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A NOTE ON p-SEMISIMPLE BCI-ALGEBRAS *

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ABSTRACT

In this note we prove some equivalent conditions for p-semisimple BCI-algebras. We also show that if X is a p-semisimple BCI-algebra then $\mathrm{Hom}(X)$, the set of all homomorphisms of X is a (p-semisimple) BCI-algebra, thus extending the class of BCI-algebras with this property as proposed in [10]. We also study some duality conditions.

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1. INTRODUCTION AND PRELIMINARIES

Recently, the theory of BCI-algebras has been enriched by several additional conditions leading to a further insight into the first order properties of BCI-algebras and their classification. For instance, we have the medial BCI-algebras, that is the BCI-algebras satisfying the condition: $(a*b)*(c*d) = (a*c)*(b*d)$. Similarly, we have another class of BCI-algebras satisfying: $a*(a*b) = b$. We refer to Iseki [9], Hoo [5, 6] and Dudek [3] for more details about BCI-algebras with these and other conditions. In fact, it is known (see [5], [3]) that for a BCI-algebra X the following conditions: (i) X is medial (ii) X satisfies: $a*(a*b) = b$ for all $a, b \in X$ (iii) X is p-semisimple, are equivalent. p-semisimple BCI-algebras form an important topic of current research (see for instance [1, 5, 6, 7]). One aim of this note is to obtain more equivalent conditions for BCI-algebras (see Section 2). Another purpose of this note is to study the set $\mathrm{Hom}(X)$ of all homomorphisms of a suitable BCI-algebra X. Iseki and Thaheem [10] proved that if X is an associative BCI-algebra then $\mathrm{Hom}(X)$ is again an associative BCI-algebra. Hoo and Ramanamurty [4] and Deeba and Goel [2] independently proved that $\mathrm{Hom}(X)$ may not, in general, be a BCI-algebra for an arbitrary BCI-algebra. However, we show here that if X is a p-semisimple BCI-algebra then $\mathrm{Hom}(X)$ is a (p-semisimple) BCI-algebra. This extends the result of Iseki and Thaheem [10] for a larger class of BCI-algebras because the class of associative BCI-algebras is properly contained in the class of p-semisimple BCI-algebras (Proposition 2.1). We prove these results in Section 3. In the end we initiate a duality theory for p-semisimple BCI-algebras and prove some basic results.

First, we recall the definition of a BCI-algebra and record some properties of BCI-algebras we require for our results.

Definition 1.1 [9] A BCI-algebra is an algebra $< X, *, 0 >$ of type $(2,0)$ satisfying the following axioms for all $x, y, z \in X$:

(BCI-1) $(x*y)*(x*z) \leq z*y$
(BCI-2) $x*(x*y) \leq y$
(BCI-3) $x \leq x$
(BCI-4) $x \leq y$ and $y \leq x$ imply $x = y$
(BCI-5) $x \leq 0$ implies $x = 0$

where $x \leq y$ if and only if $x*y = 0$.

The following properties hold in any BCI-algebra X (see [9]).

(P1) $(x*y)*z = (x*z)*y$
(P2) $x*0 = x$
(P3) $x \leq y$ implies $x*z \leq y*z$
(P4) $(a*b)*(c*b) \leq a*c$. To prove it, consider

$$((a*b)*(c*b))*(a*c)$$
$$=((a*b)*(a*c))*(c*b)$$
$$=((a*(a*c))*b)*(c*b)$$
$$\leq (c*b)*(c*b)=0 \qquad \text{(by (BCI–2) and P3).}$$

This proves property (P4).

2. SOME EQUIVALENT CONDITIONS

In this section we prove some equivalent conditions on BCI–algebras.

A BCI–algebra X is said to be associative [8] if $(x*y)*z = x*(y*z)$ for all x, y, z in X.

The following proposition gives a relationship between associative and p–semisimple BCI–algebras.

Proposition 2.1 An associative BCI–algebra is a p–semisimple BCI–algebra.

Proof Let X be an associative BCI–algebra. By [5] it is enough to prove that $a*(a*b) = b$ for all $a, b \in X$. Now $a*(a*b) \leq b$ (by BCI–2). Repeated use of associativity and (P1) shows that

$$b*(a*(a*b)) = (b*a)*(a*b) = (b*(a*b))*a$$
$$= ((b*a)*b)*a = (b*a)*(b*a) = 0.$$

This shows that $b \leq (a*(a*b))$ and hence $a*(a*b) = b$.

The converse of this proposition does not hold in general. For instance, let $X = \{0, a, b\}$ and define an operation $*$ on X as

*	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Then X is a p–semisimple BCI–algebra but it is not an associative BCI–algebra because $(0*a)*b \neq 0*(a*b)$.

The following theorem shows that with an additonal condition on X (namely $0*x = x$) the associativity and p–semisimplicity coincide. In fact, we obtain more equivalent conditions.

Theorem 2.2 Let X be a BCI–algebra with condition $0*x = x, (x \in X)$. Then the following conditions are equivalent:

(1) $a*(b*c) = (a*b)*c$
(2) $(a*b)*b = a$
(3) $(a*b)*c = (c*b)*a$
(4) $(a*b)*(c*d) = (a*c)*(b*d)$
(5) $a*(a*b) = b$
(6) X is p–semisimple.

Proof

(1) implies (2): $(a*b)*b = a*(b*b) = a*0 = a$.
(2) implies (3): consider

$$((a*b)*c)*((c*b)*a)$$
$$=((a*b)*((c*b)*a))*c \qquad \text{(by P1)}$$
$$=((a*b)*((c*a)*b))*c \qquad \text{(by P1)}$$
$$\leq (a*(c*a))*c \qquad \text{(by P4 and P3)}$$
$$=((0*a)*(c*a))*c$$
$$=((0*(c*a))*a)*c$$
$$=((c*a)*a)*c$$
$$=c*c=0 \qquad \text{(by the given condition (2)).}$$

Thus we obtain that $(a*b)*c \leq (c*b)*a$. By symmetry, $(c*b)*a \leq (a*b)*c$ and hence $(a*b)*c = (c*b)*a$.
(3) implies (4): repeated use of condition (3) gives $(x*y)*(w*z) = ((w*z)*y)*x = ((y*z)*w)*x = (x*w)*(y*z)$.
(4) implies (5) and (5) implies (6) follow from Hoo [5] (see also Dudek [3]). Finally, suppose that X is p–semisimple. Then in [11] an Abelian group operation $+$ is defined by $x+y = x*(0*y)$ and $y+x = y*(0*x)$. The condition $0*x = x$ implies that $x*y = y*x$ and hence $(a*b)*c = (b*a)*c = (b*c)*a = a*(b*c)$. This completes the proof of the theorem.

3. $Hom(X)$ FOR A BCI–ALGEBRA X

In this section we study the BCI–algebra of homomorphsims. By Iseki [9], a mapping $f : X \to Y$ between BCI–algebras X and Y is called homomorphism if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$. Define the trivial homomorphism 0 as $0(x) = 0$ for all $x \in X$. Denote by $\mathrm{Hom}(X)$ the set of all homomorphisms of X into X. Iseki and Thaheem [10] proved that if X is an associative BCI–algebra then $\mathrm{Hom}(X)$ is a BCI–algebra. Hoo and Ramanamurty [4] and Deeba and Goel [2] independently proved that this may not be the case for an arbitrary BCI–algebra. However, the BCI–algebras of their examples are not p–semisimple. Here we prove that if X is a p–semisimple then $\mathrm{Hom}(X)$ is again a BCI–algebra, thus extending the class of BCI–algebras for which $\mathrm{Hom}(X)$ is again a BCI–algebra.

Let X be a p–semisimple BCI–algebra and $f, g \in \mathrm{Hom}(X)$. Define a mapping $f*g : X \to X$ by

$$(f*g)(x) = f(x)*g(x), \quad (x \in X).$$

The following proposition is an important step to prove that Hom(X) is a BCI–algebra.

Proposition 3.1 Let X be a p-semisimple BCI–algebra and $f, g \in$ Hom(X). Then $f * g \in$ Hom(X).

Proof For any $x, y \in X$,

$$(f * g)(x * y) = f(x * y) * g(x * y) = (f(x) * f(y)) * (g(x) * g(y)).$$

By [5] X is p–semisimple if and only if X is medial. Hence

$$\begin{aligned}(f * g)(x * y) &= (f(x) * g(x)) * (f(y) * g(y)) \\ &= (f * g)(x) * (f * g)(y).\end{aligned}$$

It follows that $f * g \in$ Hom(X).

Routine calculations now lead to the following:

Theorem 3.2 If X is a p-semisimple BCI–algebra then $<$ Hom(X), $*$, $0 >$ is a (p-semisimple) BCI–algebra.

We now initiate a duality theory for BCI–algebras and investigate some properties of a BCI–algebra X via Hom(X). In what follows, X will denote a p–semisimple BCI–algebra unless mentioned otherwise. For simplicity we denote Hom(X) by X^*.

Notation It is convenient to designate the elements of X^* by x^* and write $< x, x^* >$ in place of $x^*(x) \in X$ to describe duality between X and X^*.

Definition 3.3 Let M and N be subsets of X and X^* respectively. We define orthogonal subsets $M^{\perp}$ and ${}^{\perp}N$ of M and N respectively by

$$\begin{aligned} M^{\perp} &= \{x^* \in X^* : < x, x^* > = 0 \quad \text{for all} \quad x \in M\} \\ {}^{\perp}N &= \{x \in X : < x, x^* > = 0 \quad \text{for all} \quad x^* \in N\} \end{aligned}$$

It follows immediately from the definition that if $L \subseteq M$ then $M^{\perp} \subseteq L^{\perp}$ and $M \subseteq {}^{\perp}(M^{\perp})$.

By Hoo [7] an ideal I of a BCI–algebra X is a closed ideal if $0 * x \in I$ whenever $x \in I$.

Proposition 3.4 $M^{\perp}$ and ${}^{\perp}N$ are closed ideals of X^* and X respectively.

Proof Follows easily from the definitions.

If M is a maximal ideal then it follows from the remarks following Definition 3.3 that $M = {}^{\perp}(M^{\perp})$.

An ideal I of X is called weakly implicative [7] if whenever $(x * y) * z, y * z \in I$ then $(x * z) * z \in I$.

The following proposition can be obtained from Lemma 1.5 and Lemma 1.7 of [7] but we prefer to give a direct proof.

Proposition 3.5 $M^{\perp}$ and ${}^{\perp}N$ are weakly implicative ideals.

Proof We give the proof for ${}^{\perp}N$ and the proof for $M^{\perp}$ will follow similarly. Let $(x*y)*z, y*z \in {}^{\perp}N$. Then for any $x^* \in N$,

$$\begin{aligned} < (x * y) * z, x^* > &= (< x, x^* > * < y, x^* >) * < z, x^* > \\ &= (< x, x^* > * < z, x^* >) * < y, x^* > = 0 \end{aligned}$$

This implies $\quad < x, x^* > * < z, x^* > \leq < y, x^* > . \quad$ (i)

Also $\quad < y * z, x^* > = < y, x^* > * < z, x^* > = 0 \quad$ implies $\quad < y, x^* > \leq < z, x^* > . \quad$ (ii)

It follows from (i) and (ii) that $< x, x^* > * < z, x^* > \leq < z, x^* >$. This implies that $< (x * z) * z, x^* > = 0$ for all $x^* \in N$ and hence $(x * z) * z \in {}^{\perp}N$. This proves that ${}^{\perp}N$ is a weakly implicative ideal. A similar argument works for $M^{\perp}$.

We now associate with each homomorphism $T : X \to X$ its adjoint homomorphism $T^* : X^* \to X^*$ and see how certain properties of T are reflected in terms of T^*. The following theorem furnishes the definition of T^*.

Theorem 3.6 To each homomorphism $T : X \to X$ there corresponds a unique homomorphism $T^* : X^* \to X^*$ that satisfies

$$< Tx, x^* > = < x, T^*x^* > \qquad (1)$$

for all $x \in X$ and all $x^* \in X^*$.

Proof For each $x_1^* \in X^*$ we can define a mapping $g : X \to X$ by the relation $g(x) = < Tx, x_1^* >, x \in X$. Since x_1^* and T are homomorphisms, therefore g is a homomorphism and $g \in X^*$. Denote the function defined this way by

$$T^*(x_1^*) = g.$$

Thus $T^* : X^* \to X^*$ is a mapping. Now

$$\begin{aligned} < x, T^*(x_1^* * x_2^*) > &= < Tx, x_1^* * x_2^* > = < Tx, x_1^* > * < Tx, x_2^* > \\ &= < x, T^*x_1^* > * < x, T^*x_2^* > \\ &= < x, T^*x_1^* * T^*x_2^* > \end{aligned}$$

for all $x \in X$ and $x_1^*, x_2^* \in X^*$. This shows that T^* is a homomorphism. The fact that (1) holds for all $x \in X$ obviously determines T^*x^* uniquely. This completes the proof.

Theorem 3.7 Let $T : X \to X$ be a homomorphism. Then $Ker(T^*) = R(T)^{\perp}$ and $Ker(T) = {}^{\perp}R(T^*)$ where $Ker(\ )$ and $R(\ )$ are the kernel and the range of the homomorphisms under consideration.

Proof Suppose $x^* \in Ker(T^*)$ then $T^*(x^*) = 0$ and hence $< x, T^*x^* > = < Tx, x^* >= 0$ for all $x \in X$. Thus $x^* \in R(T)^\perp$ and $Ker(T^*) \subseteq R(T)^\perp$. Similarly $R(T)^\perp \subseteq Ker(T^*)$. Therefore, $Ker(T^*) = R(T)^\perp$. This proves the first part of the theorem. The other equality holds by a simular argument.

The following corollary follows easily. We only observe that a homomorphism $T : X \to X$ is one-one if and only if $Ker(T) = \{0\}$.

Corollary 3.8 (i) $R(T) = X$ if and only if T^* is one-one. (ii) T is one-one if and only if $R(T^*) = X^*$.

We conclude this note with the following duality problem for quotient BCI-algebras.

Problem If M is an ideal of X, then what is the relationship between:

(1) $(X/M)^*$ and $M^\perp$

(2) M^* and $X^*/M^\perp$?

Are $(X/M)^*$ and $M^\perp$ (respectively $X^*/M^\perp$ and M^*) isomorphic?

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