four when the Wilson lines with full symmetry of $SU(2)_R \times SU(4)$ are used. In the table we give the possible lower-rank gauge groups obtained by our method. Massless fermions corresponding to v_f^ℓ given in ref.(8), where more detailed discussions have been given, are also listed there.

Finally a brief comment is in order. We should notice that the off-diagonal form of the background gauge fields only plays the role to reduce the rank of the gauge group as well as the number of massless particles in the untwisted sector. The non-Abelian part of the symmetry

gauge groups	Wilson lines	massless fermions
$E_6 \times U(1) \times SO(14)' \times U(1)'$	$SU(2)$	$12\ \underline{27} + 81\ \underline{1}$
$SU(6) \times SU(3) \times SO(14)' \times U(1)'$	$SU(2)_R$	$3(15,3) + 9(15,1) + 36(\bar{6},1) + 45(1,\bar{3})$
$E_6 \times SO(14)' \times U(1)'$	$SU(3)$	$3\ \underline{27} + 3\ \overline{\underline{27}} + 54\ \underline{1}$
$SU(6) \times U(1) \times E_7' \times U(1)'$	$SU(2)_R \times SU(2)$	$9\ \underline{15} + 36\ \bar{\underline{6}} + 18\ \underline{6} + 81\ \underline{1}$
$SO(10) \times SO(14)' \times U(1)'$	$SU(4)$	$3\ \underline{16} + 6\ \underline{10} + 3\ \overline{\underline{10}} + 36\ \underline{1}$
$SU(6) \times E_7' \times U(1)'$	$SU(2)_R \times SU(3)$	$3\ \underline{15} + 3\ \overline{\underline{15}} + 24\ \underline{6} + 24\ \bar{\underline{6}} + 54\ \underline{1}$
$SU(4)_c \times SU(2)_L \times E_7' \times U(1)'$	$SU(2)_R \times SU(4)$	$3(4,2) + 12(\bar{4},1) + 6(4,1) + 30(1,2) + 6(6,1) + 3(\bar{6},1) + 36(1,1)$
$SU(3)_c \times SU(2)_L \times U(1)_Y \times E_7' \times U(1)'$	$SU(2)_R \times SU(4) \times U(1)_Y$	$6(3,2) + 3(\bar{3},2) + 33(1,2) + 9(\bar{3},1) + 3(3,1) + 36\ (1,1)$

and massless spectra of the twisted sectors are determined essentially by the eigen vector v_f^{ℓ}. Different Wilson lines with the same v_f^{ℓ} give the same symmetry and massless spectra. Since the U_f depends on the continuous parameters corresponding to the Wilson lines, infinitely many Wilson lines are associated with the same eigenvalue v_f^{ℓ}. The similar situation has been found in ref.(9) for another method of non-Abelian embedding with the use of the Weyl rotations of the $E_8 \times E_8$ lattice.

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CLASSIFICATIONS OF Z_N ORBIFOLD MODELS*

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1. Introduction

The $E_8 \times E_8$ heterotic string thoery[1] has drawn much attention as unified thoery of all known interraction. But that is ten dimensoinal thoery and has unrealistic large gauge group $E_8 \times E_8$ and no matter. The toroidal compactification, which is the simplest method to reduce space-time dimensoins, however leads to 4-dimensoinal theories with N=4 space-time supersymmetry.

One of the most intersting ideas to give 4-dim theory with N=1 space-time supersymmetry, more realistic gauge group and matters is Z_N orbifold compactification, which is simpller extension of the toroidal compactification. We divide an extra 6-dimensional torus T^6 by a discrete rotation to get the Z_N orbifold.[2] It has been known that orders N of the discrete rotations preserving N=1 space-time supersymmetry are 3,4,6,7,8 and 12.[3]

Of them, the Z_3 orbifold models have been been studying in detail and classified into four types. Further, as the starting point to lead to the *real world*, the given four types of models have been been investigating with several mechanism. A recent paper[4] shows that Z_7 orbifold models are given with the same construction as one of Z_3 orbifold models. The other Z_N orbifold models can be given in the same way as the Z_3, Z_7 orbifold models. Here, among them, Z_4, Z_6 and Z_7 orbifold models are classified systematically.

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2. Z_N Orbifold

Let us start with the $E_8 \times E_8$ heterotic string in the bosonized form. That is the thoery with 10-dim supersymmetric right movers and 26-dim bosonic left movers. The mass formula of right movers is

$$\frac{1}{8}(m_R)^2 = \frac{1}{2}\sum_{j=3}^{8}(p_R^j)^2 + \frac{1}{2}\sum_{t=1}^{4}(p^t)^2 + N_R - \frac{1}{2},$$

and the left mass formula is

$$\frac{1}{8}(m_L)^2 = \frac{1}{2}\sum_{j=3}^{8}(p_L^j)^2 + \frac{1}{2}\sum_{I=1}^{16}(P^I)^2 + N_L - 1,$$

where N is the number operator and p^t's (t=1,...,4) and P^I's (I=1,...,16) are on an $SO(8)$ weight lattice and an $E_8 \times E_8$ lattice, respectively. The Z_N orbifold is obtained with the division of a 6-dim torus by a discrete rotation, or R^6 by a space group S, which consists of discrete rotations θ and discrete translations e^a, where θ should be automorphgism of a lattice spanned by e's. With this division, simultaneously, the $SO(8)$ weight lattice and the $E_8 \times E_8$ lattice must be moded, *i.e.* the discrete rotation θ is associated with shifts v^t and V^I on the $SO(8)$ weight lattice and the $E_8 \times E_8$ lattice, respectively, so that we can get a model with N=1 spce-time supersymmetry, smaller gauge group and some matters. In this embedding algebraic requirements and the modular invariance restrict v^t and V^I as

$$N\sum_{t=1}^{4} v^t = N\sum_{I=1}^{8} V^I = N\sum_{I=9}^{16} V^I = 0 \qquad \text{mod } 2,$$

$$N\sum_{t=1}^{4}(v^t)^2 = N\sum_{I=1}^{16}(V^I)^2 \qquad \text{mod } 2.$$

Under complex basis, we can always diagonalize θ, *i.e.*

$$\theta = diag[exp(2\pi i \eta^a)].$$

Discrete rotation to preserve N=1 space-time supersymmetry have been known. The η's preserving N=1 space-time supersymmetry and the e are in Tab.1. Note

that two types of Z_6, Z_8 and Z_{12} orbifolds exist. (In the following, Z_6 orbifold with $\theta = 1/6(1,1,-2)$ and $\theta = 1/6(1,2,-3)$ are called Z_6–I,Z_6–II orbifold respectivelly.) Of them, Z_3 orbifold models are studied in detail and clssified into 4 types ,which are a model with a gauge group $E_6 \times SU(3) \times E_8$, $E_6 \times SU(3) \times E_6 \times SU(3)$ model, $E_7 \times U(1) \times SO(14) \times U(1)$ one and $SU(9) \times SO(9) \times U(1)$ one. For the others, each Z_N orbifold model by only *standard embedding* has been given.[5] But with some extension, constructions of Z_N orbifold models are the same as one of Z_3 orbifold models. Here we classify Z_4, Z_6 and Z_7 orbifold models systematically.

On the orbifolds by the above construction, exist two types of closed string, which has been closed even on torus before the torus is divided by the discrete rotation, and its mass formula is the same as the previous one. The other is twisted string, whose oscilator modes of 6-dimensional parts are fractal and zero intercepts c_k are given by

$$c_k = \frac{1}{2}\sum_{a=1}^{4}\left(\mid k\eta^a \mid - Int(\mid k\eta^a \mid)\right)\left(1 - \mid k\eta^a \mid + Int(\mid k\eta^a \mid)\right).$$

In the result, mass formulae for k-twisted strings are

$$\frac{1}{8}(m_R^{(k)})^2 = \frac{1}{2}\sum_{j=3}^{8}\frac{1}{2}\sum_{t=1}^{4}(p^t + kv^t)^2 + N_R^{(k)} - \frac{1}{2} + c_k,$$

$$\frac{1}{8}(m_L^{(k)})^2 = \frac{1}{2}\sum_{j=3}^{8}\frac{1}{2}\sum_{I=1}^{16}(P^I + kV^I)^2 + N_L^{(k)} - 1 + c_k,$$

Massless states can be given from the above mass formulae of the two types string. Futher, physical states are selected by the generalised GSO projection.[5][6] Here we don't study the projection and matters in detail. Of massless physical states, gauge bosons are states with P^I, p^t satisfying $\sum P^I V^I, \sum p^t v^t \in Z$.

3. Example and Classification

Here, let us demonstrate the above construction of Z_N orbifold models and take a Z_7 orbifold model with shifts v^i=1/7(1,2,-3) and V^I=1/7(2,2,2,0,0,0,0,0), 1/7(1,1,0,0,0,0,0,0) as an example. We use a representation where E_8 roots are $(0,\ldots,0,\pm1,0,\ldots,0,\pm1,0,\ldots0)$ and $(\pm\frac{1}{2},\ldots,\pm\frac{1}{2})$. As said in the previous section, gauge bosons are states with P^I satisfying $\sum P^I V^I \in Z$ in the $E_8 \times E_8$ roots. In this case, the P^I's($I = 1 \sim 8$) with $P^J = 0(J = 9 \sim 16)$ are

$$(0,0,0,\ldots,\pm1,\ldots,\pm1,\ldots)$$

$\pm(1,-1,0,0,\ldots,0), \pm(1,0,-1,0,\ldots,0), \pm(0,1,-1,0,\ldots,0)$,

which are $SO(10) \times SU(3) \times U(1)$ roots. In the same way, P^I's($I = 19 \sim 26$) (with $P^J = 0(J = 11 \sim 18)$) satisfying the above condition are $E_7 \times U(1)$ roots. In the result, agauge group of this model is $SO(10) \times SU(3) \times U(1) \times E_7 \times U(1)$. Further, using mass formulae and the generalized GSO projection, matters are given and that in each fixed point are

$(16_c,\bar{3};1),(10_v,3;1),(16_s,3;1),(1,1;56),(1,1;1)$	in	k=0	sector
$(16_c,1;1),(1,3;1),(1,1;1)$	in	k=1	sector
$(16_c,1;1),(1,3;1),(1,1;1)$	in	k=2	sector
$(10_v,1;1),(1,3;1),(1,3;1)$	in	k=3	sector

under the above gauge group.

If we do the same thing for all possible shifts V^I's, we get all possible Z_7 orbifold models. Gauge groups broken from one E_8 of $E_8 \times E_8$ by this procedure are in Tab.3 and a complete classification of Z_7 orbifold models are in Tab.4. But matters are omitted in Tab.4.*

In the same way, Z_4 and Z_6 orbifold models are given and classifications of these are in Tab.2,5 and 6. In these tables matters are omitted, too.†

* see Ref [7]
† see Ref [8]

4. conclusion

We have discussed Z_4, Z_6 and Z_7 orbifold models and classified them. Finally we have got ten Z_4 models,fifty four $Z_6 - I$ ones,fifty six $Z_6 - II$ ones and thirty nine Z_7 ones. But given models are still unrealistic, *i.e.* they have large gauge groups and many matters. To get a more realistic model from given models, we need some mechanism as well as Z_3 orbifold models.

Further, Z_8 and Z_{12} orbifold models can be given by the same construction as the above. These are been investgating.

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Table 1. Z_N orbifolds

	η	Lattice
Z_3	1/3(1,1,-2)	$SU(3)^3$
Z_4	1/4(1,1,-2)	$SU(4)^2$
Z_6	1/6(1,1,-2)	$SU(3) \times G_2^2$
Z_6	1/6(1,2,-3)	$SU(6) \times SU(2)$
		$SU(3) \times SO(8)$
Z_7	1/7(1,2,-3)	$SU(7)$

	η	Lattice
Z_8	1/8(1,2,-3)	$SO(5) \times SO(8)$
		$SO(5) \times SO(9)$
Z_8	1/8(1,3,-4)	$SO84) \times SO(8)$
		E_6
Z_{12}	1/12(1,4,-5)	$SU(3) \times F_4$
		$SU(3) \times SO(8)$
Z_{12}	1/12(1,5,-6)	$SO(4) \times F_4$

Table 2. Gauge Groups in Z_4 orbifold models

No.	Gauge Group (U_1's are omitted)
1	$(E_6, SU_2; E_8)$
2	$(E_6, SU_2; E_7, SU_2)$
3	$(SO_{14}; E_7)$
4	$(SO_{12}, SU_2; SO_{14})$
5	$(SO_{10}, SU_4; E_7)$
6	$(SO_{10}, SU_4; SO_{12}, SU_2)$
7	$(SU_8, SU_2; E_8)$
8	$(SU_8, SU_2; E_7, SU_2)$
9	$(SU_8; E_6, SU_2)$
10	$(SU_8; SU_8, SU_2)$

Table 3. Gauge groups in Z_3, Z_4, Z_6 and Z_7 orbifold models

No.	Gauge Group	Z_3	Z_4	Z_6	Z_7
0	E_8	*	*	*	*
1	$E_7 \times SU_2$		*	*	
2	$E_7 \times U_1$	*	*	*	*
3	$E_6 \times SU_3$	*		*	
4	$E_6 \times SU_2 \times U_1$		*	*	*
5	$E_6 \times U_1^2$			*	*
6	$SO_{14} \times U_1$	*	*	*	*
7	$SO_{12} \times SU_2 \times U_1$		*	*	
8	$SO_{12} \times U_1^2$			*	*
9	$SO_{10} \times SU_4$		*		
10	$SO_{10} \times SU_3 \times U_1$			*	*
11	$SO_{10} \times SU_2^2 \times U_1$			*	
12	$SO_{10} \times SU_2 \times U_1^2$			*	*
13	$SO_8 \times SU_4U_1$			*	
14	$SO_8 \times SU_3 \times U_1^2$				*
15	SU_9	*		*	
16	$SU_8 \times SU_2$		*		
17	$SU_8 \times U_1$		*	*	*
18	$SU_7 \times SU_2 \times U_1$			*	*
19	$SU_7 \times U_1^2$			*	*
20	$SU_6 \times SU_3 \times SU_2$			*	
21	$SU_6 \times SU_3 \times U_1$			*	
22	$SU_6 \times SU_2^2 \times U_1$			*	
23	$SU_6 \times SU_2 \times U_1^2$				*
24	$SU_5 \times SU_4 \times U_1$			*	*
25	$SU_5 \times SU_3 \times SU_2 \times U_1$				*

Table 4. Gauge groups in Z_7 orbifold models

No.	Gauge Group (U_1's are omitted)	No.	Gauge Group (U_1's are omitted)
1	$(E_7; E_8)$	21	$(SO_{10}, SU_2; SO_{12})$
2	$(SO_8, SU_3; E_8)$	22	$(SU_8; SO_{12})$
3	$(SU_7, SU_2; E_8)$	23	$(SU_7; SO_{12})$
4	$(E_6, SU_2; E_7)$	24	$(SU_5, SU_3, SU_2; SO_{12})$
5	$(SO_{12}; E_7)$	25	$(SO_{10}, SU_2; SO_{10})$
6	$(SO_{10}, SU_3; E_7)$	26	$(SU_8; SO_{10}, SU_3)$
7	$(SU_6, SU_2; E_7)$	27	$(SU_7; SO_{10}, SU_3)$
8	$(SU_5, SU_4; E_7)$	28	$(SU_5, SU_3, SU_2; SO_{10}, SU_3)$
9	$(SO_{14}; E_6, SU_2)$	29	$(SU_6, SU_2; SO_{10}, SU_2)$
10	$(SO_{10}, SU_2; E_6, SU_2)$	30	$(SU_5, SU_4; SO_{10}, SU_3)$
11	$(SU_8; E_6, SU_2)$	31	$(SO_8, SU_3; SO_8, SU_3)$
12	$(SU_7; E_6, SU_2)$	32	$(SU_7, SU_2; SO_8, SU_3)$
13	$(SU_5, SU_3, SU_2; E_6, SU_2)$	33	$(SU_6, SU_2; SU_8)$
14	$(E_6; E_6)$	34	$(SU_5, SU_4; SU_8)$
15	$(SO_8, SU_3; E_6)$	35	$(SU_7, SU_2; SU_7, SU_2)$
16	$(SU_7, SU_2; E_6)$	36	$(SU_6, SU_2; SO_7)$
17	$(SO_{12}; SO_{14})$	37	$(SU_5, SU_4; SU_7)$
18	$(SO_{10}, SU_3; SO_{14})$	38	$(SU_5, SU_3, SU_2; SU_6, SU_2)$
19	$(SU_6, SU_2; SO_{14})$	39	$(SU_5, SU_3, SU_2; SU_5, SU_4)$
20	$(SU_5, SU_4; SO_{14})$		

Table 5. Gauge groups in Z_6-I orbifold models

No.	Gauge Group (U_1's are omitted)	No.	Gauge Group (U_1's are omitted)
1	$(E_7, SU_2; E_8)$	28	$(SU_8; SO_{12})$
2	$(E_6, SU_3; E_7, SU_2)$	29	$(SU_8; SO_{10}, SU_2^2)$
3	$(E_6, SU_2; E_8)$	30	$(SU_7, SU_2; SO_{14})$
4	$(E_6, SU_2; E_6, SU_3)$	31	$(SU_7, SU_2; SO_{14})$
5	$(SO_{14}; E_7)$	32	$(SU_7, SU_2; SO_8, SO_4)$
6	$(SO_{14}; E_6)$	33	$(SU_7; E_7)$
7	$(SO_{14}; E_7)$	34	$(SU_7; E_6)$
8	$(SO_{14}; E_6)$	35	$(SU_7; SO_{12}, SU_2)$
9	$(SO_{12}, SU_2; SO_{14})$	36	$(SU_7; SU_7, SU_2)$
10	$(SO_{12}, SU_2; SO_{14})$	37	$(SU_7; E_7)$
11	$(SO_{12}; E_7)$	38	$(SU_7; SO_{12}, SU_2)$
12	$(SO_{12}; SO_{12}, SU_2)$	39	$(SU_7; SU_9)$
13	$(SO_{10}, SU_3; E_7, SU_2)$	40	$(SU_7; SU_8)$
14	$(SO_{10}, SU_3; E_6, SU_2)$	41	$(SU_7; SU_8)$
15	$(SO_{10}, SU_2^2; E_7, SU_2)$	42	$(SU_6, SU_3, SU_2; E_8)$
16	$(SO_{10}, SU_2^2; E_7)$	43	$(SU_6, SU_3, SU_2; E_6, SU_3)$
17	$(SO_{10}, SU_2^2; SO_{12}, SU_2)$	44	$(SU_6, SU_3, SU_2; SO_{10}, SU_3)$
18	$(SO_{10}, SU_2; E_8)$	45	$(SU_6, SU_3; SO_{14})$
19	$(SO_{10}, SU_2; E_6, SU_3)$	46	$(SU_6, SU_3; SO_{14})$
20	$(SO_{10}, SU_2; SO_{10}, SU_3)$	47	$(SU_6, SU_3; SO_8, SU_4)$
21	$(SO_8, SU_4; E_7)$	48	$(SU_6, SU_3; SU_7)$
22	$(SO_8, SU_4; E_6)$	49	$(SU_6, SU_2^2; E_6, SU_2)$
23	$(SO_8, SU_4; SO_{12}, SU_2)$	50	$(SU_6, SU_2^2; SO_{10}, SU_2)$
24	$(SU_9; SO_{12})$	51	$(SU_6, SU_2^2; SU_6, SU_3, SU_2)$
25	$(SU_9; SO_{10}, SU_2^2)$	52	$(SU_5, SU_4; SO_{12})$
26	$(SU_8; SO_{12})$	53	$(SU_5, SU_4; SO_{10}, SU_2^2)$
27	$(SU_8; SO_{10}, SU_2^2)$	54	$(SU_5, SU_4; SU_7)$

Table 6. Gauge groups in Z_6-II orbifold models

No.	Gauge Group (U_1's are omitted)	No.	Gauge Group (U_1's are omitted)
1	$(E_7; E_8)$	29	$(SU_8; E_6, SU_2)$
2	$(E_7; E_7, SU_2)$	30	$(SU_8; SO_{10}, SU_2)$
3	$(E_6, SU_3; E_7)$	31	$(SU_7, SU_2; E_8)$
4	$(E_6, SU_2; E_7)$	32	$(SU_7, SU_2; E_6, SU_3)$
5	$(E_6; E_8)$	33	$(SU_7, SU_2; SO_{10}, SU_3)$
6	$(E_6; E_6, SU_3)$	34	$(SU_7; SO_{12})$
7	$(SO_{12}, SU_2; E_6, SU_2)$	35	$(SU_7; SO_{12}, SU_2^2)$
8	$(SO_{12}, SU_2; E_8)$	36	$(SU_7; SO_{14})$
9	$(SO_{12}; SO_{14})$	37	$(SU_7; SO_{14})$
10	$(SO_{12}; SO_{14})$	38	$(SU_7; SO_8, SU_4)$
11	$(SO_{10}, SU_3; E_7)$	39	$(SU_7; SU_7)$
12	$(SO_{10}, SU_3; E_6)$	40	$(SU_6, SU_3, SU_2; E_6)$
13	$(SO_{10}, SU_3; SO_{10}, SU_2)$	41	$(SU_6, SU_3, SU_2; SO_{12}, SU_2)$
14	$(SO_{10}, SU_2^2; SO_{14})$	42	$(SU_6, SU_3, SU_2; SU_9)$
15	$(SO_{10}, SU_2^2; SO_{14})$	43	$(SU_6, SU_3, SU_2; SU_8)$
16	$(SO_{10}, SU_2; E_7, SU_2)$	44	$(SU_6, SU_3, SU_2; SU_8)$
17	$(SO_{10}, SU_2; E_7)$	45	$(SU_6, SU_3; E_8)$
18	$(SO_{10}, SU_2; E_6, SU_3)$	46	$(SU_6, SU_3; E_6, SU_3)$
19	$(SO_{10}, SU_2; SO_{12}, SU_2)$	47	$(SU_6, SU_3; SO_{10}, SU_3)$
20	$(SO_8, SU_4; SO_{12})$	48	$(SU_6, SU_2^2; E_7)$
21	$(SO_8, SU_4; SO_{10}, SU_2^2)$	49	$(SU_6, SU_2^2; E_6)$
22	$(SU_9; E_7, SU_2)$	50	$(SU_6, SU_2^2; SO_{12}, SU_2)$
23	$(SU_9; E_6, SU_2)$	51	$(SU_6, SU_2^2; SU_7, SU_2)$
24	$(SU_9; SO_{10}, SU_2)$	52	$(SU_5, SU_4; E_7, SU_2)$
25	$(SU_8; E_7, SU_2)$	53	$(SU_6, SU_2^2; SU_6, SU_3)$
26	$(SU_8; E_6, SU_2)$	54	$(SU_5, SU_4; E_6, SU_2)$
27	$(SU_8; SO_{10}, SU_2)$	55	$(SU_5, SU_4; SO_{10}, SU_2)$
28	$(SU_8; E_7, SU_2)$	56	$(SU_5, SU_4; SU_6, SU_3, SU_2)$

Superpotential in Calabi-Yau Compactification

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We study the structure of superpotential by relating Calabi-Yau compactification to the string construction with the tensor product of $N = 2$ superconformal field theory. In the case of 4-generation $Y(4;5)$ Calabi-Yau manifold we completely calculate the generation dependent Yukawa coupling constants in the base of physical quarks and leptons. Proton decay mediating coupling can not be zero in this model.

1. Introduction

Superstring theory is very promising as a unification theory containing the gravity.[1] But this is the theory at Planck scale and if we want to relate it to the low energy physics, it is necessary to construct the effective theory of it at Weinberg-Salam scale. Then what is the low energy effective theory? To construct it we start from the heterotic string theory and reduce to 4-dim superstring or the effective field theory. These theories are characterized by $M_4 \times K$ where M_4 is the flat Minkowski space and K is some theory constructed from the extra degrees of freedoms. Whatever method we would adopt to describe K, we can expect that the 4-dim effective field theory is the $N = 1$ supergravity coupled with super Yang-Mills fields and chiral superfields so far as we impose the $N = 1$ space-time supersymmetry to the heterotic string theory.[2] In this system there remain some freedoms, that is, the gauge group, the representations and numbers of chiral superfields, Kähler potential (containing superpotential) and the gauge kinetic term normarization function. These are dependent on the feature of K. Up to now many construction methods of K are proposed, *e.g.* Calabi-Yau (C-Y) compactification,[3] [4] the orbifold compactification,[5] the toroidal compactification,[6] the free fermion construction[7] and the construction with the tensor product of $N = 2$ superconformal field theory with level three[8] *etc* .

Recently Gepner conjectured that C-Y compactification and the tensor product of $N = 2$ superconformal field theory with $C = 9$ might be equivalent.[9] In this talk we utilize this conjecture to construct the effective field theory, in particular to calculate Yukawa couplings in the superpotential.[10]

Before proceeding to the calculation we briefly review Yukawa couplings in both construction methods of K.

(i) Calabi-Yau compactification

In this scheme the massless matter fields are the elements of cohomology group $H^{2,1}$ and $H^{1,1}$ of C-Y manifold. The former corresponds to **27** of E_6 and the latter to **27***. The generation number is represented by $|h^{2,1} - h^{1,1}|$. On C-Y manifold the elements of $H^{2,1}$ can be represented by the elements of $H^1(T)$.[11] [12] [13] Moreover in the case of complete intersection C-Y manifold ($CICY$) the elements of $H^1(T)$ can be described by the polynomials. If we note this fact, $\mathbf{27}^3$ Yukawa coupling constants λ_{ijk} are written as

$$\lambda_{ijk} = \int A_i^a \wedge A_j^b \wedge A_k^c \epsilon_{abc} = \int P_i(z) P_j(z) P_k(z),$$

where A_i^a is the element of $H^1(T)$ and $P_i(z)$ is its polynomial representation. Physical Yukawa couplings should be of course understood after taking account of the kinetic term normarization.

(ii) the tensor product of $N = 2$ superconformal field theory

The massless fields in this construction are described as $N = 2$ superconformal fields. Then Yukawa coupling constants are represented as the three point correlation function under properly normarized superconformal fields $\phi_i(z_i)$,

$$\lambda_{ijk} = \langle \phi_i(z_i) \phi_j(z_j) \phi_k(z_k) \rangle$$

Following the detailed study of $N = 2$ superconformal field theory, the primary fields of this theory are, using the parafermion theory, related to the primary fields of $SU(2)$WSW theory up to free bosons.[14] Fortunately $SU(2)$WSW theory is exactly solvable and the correlation function can be completely determined.[15]

Now we concentrate on these features of Yukawa couplings and try to determine Yukawa couplings completely. In C-Y compactification the effective theory has some phenomenologically interesting problems. One is the fast proton decay mediated by the extra color triplets which in general remain massless in this scheme. Then if Yukawa couplings related to this process are not zero, these theories can not be realistic. The other is the determination of quarks and leptons mass matrics. Complete determination of Yukawa couplings give the answer to these phenomenological questions. Moreover the calculation of Yukawa couplings in both schemes makes it possible to check Gepner's conjecture.

In the later sections we would try to carry out this scenario for the typical example, that is, $Y(4;5)$ C-Y manifold (zeros of quintic polynomial in CP^4) and 3^5 model (the five tensor product of $N=2$ superconformal field theory with level three). These are conjectured to be equivalent by Gepner.[9] To answer our phenomenological motivation, we must calculate Yukawa couplings not in the **27** basis of E_6 but in quark and lepton bases of the standard gauge group. For this purpose we need to introduce the Wilson line.[16]

2. $Y(4;5)$ model

$Y(4;5)$ C-Y manifold is the typical $CICY$.[12,13] [17] Here we adopt the defining polynomial in CP^4 as follows,

$$P(z)=\frac{1}{5}\sum_{i=1}^{5} z_i^5 - cz_1z_2z_3z_4z_5=0$$

where c is a complex number. In the case of c $=0$ there are fruitful discrete symmetries on this manifold; the permutation $z_i \rightarrow z_j$ and the phase transformation $z_i \rightarrow \alpha^{n_i} z_i$ ($\sum_{i=1}^5 n_i = 0\ mod 5$). These compose $S_5 \times (Z_5)^5/Z_5$. As the freely acting ones in these symmetries we can take

$$\begin{cases} S : z_i \rightarrow z_{i+1}, \\ T : z_i \rightarrow \alpha^i z_i, \qquad \alpha = exp\left(\dfrac{2\pi i}{5}\right). \end{cases}$$

Clearly these generate $Z_5 \times Z_5$. As mentioned in introduction the fields **27** of E_6

are represented by the independent monomials on this manofold ;

$z_1z_2z_3z_4z_5$	1
$z_i^3z_j^2(i \neq j)$	20
$z_i^3z_jz_k(i \neq j \neq k)$	30
$z_i^2z_j^2z_k(i \neq j \neq k)$	30
$z_i^2z_jz_kz_l(i \neq j \neq k \neq l)$	20.

Here $h^{2,1} = 101$ and $h^{1,1} = 1$ so that this has 100 generations. To reduce the genaration number and to break down E_6 to its subgroup, we consider the quotient manifold $Y(4;5)/Z_5 \times Z_5$ by using the discrete symmetries generated by S and T and the background gauge field on it. Then we get the 4-generation model and due to the Wilson line mechanism E_6 breaks down to G which commutes with this Wilson line.[4] [16]

On $Y(4;5)/Z_5 \times Z_5$ there remain many discrete symmetries $Z_5 \times Z_4$ which are generated by,

$$\begin{cases} B : z_i \rightarrow \alpha^{2i^2} z_i, \\ Y : z_i \rightarrow z_{2i}. \end{cases}$$

The massless spectrum on this manifold should be represented as (S,T) eigenstates. We denote them as $T_{nm}^{(i)}$ where their (S,T) eigenvalues are (α^n, α^m). i is generation index ($i = 1 \sim 4$). The transformation properties of $T_{nm}^{(i)}$ under B and Y are easily determined. The gauge symmetry and the discrete symmetry are dependent on how we embed the S and T discrete symmetries into the gauge group E_6 . From the phenomenological point of view we select the embedding under which (B,Y) all remain unbroken and this corresponds to the embedding of S only. The gauge group and the correspondence of polynomials and fields are completely determined by this embedding.[13]

Now we can proceed to calculation of Yukawa couplings. There are severe constraints on Yukawa couplings

$$\lambda_{ijk} = \int T_{nn'}^{(i)} T_{mm'}^{(j)} T_{ll'}^{(k)}.$$

· (S,T)-invariance (gauge invariance)

$$n+m+l=0\ (mod5),$$
$$n'+m'+l'=0\ (mod5).$$

· (B,Y)-invariance

$$i+j+k=0\ (mod5).$$

· contribution to the integral (which is constrained by the psuedosymmetry)

$$\int (z_1z_2z_3z_4z_5)^3 = \frac{1}{c}\int (z_1z_2z_3z_4z_5)^2 z_i^5 = \ldots = \frac{1}{c^3}\int z_i^5 z_j^5 z_k^5 = \mu \neq 0.$$

And to get the physical Yukawa couplings we must normarize the kinetic terms. In our case as the result of the discrete symmetries there are only two normarization parameters,

$$||T_{00}^{(0)}||/||T_{n0}^{(1)}|| = ||T_{00}^{(0)}||/||T_{n0}^{(4)}|| = \zeta, \quad ||T_{00}^{(0)}||/||T_{n0}^{(2)}|| = ||T_{00}^{(0)}||/||T_{n0}^{(3)}|| = \xi.$$

Taking account of these conditions we can calculate Yukawa couplings as shown in Table.[13]

3. 3^5 model

Starting from the heterotic string we construct the 4-dim superstring as

$$(right\ secter) = M_4 \times K,$$
$$(left\ secter) = M_4 \times K \times (gauge\ group).$$

K is now taken as $N=2$ minimal theory. The central charge of $N=2$ minimal theory with level k is $C=3k/(k+2)$.[14] [19] The central charge matching requires that the contribution of K should be $C_K = 15-(1+1/2)\times 4 = 9$ and the rank of gauge group is $26-4-9=13$. Here we take this gauge group as $SO(10)\times E_8$.

Following to the representation theory of $N=2$ superconformal field theory the primary fields are represented by three integer quantum numbers (l,q,S). These satisfy the conditions: $0 \leq l \leq k$, q is $mod2(k+2)$, S is $mod4$ ($S=0,2$ correspond to NS sector and $S=\pm 1$ to R sector) and $l+q+S=0$ ($mod2$). With these quantum numbers the conformal dimension h and $U(1)$ charge Q are represented in case of $|q-S| \leq l$,

$$h = \frac{l(l+2)}{4(k+2)} - \frac{q^2}{4(k+2)}, \quad Q = -\frac{q}{k+2} \quad \text{for} S = 0,$$
$$h = \frac{l(l+2)}{4(k+2)} - \frac{q^2}{4(k+2)} + \frac{S^2}{8}, \quad Q = -\frac{q}{k+2} + \frac{S}{2} \quad \text{for} S = \pm 1.$$

In other cases taking account of the identification

$$\phi^l_{q,S} = \phi^{k-l}_{q+k+2,S+2} = \phi^l_{q+2(k+2),S} = \phi^l_{q,S+4},$$

we can calculate h and Q with the previous formura up to integers.

The spectrum of this model can be read off from the partition function which is modular invariant and $N=1$ space-time supersymmetric. The construction method of such a partition function is proposed by Gepner.[6] It is constructed by a set of certain conformal fields which satisfy the appropriate conditions. We represent a conformal field by a vector μ which is defined by $\mu = (\lambda; q_1, ..., q_5, S_1, ..., S_5)$. Using these μ the partition function is described

$$Z = \sum_{\{\mu\}} (\text{affine} A^1_1 \text{part}) \Theta_\mu \Theta^*_{\mu+v},$$
$$\Theta_\mu = \prod_{i=1}^{5} \Theta_{q_i,5} \Theta_{S_i,2},$$

where $\Theta_{q_i,5}$ and $\Theta_{S_i,2}$ are theta functions with level five and two respectively and μ satisfies $\beta_0 \cdot \mu \in Z + \frac{1}{2}$ and $\beta_i \cdot \mu \in Z$ for $\beta_0 = (\bar{s}; 1, \ldots, 1, 1, \ldots, 1)$ and $\beta_i = (v; 0, \ldots, 0, 0, .., 2, .., 0)$.

The solution for such vectors μ is

$(0,0,0)^5$	1
$A(2,-2,0)^5$	1
$B(2,-2,0)^3(3,-3,0)(1,-1,0)$	20
$C(3,-3,0)^2(1,-1,0)^2(2,-2,0)$	30
$D(0,0,0)(1,-1,0)(3,-3,0)^3$	20
$E(0,0,0)(3,-3,0)(2,-2,0)^2$	30.

The total number of these conformal fields is 101 and these just correspond to 101 monomials in $Y(4;5)$ model.

The discrete symmetries of this model are easily studied and found to be $S_5 \times (Z_5)^5/Z_5$. These completely coinside with them of $Y(4;5)$ model. Noting this fact we can correlate the discrete symmetries of both models as follows. In $Y(4;5)$ model the discrete charges $(Z_5)^5/Z_5$ of monomial $Z_1^{l_1}Z_2^{l_2}Z_3^{l_3}Z_4^{l_4}Z_5^{l_5}$ are understood as (l_1,l_2,l_3,l_4,l_5) where l_i is positive integer such that $\sum_i l_i = 5$. In 3^5 model by using the symmetry of the model (q_1,q_2,q_3,q_4,q_5) can reduce to $(\bar{q}_1,\bar{q}_2,\bar{q}_3,\bar{q}_4,\bar{q}_5)$ where $\bar{q}_i$ is positive integer and $\sum_i \bar{q}_i = 5$. Then if $(l_1,l_2,l_3,l_4,l_5) = (\bar{q}_1,\bar{q}_2,\bar{q}_3,\bar{q}_4,\bar{q}_5)$ we identify μ with $Z_1^{l_1}Z_2^{l_2}Z_3^{l_3}Z_4^{l_4}Z_5^{l_5}$.

Making use of this correspondence we can easily find the discrete symmetries in 3^5 model;

$$\begin{cases} \bar{S} : (l, l+2n, 0)_i \to (l, l+2n, 0)_{i+2}, \\ \bar{T} : (l, l+2n, 0)_i \to \alpha^{\bar{q}_i i}(l, l+2n, 0)_i, \end{cases}$$

$$\begin{cases} \bar{B} : (l, l+2n, 0)_i \to \alpha^{2i^2 \bar{q}_i}(l, l+2n, 0)_i, \\ \bar{Y} : (l, l+2n, 0)_i \to (l, l+2n, 0)_{2i}. \end{cases}$$

These correspond to S, T, B and Y in $Y(4;5)$ respectively. The parallel arguments to $Y(4;5)$ model make it easy to construct the (S,T) eigenstates and its transformation properties in 3^5 model. The constraints on Yukawa couplings are almost same as $Y(4;5)$ model. The only change is that the polynomial integration is replaced with the three point correlation of superconformal fields which is exactly

calculable. Refering to Zamolodchikov-Fateev formula,[15] the nonzero three point correlations are[20]

$$
\begin{aligned}
&A^3 = \omega^5, \quad AB^2 = \omega^3, \quad B^3 = B^2E = E^3 = \omega^2, \\
&AC^2 = CE^2 = B^2C = BCE = \omega, \\
&BC^2 = C^2E = D^2E = BCD = CED = 1.
\end{aligned}
$$

Here $A \sim E$ should be combined to be supersymmetric and gauge invariant. ω is a constant about 1.09. Summarizing these results we can completely determine Yukawa couplings as listed in Table.

4. Summary

We calculated Yukawa coupling constants after introducing the Wilson line for both $Y(4;5)$ and 3^5 models. This makes it possible to interpret Yukawa coupling constants in the basis of physical quarks and leptons. The generation structure is understood as the internal structure of the model, more concretely $(Z_5)^5/Z_5$ charges in both models.

As easily seen in Table if we take the parameters in $Y(4;5)$ model as

$$
\mu = 5\omega^5, \quad \xi = \omega^{-2}, \quad \zeta = \omega^{-1},
$$

the results in both models completely agree in $c \rightarrow 0$ limit. This fact supports Gepner's conjecture from Yukawa coupling constants. Phenomenologically our results show that the couplings mediating proton decay remain nonzero unfortunately. This is a very severe problem for our model. The quarks and leptons mass matrices were completely determined at Planck scale. If we want to know the phenomenological features of them we must carry out the renormalization group study.

The four generation model adopted in this study may or may not be realistic . But it is a very interesting and useful example in which we can concretely calculate various quantities. It is worthy to study the structure of the effective action of this model in more details.

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Table. Values of Yukawa couplings.

Type of couplings	$Y(4;5)/Z_5 \times Z_5$ model	3^5 model
$T^{(0)}_{00}T^{(0)}_{00}T^{(0)}_{00}$	$\mu/5$	ω^5
$T^{(0)}_{00}T^{(2)}_{n0}T^{(3)}_{m0}$	$\mu\xi^2$	5ω
$T^{(0)}_{00}T^{(1)}_{n0}T^{(4)}_{m0}$	$c\mu\zeta^2$	0
$T^{(3)}_{n0}T^{(3)}_{n0}T^{(4)}_{l0}$ $T^{(2)}_{n0}T^{(2)}_{n0}T^{(1)}_{l0}$	$(2+2c+c^3)\mu\xi^2\zeta$	10
$T^{(3)}_{n0}T^{(3)}_{m0}T^{(4)}_{l0}\,\lvert n-m\rvert=2$ $T^{(2)}_{n0}T^{(2)}_{m0}T^{(1)}_{l0}\,\lvert n-m\rvert=1$	$\{(\alpha^{-1}+\alpha)+(\alpha^{-2}+\alpha^2)c+c^3\}\mu\xi^2\zeta$	$5(\alpha^{-1}+\alpha)$
$T^{(3)}_{n0}T^{(3)}_{m0}T^{(4)}_{l0}\,\lvert n-m\rvert=1$ $T^{(2)}_{n0}T^{(2)}_{m0}T^{(1)}_{l0}\,\lvert n-m\rvert=2$	$\{(\alpha^{-2}+\alpha^2)+(\alpha^{-1}+\alpha)c+c^3\}\mu\xi^2\zeta$	$5(\alpha^{-2}+\alpha^2)$
$T^{(1)}_{n0}T^{(1)}_{n0}T^{(3)}_{l0}$ $T^{(4)}_{n0}T^{(4)}_{n0}T^{(2)}_{l0}$	$(2+c+2c^2)\mu\xi\zeta^2$	10ω
$T^{(1)}_{n0}T^{(1)}_{m0}T^{(3)}_{l0}\,\lvert n-m\rvert=2$ $T^{(4)}_{n0}T^{(4)}_{m0}T^{(2)}_{l0}\,\lvert n-m\rvert=1$	$\{(\alpha^{-1}+\alpha)+c+(\alpha^{-2}+\alpha^2)c^2\}\mu\xi\zeta^2$	$5(\alpha^{-1}+\alpha)\omega$
$T^{(1)}_{n0}T^{(1)}_{m0}T^{(3)}_{l0}\,\lvert n-m\rvert=1$ $T^{(4)}_{n0}T^{(4)}_{m0}T^{(2)}_{l0}\,\lvert n-m\rvert=2$	$\{(\alpha^{-2}+\alpha^2)+c+(\alpha^{-1}+\alpha)c^2\}\mu\xi\zeta^2$	$5(\alpha^{-2}+\alpha^2)\omega$

In the case that unbroken gauge group is $SU(3)_C \times SU(2)_L \times SU(2)' \times U(1)$, $T^{(i)}_{n0}$ corresponds to the fields as follows :
$T^{(i)}_{00}=(h',h)(S_1)$, $T^{(i)}_{10}=(\bar{d},\bar{u})$, $T^{(i)}_{20}=(g)(S_2,\bar{e})$, $T^{(i)}_{30}=(\bar{g})(l)$, $T^{(i)}_{40}=(Q)$,
where $Q, \bar{u}, \bar{d}, l$, and $\bar{e}$ denote the ordinary quarks and leptons and h, h' are doublet Higgs fields. $g, \bar{g}$ and S_1, S_2 are the color triplet and neutral extra fields respectively.

Zero Mode and Modular Invariance in String on Non-Abelian Orbifold*)

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§ 1 Introduction

Orbifolds are interesting ones as the candidates of the compactified space in string theories. Many attempts have been done on this subject and the quantum properties (modular invariance, etc.) of the string on Z-orbifold have been thoroughly clarified. However, in the non-abelian case, these properties have not been sufficiently clarified yet while this case may be more important for constructing realistic model. The main complexity which arises in the non-abelian orbifolds is that the action of the dividing group on the torus is, in general, considerably nontrivial compared with the case of Z-orbifold, leaving, sometimes, a nontrivial subspace as a invariant subspace. This requires a special care for the treatment of the zero mode part of string. It is thus important to investigate the quantum structure of the zero modes of the string on non-abelian orbifold.

In this report, we discuss the operator formalism of the zero mode part of closed Bosonic string on orbifold following the method in Ref.2). The dividing group of orbifold which we treat here may be non-abelian and have invariant subspace.

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*)This report is based on the work[1] with K.Inoue and H.Takano**).

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§2 Operator Formalism of zero mode

The orbifold we deal with is the quotient space T^d/G . T^d is the d-dimensional torus. T^d is described by the identification of x^I and $x^I + \pi L^I$ on flat space where L^I is a point on a lattice Λ. G is a symmetry of Λ. We take the discrete group of rotation and reflection as G which is, in general, non-abelian. Thus its element R is a orthogonal matrix;

$$R^T R = 1 \text{ and } L'^I \equiv R^{IJ} L^J \in \Lambda \quad \text{for any } L^I \in \Lambda.$$

In this way, the following identification exists on orbifold.

$$x^I \sim R^{IJ} x^J + \pi L^I \quad \text{for any } R \in G \text{ and } L^I \in G.$$

Our plan for treating the zero mode part of the string on orbifold (= T^d/G) is as follows. In order to express the quantum theory on orbifold, we construct at first the Hilbert space of a dynamical system on T^d. This Hilbert space is a direct sum of the Hilbert spaces of some constrained systems and each constraint is connected with each boundary condition of string. Next we project this Hilbert space into the subspace invariant under G-transformation. In this report, we mainly concentrate on the construction of the Hilbert space of the constrained system mentioned above and briefly discuss the one-loop vacuum amplitude and its modular invariance.

2-1 The Hilbert space of the string in R-sector

We start from the following action S

$$S = \frac{1}{2\pi}\int_0^{\pi} d\sigma \int d\tau \{ \eta^{\alpha\beta}\partial_\alpha X^I(\sigma,\tau)\partial_\beta X^I(\sigma,\tau) + B^{IJ}\varepsilon^{\alpha\beta}\partial_\alpha X^I(\sigma,\tau)\partial_\beta X^J(\sigma,\tau) \} \quad (2.1.a)$$

$$= \int_0^{\pi} d\sigma \int d\tau \; L(X) \, , \quad (2.1.b)$$

where B^{IJ} is antisymmetric constant background field. Now we require that the Lagrangean density L is single valued on orbifold. Since $R^{IJ}X^J + \pi L^I$ and X^I denote the same point on orbifold for any $R \in G$ and $L^I \in \Lambda$, $L(RX+\pi L)$ and $L(X)$ must have the same value. This implies.

$$R^T B R = B \quad \text{for any } R \in G \quad (2.2)$$

The canonical momenta. the total Hamiltonian and the generator of σ-translation are

$$P^I = \frac{1}{\pi}\{ \partial_\tau X^I + B^{IJ}\partial_\sigma X^J \}. \quad (2.3)$$

$$H = \frac{1}{2\pi}\int_0^{\pi} d\sigma \, \{(\pi P^I - B^{IJ}\partial_\sigma X^J)^2 + (\partial_\sigma X^I)^2\} \, , \quad (2.4)$$

$$T^\sigma = \int_0^{\pi} d\sigma \, \{ P^I \partial_\sigma X^I \} \, . \quad (2.5)$$

Since $R^{IJ}X^J + \pi L^I$ is identified with X^I on orbifold for any $R \in G$ and $L^I \in \Lambda$, there exist many types of "closed" string with the boundary condition;

$$X_R^I(\sigma+\pi,\tau) = R^{IJ}X_R^J(\sigma,\tau) + \pi L^I \quad (2.6)$$

$$R \in G \, , \; L^I \in \Lambda \, .$$

We call this X_R^I the string in R-sector. We will treat this dynamical system following the method in Ref.2).

It is convenient to take the following coordinate system.

$$R^{IJ} = \begin{pmatrix} \delta_{A,B} & 0 \\ 0 & \bar{R}^{\alpha\beta} \end{pmatrix} \quad (2.7)$$

In this coordinate system, it is found from (2.2) that

$$B^{A\alpha} = -B^{\alpha A} = 0. \tag{2.8}$$

Then, we obtain the mode expansion of X_R^I,

$$X_R^A = x^A + (p^A - B^{AB}L^B)\tau + L^A\sigma + [\text{oscillator term}], \tag{2.9.a}$$

$$X_R^\alpha = x^\alpha + [\text{oscillator term}] \tag{2.9.b}$$

and the following equations ;

$$(1 - \bar{R})^{\alpha\beta}x^\beta - \pi L^\alpha = 0, \tag{2.10,a}$$

$$p^\alpha = 0, \tag{2.10,b}$$

where x^I is center of mass coordinate of the string and p^I is momentum conjugate to x^I. x^I is on the torus T^d.

Following Ref.2), we treat L^I as dynamical variables and introduce Q^I conjugate to L^I. Then the zero modes compose a dynamical system with the dynamical variables x, p, L and Q. This dynamical system is a constrained system which has following Poisson brackets and second class constraints.

$$\{ x^I , p^J \}_P = \delta^{IJ} \tag{2.11.a}$$

$$\{ Q^I , L^J \}_P = \delta^{IJ} \tag{2.11.b}$$

$$(1 - \bar{R})^{\alpha\beta}x^\beta - \pi L^\alpha \approx 0 \tag{2.12.a}$$

$$p^\alpha \approx 0 \quad . \tag{2.12.b}$$

In order to quantize this system, we use Dirac bracket and obtain the following commutation relations,

$$[\hat{x}^A , \hat{p}^B] = i\delta^{AB}, \tag{2.13.a}$$

$$[\hat{Q}^I , \hat{L}^J] = i\delta^{IJ}, \tag{2.13.b}$$

$$[\hat{x}^\alpha , \hat{Q}^\beta] = -i\pi(1-\bar{R})^{-1\alpha\beta}, \tag{2.13.c}$$

$$[\hat{x}^\alpha , \hat{p}^\beta] = 0 \quad . \tag{3.13.d}$$

Let us define $\hat{k}^I$ as follows

$$\hat{k}^I = (\hat{k}^A , \hat{k}^\alpha)$$

$$= (\hat{p}^A , - \frac{1}{\pi}\hat{Q}^\beta (1 - \bar{R})^{\beta\alpha}) \qquad (2.14.a)$$

$$= (\hat{p}^A , - \frac{1}{\pi}(1 - \bar{R}^{-1})^{\alpha\beta} \hat{Q}^\beta) \qquad (2.14.b)$$

Then,

$$[\hat{x}^I , \hat{k}^J] = i\delta^{IJ}, \qquad (2.15.a)$$

$$[\hat{L}^\alpha , \hat{k}^\beta] = \frac{i}{\pi}(1-\bar{R})^{\alpha\beta} . \qquad (2.15.b)$$

In this way, the translation generators of $\hat{x}^I$ are not $\hat{p}^I$ but $\hat{k}^I$. From the periodicity of the wave function;

$$\psi(x^I + \pi N^I) = \psi(x^I) \quad \text{for any } N^I \in \Lambda ,$$

the eigenvalue of the operator $\hat{k}^I$ is on $2\Lambda^*$, where Λ^* is the dual lattice of Λ. (We denote by $b\Lambda$ the lattice whose base vectors are $b \times E^I{}_i$, where b is a constant and $E^I{}_i$ are the base vectors of Λ.) The eigenvalue of $\hat{L}^I$ is on Λ by definition.

Let us fix a representation. The commutation relations among independent variables are

$$[\hat{x}^A , \hat{k}^B] = i\delta^{AB}, \qquad (2.16.a)$$

$$[\hat{L}^\alpha , \hat{k}^\beta] = \frac{i}{\pi}(1-\bar{R})^{\alpha\beta} , \qquad (2.16.b)$$

$$[\hat{Q}^A , \hat{L}^A] = i\delta^{AB}. \qquad (2.16.c)$$

We take a representation diagonal with respect to each ones of these conjugate pairs. It will be convenient to diagonalize k and L because Hamiltonian is a function of k^A and L^A. However, we should notice that $\hat{k}^\alpha$ and $\hat{L}^\alpha$ cannot be simultaneously diagonalized due to (2.16.b). Thus we have the following two types of "momentum" representation for this quantum system;

$$| k^A , L^I \rangle \text{ and } | k^I , L^A \rangle.$$

This system has two types of periodicity.

$$x^I \sim x^I + \pi N^I \quad \text{for any } N^I \in \Lambda, \tag{2.17.a}$$

$$Q^I \sim Q^I + \pi \ell^I \quad \text{for any } \ell^I \in 2\Lambda^*. \tag{2.17.b}$$

The relation (2.17.b) comes from the fact that $\hat{Q}^I$ is conjugate to $\hat{L}^I$ and the eigenvalue of $\hat{L}^I$ is on Λ.

Since $L^\alpha = \frac{1}{\pi}(1 - \bar{R})^{\alpha\beta} x^\beta$ and $k^\alpha = -\frac{1}{\pi}(1 - \bar{R}^{-1})^{\alpha\beta} Q^\beta$,

we obtain the following periodicity for L^I and k^I.

$$L^\alpha \sim L^\alpha + (1 - \bar{R})^{\alpha\beta} N^\beta$$

$$\rightarrow L^I \sim L^I + (1 - R)^{IJ} N^J \quad \text{for any } N^I \in \Lambda \tag{2.18.a}$$

$$k^\alpha \sim k^\alpha - (1 - \bar{R}^{-1})^{\alpha\beta} \ell^\beta$$

$$\rightarrow k^I \sim k^I - (1 - R^{-1})^{IJ} \ell^J \quad \text{for any } \ell^I \in 2\Lambda^* \tag{2.18.b}$$

In terms of the state vectors, such identification means that the state vectors for any $L^I \in \Lambda$ and $k^I \in 2\Lambda^*$ are not independent. For example,

$$\begin{aligned} | k^A, L^I + (1-R)^{IJ} N^J \rangle &= e^{i\hat{Q}^I (1-R)^{IJ} N^J} | k^A, L^I \rangle \\ &= e^{i\hat{Q}^\alpha (1-\bar{R})^{\alpha\beta} N^\beta} | k^A, L^I \rangle \\ &= e^{-i\pi \hat{k}^\beta N^\beta} | k^A, L^I \rangle \\ &= e^{-i\pi \hat{k}^I N^I + i\pi \hat{k}^A N^A} | k^A, L^I \rangle \\ &= e^{i\pi k^A N^A} | k^A, L^I \rangle, \end{aligned} \tag{2.19}$$

where we have used the fact that in this system $e^{-i\pi \hat{k}^I N^I} = 1$ for any $N^I \in \Lambda$. Note that $| k^A, L^I + (1-R)^{IJ} N^J \rangle$ is not just equal to $| k^A, L^I \rangle$ but with a phase factor $e^{i\pi k^A N^A}$. (However in the x-representation, of course, $| x^I + \pi N^I \rangle = | x^I \rangle$.)

Similarly,

$$| k^I - (1-R^{-1})^{IJ} \ell^J, L^A \rangle = e^{i\pi L^A \ell^A} | k^I, L^A \rangle \tag{2.20}$$

In this way, the properties of this system on these representations are summarised as follows.

•commutation relations

$$[\ \hat{x}^A\ ,\ \hat{k}^B\ ] = i\delta^{AB},$$

$$[\ \hat{L}^\alpha\ ,\ \hat{k}^\beta\ ] = \frac{1}{\pi}i(1-\bar{R})^{\alpha\beta}\ ,$$

$$[\ \hat{Q}^A\ ,\ \hat{L}^A\ ] = i\delta^{AB}.$$

•representation

$$|\ k^A.\ L^I\ > \quad \text{and} \quad |\ k^I,\ L^A\ >.$$

•complete sets

$$1 = \sum_{\substack{k^A\in [2\Lambda^*/(1-R^{-1})2\Lambda^*] \\ L^I\in \Lambda/(1-R)\Lambda}} |\ k^A\ ,\ L^I\ >< k^A\ ,\ L^I\ | = \sum_{\substack{L^A\in [\Lambda/(1-R)\Lambda] \\ k^I\in 2\Lambda^*/(1-R^{-1})2\Lambda^*}} |\ k^I\ ,\ L^A\ >< k^I\ ,\ L^A\ |. \tag{2.21}$$

•inner products

$$<\ k^A,\ L^I|k'^I,L'^A\ > = \frac{1}{\sqrt{V}}\ \delta_{k^A,k'^A}\ \delta_{L^A,L'^A}\ e^{i\pi k'^\alpha(1-\bar{R})^{-1\alpha\beta}L^\beta}, \tag{2.22.a}$$

$$<\ k^A,\ L^I|k'^A,L'^I\ > = \delta_{k^A,k'^A}\sum_{N^I\in\Lambda/\Lambda_R}\delta_{L^I,\ L'^I+(1-R)^{IJ}N^J}\ e^{-i\pi N^A k^A}, \tag{2.22.b}$$

$$<\ k^I,\ L^A|k'^I,L'^A\ > = \delta_{L^A,L'^A}\sum_{\ell^I\in 2\Lambda^*/2\Lambda_R^*}\delta_{k^I,\ k'^I-(1-R^{-1})^{IJ}\ell^J}\ e^{-i\pi\ell^A L^A}. \tag{2.22.c}$$

$\Lambda/(1-R)\Lambda$ $(2\Lambda^*/(1-R^{-1})2\Lambda^*)$ in (2.21) denotes the set of the independent lattice points of Λ $(2\Lambda^*)$ up to the relation (2.18). $[\Lambda/(1-R)\Lambda]$ $([2\Lambda^*/(1-R^{-1})2\Lambda^*])$ means that the sum of L^A (k^A) is taken over the possible values of L^A (k^A) in $\Lambda/(1-R)\Lambda$ $(2\Lambda^*/(1-R^{-1})2\Lambda^*)$ without duplication if any. Note that $[\Lambda/(1-R)\Lambda]$

$([2\Lambda^*/(1-R^{-1})2\Lambda^*])$ is ,in general, not equal to the R-invariant sublattice of Λ $(2\Lambda^*)$.

The factor $e^{i\pi k'^{\alpha}(1-\bar{R})^{-1\alpha\beta}L^{\beta}}$ in (2.22.a) comes from the commutation relation (3.16.b). V is a constant given by normalization of vectors and

$$V = \left| \frac{\det(1-\bar{R})^{\alpha\beta}}{V_{\Lambda_R} \; V_{\Lambda_R^*}} \right| . \tag{2.23}$$

where V_{Λ_R} and $V_{\Lambda_R^*}$ are the one-unit volumes of Λ_R and Λ_R^* which are R-invariant sublattices of Λ and Λ^* respectively.

The right hand sides in (2.22.b,c) result from (2.19) and (2.20). As far as $L^I(k^I)$ and $L'^I(k'^I)$ are belonging to the same region $\Lambda/(1-R)\Lambda$ $(2\Lambda^*/(1-R^{-1})2\Lambda^*)$, the inner product is a mere Kronecker's delta. The reason why the form of the inner products such as (2.22.b,c) is needed is that we must deal with the G-transfomation; $L^I \to U^{IJ}L^J$ $(k^I \to U^{IJ}k^J)$, $U \in G$, and for $L^I \in \Lambda/(1-R)\Lambda$ $(k^I \in 2\Lambda^*/(1-R^{-1})2\Lambda^*)$, $U^{IJ}L^J$ $(U^{IJ}k^J)$ is, in general, out of $\Lambda/(1-R)\Lambda$ $(2\Lambda^*/(1-R^{-1})2\Lambda^*)$.

<u>example</u>

Let us consider the root lattice of SU(3), fig.1, as an example of Λ. This lattice has a symmetry G = { U , 1 }, where

$$U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Then we can take the points denoted by ○ in fig.1 as $\Lambda/(1-U)\Lambda$.

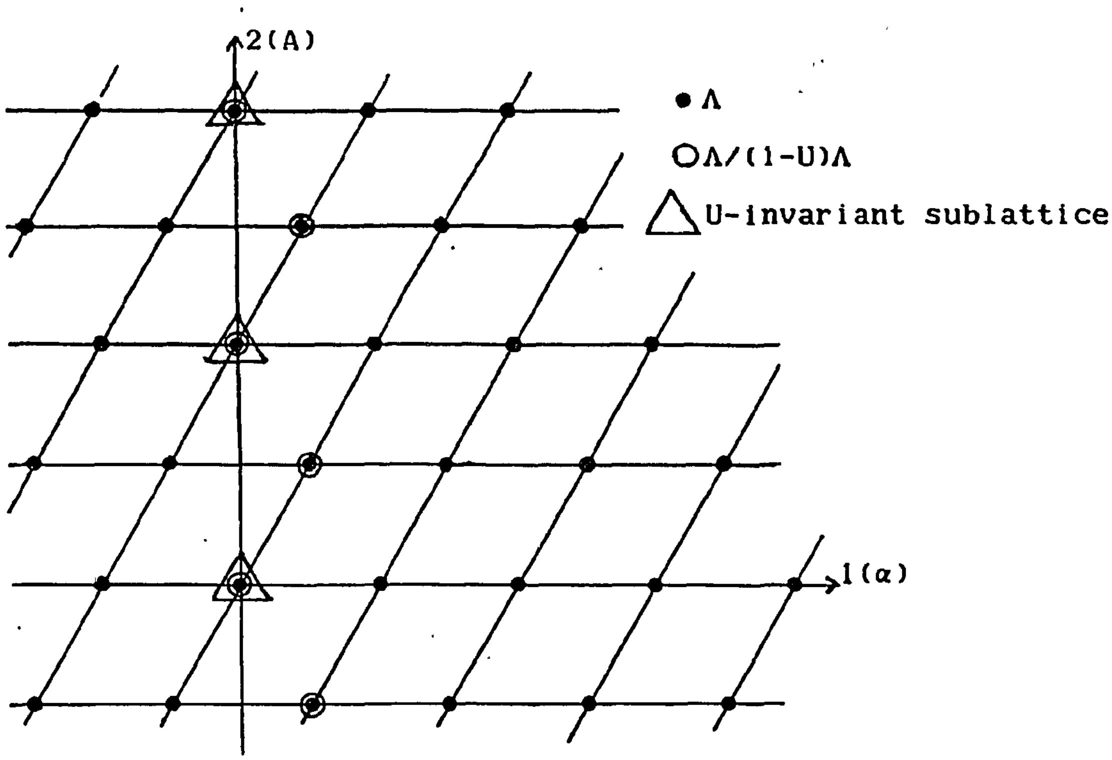

fig.1

2-2 The one-loop vacuum amplitude and modular invariance

Let $\mathcal{H}$ be the total Hilbert space of all strings with such types of the boundary condition as (3.6). Then the operator on $\mathcal{H}$ is written by matrix form with sector indices. The Hilbert space of string on orbifold is the G-invariant subspace of $\mathcal{H}$. The G-invariant subspace is the projection space $\hat{P}\mathcal{H}$ with the projection operator[1),3)]

$$\hat{P}_{R',R} = \frac{1}{N} \sum_{U\in G} \hat{g}_U(R)\, \delta_{R',URU^{-1}} \; . \tag{2.24}$$

where $\hat{g}_U(R)$ is the operator which connects the string operator in R-sector, $\hat{X}^I_R$, to that in URU^{-1}-sector;

$$\hat{g}_U^{-1}(R)\, \hat{X}^I_{URU^{-1}}\, \hat{g}_U(R) = U^{IJ}\, \hat{X}^J_R \; . \tag{2.25}$$

The one-loop vacuum amplitude is given as

$$I_{OP} = \mathrm{Tr}[\int\frac{d\tau_2}{\tau_2}\int d\tau_1 \; e^{-\pi\tau_2\hat{H}} \; e^{-i\pi\tau_1\hat{T}^\sigma} \; \hat{P}] \tag{2.26}$$

The trace is taken over the total Hilbert space $\mathcal{H}$. Since $\hat{H}$ and $\hat{T}_\sigma$ are diagonal with respect to sector indices, only the diagonal elements of $\hat{P}$ contribute to the trace. Thus,

$$I_{OP} = \int\frac{d\tau_2}{\tau_2}\int d\tau_1 \sum_R \mathrm{Tr}[e^{-\pi\tau_2\hat{H}} \; e^{-i\pi\tau_1\hat{T}^\sigma} \; \hat{P}_{R,R}]_R$$

$$= \frac{1}{N}\int\frac{d\tau_2}{\tau_2}\int d\tau_1 \sum_{R,U} \delta_{R,URU^{-1}} \mathrm{Tr}[e^{-\pi\tau_2\hat{H}} \; e^{-i\pi\tau_1\hat{T}} \; \hat{g}_U(R)]_R. \tag{2.27}$$

The factor $\delta_{R,URU^{-1}}$ means that the sum is taken over all R and U commuting with each other.

Let $A_U(R)$ be the zero mode part of R-sector in the trace. It is given as

$$A_U(R) = \mathrm{Tr}\,[e^{-\pi\tau_2\hat{H}_0 - i\pi\tau_1\hat{T}_0{}^\sigma} \; \hat{g}_U(R)], \tag{2.28}$$

where

$$H_0 = \frac{1}{2}(p^A - B^{AB}L^B)^2 + \frac{1}{2}(L^A)^2 , \tag{2.29.a}$$

$$T_0{}^\sigma = p^A L^A . \tag{2.29.b}$$

$$g_U(R) \mid k^A, L^I \rangle = \mid U^{AB}k^B, U^{IJ}L^J \rangle \quad (URU^{-1} = R). \tag{2.29.c}$$

Then,

$$A_U(R) = \sum \langle k^A, L^I \mid e^{-\pi\tau_2\hat{H}_0 - i\pi\tau_1\hat{T}_0{}^\sigma} \; \hat{g}_U(R) \mid k^A, L^I \rangle$$

$$= \sum_{\substack{N^I \in \Lambda/\Lambda_R \\ L^I \in \Lambda/(1-R)\Lambda \\ k^A \in [2\Lambda^*/(1-R^{-1})2\Lambda^*]}} \delta_{k^A, U^{AB}k^B} \, \delta_{(1-U)^{IJ}L^J, (1-R)^{IJ}N^J} \; F(k^A, L^A) e^{-i\pi k^A L^A}, \tag{2.30}$$

where $F(k^A, L^A) \equiv \exp\{ -\frac{\pi}{2}\tau_2[(k^A - B^{AB}L^B)^2 + (L^A)^2] - i\pi\tau_1 k^A L^A \}$.

For this result, some remarks are needed. The sum of k and L in (2.30) is, in general, different from the sum of the U-invariant

points on the R-invariant sublattice of Λ and $2\Lambda^*$. Furthermore the form of the function in the sum is not mere $F(k,L)$ but with phase factor.

As far as we calculate the amplitude, we have another method, Polyakov's path integral. The one-loop vacuum amplitude of closed Bosonic string on orbifold is given as[1)]

$$\sum_{\substack{R,U\\L,M}} \int \frac{[dg][dX]}{V} e^{-S[g,X]} . \tag{2.31}$$

The functional integral runs over each X^I and Euclidean metric $g_{\alpha\beta}$ with boundary condition

$$X^I(\sigma^1+1,\sigma^2) = R^{IJ}X^J(\sigma^1,\sigma^2) + \pi L^I , \tag{2.32.a}$$

$$X^I(\sigma^1,\sigma^2+1) = U^{IJ}X^J(\sigma^1,\sigma^2) + \pi M^I , \tag{2.32.b}$$

$$g_{\alpha\beta}(\sigma^1+1,\sigma^2) = g_{\alpha\beta}(\sigma^1,\sigma^2+1) = g_{\alpha\beta}(\sigma^1,\sigma^2) , \tag{2.32.c}$$

and the sum is taken over all R , $U \in G$ and L^I, $M^I \in \Lambda$ under the condition

$$RU - UR = 0, \tag{2.33.a}$$

$$(1 - R)^{IJ}M^J = (1 - U)^{IJ}L^J . \tag{2.33.b}$$

Let $A\binom{R}{U}(\tau)$ be the zero mode part in (2.31) corresponding to $A_U(R)$; the part of the sum of the winding L^I and M^I for fixed R and U. For modular transformation, we can show that

$$A\binom{R}{U}(-\tfrac{1}{\tau}) = A\binom{U^{-1}}{R}(\tau) \tag{2.34.a}$$

$$A\binom{R}{U}(\tau+1) = A\binom{R}{RU}(\tau) \tag{2.34.b}$$

It is known that the oscillator part in (2.31) obeys the definite transformation rule, namely same as (2.34). Thus the total amplitude $\sum_{R,U}\{[\text{oscillator}] \times A\binom{R}{U}\}$ is modular invariant.

Furthermore we can also show that

$$A_U(R) = (2\tau_2)^{-n/2}\pi^{-n}|\det(1-\hat{U})^{ab}|^{-1}A\binom{R}{U} \tag{2.35}$$

and that the corresponding relation for the oscillator part has the reciprocal factor, so the total amplitude $\sum_{R,U}\{[\text{oscillator}] \times A_U(R)\}$ is equal to the one in the path integral and modular invariant.

For the details of the discussions in this subsection, see Ref.1).

References

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