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ON THE QUANTUM INVERSE PROBLEM FOR A NEW TYPE OF NONLINEAR SCHRÖDINGER EQUATION FOR ALFVEN WAVES IN PLASMA *

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ABSTRACT

The nonlinear Alfven waves are governed by the Vector Derivative nonlinear Schrödinger (VDNLS) equation, which for parallel or quasi parallel propagation reduces to the Derivative Nonlinear Schrödinger (DNLS) equation for the circularly polarized waves [1].

We have formulated the Quantum Inverse problem for a new type of Nonlinear Schrödinger Equation of the form

$$\psi_{1t} = i\,\psi_{1xx} - 2c\,\psi_1\psi_2\psi_{1x}$$
$$\psi_{2t} = -i\,\psi_{2xx} - 2c\,\psi_1\psi_2\psi_{2x}$$

This set of evolution equations have many properties similar to the usual NLS problem but the structure of classical and quantum R matrix are distinctly different. The commutation rules of the scattering data are obtained and the Algebraic Bethe Ansatz is formulated to derive the eigenvalue equation for the energy of the excited states.

1. INTRODUCTION

The quantization of nonlinear systems is one of the most important programmes in the study of solitons. Apart from the semiclassical approach of Gutzwiller [2], the most important formulation was that of Faddeev [3]. The basic ingredient of the formulation is the classical $r(\lambda, u)$ and quantum $R(\lambda, u)$ matrix. Theories with some particular form of r and R have been classified by Tarasov [4] and Sklyanin [5]. Here we have tried to formulate the Quantum Inverse Problem for a new integrable nonlinear system, similar to the NLS problem.

2. FORMULATION

The equations under consideration can be written as

$$\begin{aligned} \psi_{1t} &= i\,\psi_{1xx} - 2c\,\psi_1\psi_2\psi_{1x} \\ \psi_{2t} &= i\,\psi_{2xx} - 2c\,\psi_1\psi_2\psi_{2x} \end{aligned} \tag{1}$$

Eq.(1) is an integrable system both in the sense of Painleve analysis and in having a Lax pair. The Lax pair has a similar form regarding its dependence on the field variables. But with respect to the eigenvalue parameter λ, it resembles the Kaup Nevell problem. The space part of the Lax pair for (1) can be written as;

$$\left[\frac{\partial}{\partial x} + \begin{pmatrix} -\frac{i}{2}(c\psi_1\psi_2 + \lambda^2) & -\lambda\psi_1\sqrt{c} \\ -\lambda\psi_2\sqrt{c} & \frac{i}{2}(\lambda^2 + \psi_1\psi_2 c) \end{pmatrix}\right]\psi = 0 \tag{2}$$

At present there exists two approaches to the quantization.

a) The differential equation approach: this method has been critized severely by Gutkin [6].

b) The space discretization approach: this does not suffer from difficulties of (a) but is still considered to be an approximate one unless a limit $N \to \infty$ is properly taken, where N is the number of subdivision of the interval $(0, L)$ of the real axis in subintervals of length Δ. We pursue here the second line of working.

By converting the linear problem into a Riccati system, we can generate an infinite number of conservation laws, C_n. From these it is easy to pick up the Hamiltonian given as

$$H = \int_{x_0}^{x_0+L} C_2\,dx = \frac{1}{2}\int_{x_0}^{x_0+L}\Big[i(\psi_1\psi_{2x} - \psi_{1x}\psi_{2x}) - \psi_1\psi_2(\psi_1\psi_{2x} - \psi_{1x}\psi_2)\Big]dx \tag{3}$$

from which Eq.(1) can be generated via

$$\begin{aligned} \frac{\partial\psi_1}{\partial t} &= \{H, \psi_1\}_{\mathcal{L}} \\ \frac{\partial\psi_2}{\partial t} &= \{H, \psi_2\}_{\mathcal{L}} \end{aligned} \tag{4}$$

where $\mathcal{L}$ is the symplectic operator

$$\mathcal{L} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{5}$$

yielding the Poisson Bracket

$$\{f, g\}_{\mathcal{L}} = i\int_{x_0}^{x_0+L}\left(\frac{\delta f}{\delta\psi_2}\frac{\delta g}{\delta\psi_1} - \frac{\delta f}{\delta\psi_1}\frac{\delta g}{\delta\psi_2}\right)dx \tag{6}$$

Thus we proceed with the discretization of the interval $(x_0, x_0 + L)$ and rewrite the Lax Eq.(2) in the form:

$$\psi_{n+1} = L_n\,\psi_n \tag{7}$$

where L_n is defined via

$$L_n \approx 1 + \Delta\int_{x_0}^{x_0+L} L\,dx \tag{8}$$

and

$$\begin{aligned} \psi_{1n} &= \int_{x_0}^{x_0+L}\psi_1\,dx \\ \psi_{2n} &= \int_{x_0}^{x_0+L}\psi_2\,dx \end{aligned} \tag{9}$$

The Poisson bracket is defined through

$$[\psi_{in}, \psi_{2n}] = \frac{-1}{\Delta} \tag{10}$$

and

$$L_n = \begin{pmatrix} 1 - \frac{i\Delta}{2}(c\psi_{1n}\psi_{2n} + \lambda^2) & -\Delta\lambda\,\psi_{1n}\sqrt{c} \\ -\Delta\lambda\,\psi_{2n}\sqrt{c} & 1 + \frac{i\Delta}{2}(\psi_{1n}\psi_{2n}c + \lambda^2) \end{pmatrix} \tag{11}$$

A. The classical r–matrix

The classical r–matrix is defined following Faddeev via the equation

$$\{L(\lambda, x) \underset{j}{\otimes} L(u, y)\} = \left[r(\lambda, u), L(\lambda, x) \otimes 1 + 1 \otimes L(u, y)\right] \tag{12}$$

Computing the 16 Poisson brackets on the LHS we can easily solve for $r(\lambda, u)$, and

$$r(\lambda, u) = \begin{pmatrix} \frac{-(\lambda^2+u^2)}{2(\lambda^2-u^2)} & 0 & 0 & 0 \\ 0 & 0 & \frac{-\lambda u}{\lambda^2-u^2} & 0 \\ 0 & \frac{-\lambda u}{\lambda^2-u^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{-(\lambda^2+u^2)}{2(\lambda^2-u^2)} \end{pmatrix} \tag{13}$$

This form of r–matrix is distinctly different from that of the NLS model, having poles at two positions $\lambda = \pm u$, but the symmetry remains the same.

B. Quantum R matrix

For the quantum mechanical case we start with the discrete form of L i.e. L_n and observe that πL_n can be interpreted as the transition matrix. We now evaluate the equation [7]:

$$R(\lambda,u)[L_n(\lambda)\otimes L_n(u)] = [L_n(u)\otimes L_n(\lambda)]R(\lambda,u) \tag{14}$$

keeping in mind the ordering of ψ_{1n}, ψ_{2n} and interpreting Eq.(10) as a commutator and obtain

$$R(\lambda,u) = \begin{pmatrix} 1+c\frac{\lambda^2+u^2}{2(\lambda^2-u^2)} & 0 & 0 & 0 \\ 0 & 1 & \frac{c\lambda u}{\lambda^2-u^2} & 0 \\ 0 & \frac{c\lambda u}{\lambda^2-u^2} & 1 & 0 \\ 0 & 0 & 0 & 1+c\frac{\lambda^2+u^2}{2(\lambda^2-u^2)} \end{pmatrix}. \tag{15}$$

It is interesting to note that the form of Quantum R–matrix follows the same rule as that of the NLS equation, i.e. if one thinks of 'c' to be proportional to $\hbar$, the Eq.(15) is true for calculation up to Δ.

C. Commutation rule for the scattering data

Since written in discrete variable, the space part of the Lan equation becomes

$$\psi_{n+1} = L_n\,\psi_n \tag{16}$$

One can interpet L_n as the transfer matrix over one lattice site.

We define the transition matrix

$$T(n,m/\lambda) = L_n(\lambda)\;L_{n-1}(\lambda)\ldots L_m(\lambda) \tag{17}$$

and the monodromy matrix

$$\begin{aligned} T(\lambda)\text{as} &= L_n(\lambda)\;L_{n-1}(\lambda)\ldots L_1(\lambda) \\ &= \prod_{i=1}^{n} L_i(\lambda). \end{aligned} \tag{18}$$

So from the basic relation (14) we get

$$R(\lambda,u)\big(T(\lambda)\otimes T(u)\big) = \big(T(u)\otimes T(\lambda)\big)R(\lambda,u). \tag{19}$$

We define

$$T(\lambda) = \begin{pmatrix} a(\lambda) & \bar{b}(\lambda) \\ b(\lambda) & -\bar{a}(\lambda) \end{pmatrix} \tag{20}$$

So coupling (19) we obtain

$$\begin{aligned} a(\lambda)a(u) &= a(u)a(\lambda) \\ \bar{b}(\lambda)\bar{b}(u) &= \bar{b}(u)\bar{b}(\lambda) \\ b(\lambda)b(u) &= b(u)b(\lambda) \\ \bar{a}(\lambda)\bar{a}(u) &= \bar{a}(u)\bar{a}(\lambda) \\ a(u)\bar{b}(\lambda) &= \alpha(\lambda u)\bar{b}(\lambda)a(u) - \beta(\lambda u)\bar{b}(u)a(\lambda) \\ b(\lambda)a(u) &= \alpha(\lambda u)a(u)b(\lambda) - \beta(\lambda u)a(\lambda)b(u) \\ b(u)\bar{a}(\lambda) &= \alpha(\lambda u)\bar{a}(\lambda)b(u) - \beta(\lambda u)\bar{a}(u)b(\lambda) \\ \bar{a}(\lambda)\bar{b}(u) &= \alpha(\lambda u)\bar{b}(u)\bar{a}(\lambda) - \beta(\lambda u)\bar{b}(\lambda)\bar{a}(u) \end{aligned} \tag{21}$$

where

$$\alpha(\lambda u) = 1 + \frac{c\lambda^2+u^2}{2(\lambda^2-u^2)}$$

$$\beta(\lambda u) = \frac{c\lambda u}{\lambda^2-u^2}$$

D. Construction of the Eigenstates

The eigenstates of the quantized system can be constructed by starting with a postulated vacuum. Let us designate the vacumm state by

$$\psi_{2n}|0> = 0$$

Then the operator L_n acts on $|0>$ as a triangular matrix, i.e.

$$L_n(\lambda)|0> + \begin{pmatrix} 1-\frac{i\Delta\lambda^2}{2} & -\sqrt{c}\,\lambda\Delta\psi_{in} \\ 0 & 1+\frac{i\Delta\lambda^2}{2} \end{pmatrix}|0>. \tag{22}$$

From the property of triangular matrices we immediately get

$$\begin{aligned} T(\lambda)|0> &= \prod_{i=1}^{n} L_i(\lambda)|0> \\ &= \begin{pmatrix} \left(1-\frac{i\Delta\lambda^2}{2}\right)^n & \sigma(\lambda)\psi_{1n} \\ 0 & \left(1+\frac{i\Delta\lambda^2}{2}\right)^n \end{pmatrix}|0> \end{aligned} \tag{23}$$

Making n very large so that $\Delta = \frac{L}{N}$ is small we get in the limit

$$T(\lambda)|0> = \begin{pmatrix} e^{-iL\lambda^2/2} & \sigma(\lambda)\psi_{1n} \\ 0 & e^{iL\lambda^2/2} \end{pmatrix}|0> \tag{24}$$

So ψ_{1n} is to be used as the creation operator and also we get

$$\begin{aligned} a(\lambda)|0> &= e^{-iL\lambda^2/2}|0> \\ \bar{b}(\lambda)|0> &= \sigma(\lambda)|0> \\ b(\lambda)|0> &= 0 \\ \bar{a}(\lambda)|0> &= -e^{iL\lambda^2/2}|0> \end{aligned} \tag{25}$$

as the basic relation to be used in the construction of the Algebraic Bethe Ansatz.

We now define a series of physical states of the form [8]

$$\Omega_1(\lambda_1) = \bar{b}(\lambda_1)|0>$$
$$\Omega_2(\lambda_1\lambda_2) = \bar{b}(\lambda_1)\bar{b}(\lambda_2)|0>$$
$$\Omega_n(\lambda_1\ldots\lambda_n) = \bar{b}(\lambda_1)\ldots\bar{b}(\lambda_n)|0> \qquad (26)$$

To ascertain the eigenmomenta and eigenenergies of $\Omega_1, \Omega_2 \ldots$, we operate with $a(u)$ and $\bar{a}(u)$ on $\Omega_1, \Omega_2 \ldots$, and utilize the commutation rules (21) to shift $a(u)$ so as to operate on $|0>$. Since in the sequel we define the Hamiltonian by $Tr\,T(\lambda) = a(u) + \bar{a}(u)$, we demand $\Omega_1, \Omega_2 \ldots$, to be an eigenstate of $a(u) + \bar{a}(u)$ which leads to the following equations determining the eigenmomenta and energy eigenvalues of the two particle state

$$\begin{aligned}&[a(u) + \bar{a}(u)]\Omega_2(\lambda_1, \lambda_2)\\ &\quad = [\alpha(\lambda_1, u)\alpha(\lambda_2, u)e^{-iu^2 L/2} - \alpha(u\lambda_1)\alpha(u\lambda_2)e^{iu^2 L/2}]\Omega_2(\lambda_1\lambda_2)\end{aligned} \qquad (27)$$

along with the condition

$$e^{i\lambda_1^2 L} = \frac{\beta(\lambda_1, u)\alpha(\lambda_2\lambda_1)}{\beta(u, \lambda_1)\alpha(\lambda_1\lambda_2)}$$
$$e^{i\lambda_2^2 L} = \frac{\alpha(\lambda_1, u)\beta(\lambda_2, u) - \beta(\lambda, u)\beta(\lambda_2\lambda_1)}{\alpha(u, \lambda_1)\beta(u, \lambda_2) - \beta(u\lambda_1)\beta(\lambda_1\lambda_2)}$$

By using the expressions for $\alpha(\lambda u)$ and $\beta(\lambda u)$ we get

$$e^{i\lambda_1^2 L} = \frac{\lambda_2^2(c+2) + \lambda_1^2(c-2)}{\lambda_1^2(c+2) + \lambda_2^2(c-2)}$$
$$e^{i\lambda_2^2 L} = \frac{\lambda_1^2(c+2) + \lambda_2^2(c-2)}{\lambda_2^2(c+2) + \lambda_1^2(c-2)}$$

The same construction can easily be extendxed to the n particle states and equations are trascendental equations determining the eigenvalues (momenta) of λ excited states.

E. Refinement of the Yang–Baxter relation

The above calculations are confined to the periodic case $n \to -L$ to $+L$. But for $n \to \infty$ the commutation relations get changed due to the modified Yang–Baxter relation.

The expectation value of $L_n(\lambda) \otimes L_n(u)$ between the vacuum states are [9]

$$W(\lambda, u) = <0|L_n(\lambda) \otimes L_n(u)|0> .$$

Thus

$$W(\lambda, u) = \begin{pmatrix} 1 - \frac{i\Delta}{2}(\lambda^2 + u^2) & 0 & 0 & 0 \\ 0 & 1 - \frac{i\Delta}{2}(\lambda^2 - u^2) & -ic\lambda u\Delta & 0 \\ 0 & 0 & 1 + \frac{i\Delta}{2}(\lambda^2 - u^2) & 0 \\ 0 & 0 & 0 & 1 + \frac{i\Delta}{2}(\lambda^2 + u^2) \end{pmatrix} \qquad (28)$$

The normalized monodromy matrix $T(\lambda)$ is defined by [7]

$$T(\lambda) = \lim_{N\to\infty} v(\lambda)^{-N} L_n(\lambda) \ldots L_{-N+1}(\lambda) v(\lambda)^{-N}$$

where

$$v(\lambda) = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}; \quad \begin{aligned} w_1 &= 1 - \tfrac{i\Delta}{2}\lambda^2 \\ w_2 &= 1 + \tfrac{i\Delta}{2}\lambda^2 \end{aligned} \qquad (29)$$

The Yang–Baxter relation for $T(\lambda)$ is modified to

$$R_1(\lambda, u)\big(T(\lambda) \otimes T(u)\big) = \big(T(u) \otimes T(\lambda)\big)R_2(\lambda u) \qquad (30)$$

where

$$R_1(\lambda u) = U_1(u\lambda)^{-1} R(\lambda u) U_1(\lambda u)$$
$$R_2(\lambda u) = U_2(u\lambda) R(\lambda u) U_2(\lambda u)^{-1}$$
$$U_1(\lambda u) = \lim_{N\to\infty} W(\lambda u)^{-N}\big(v(\lambda)^N \otimes v(u)^N\big)$$
$$U_2(\lambda u) = \lim_{N\to\infty} \big(v(\lambda)^N \otimes v(u)^N\big) W(\lambda u)^{-N}.$$

We computed $R_1(\lambda, u)$ and $R_2(\lambda, u)$ as

$$R_1(\lambda u) = \begin{pmatrix} 1 + \frac{c(\lambda^2+u^2)}{2(\lambda^2-u^2)} & 0 & 0 & 0 \\ 0 & 1 - f(\lambda u) & \frac{c\lambda u}{(\lambda^2-u^2)}(1 + h(\lambda u)) & 0 \\ 0 & \frac{c\lambda u}{(\lambda^2-u^2)} & 1 - g(\lambda u) & 0 \\ 0 & 0 & 0 & 1 + \frac{c(\lambda^2+u^2)}{2(\lambda^2-u^2)} \end{pmatrix}$$

$$R_2(\lambda u) = \begin{pmatrix} 1 + \frac{c(\lambda^2+u^2)}{2(\lambda^2-u^2)} & 0 & 0 & 0 \\ 0 & 1 - g(\lambda u) & \frac{c\lambda u}{(\lambda^2-u^2)}(1 + h(\lambda u)) & 0 \\ 0 & \frac{c\lambda u}{(\lambda^2-u^2)} & 1 - f(\lambda u) & 0 \\ 0 & 0 & 0 & 1 + \frac{c(\lambda^2+u^2)}{2(\lambda^2-u^2)} \end{pmatrix} \qquad (31)$$

where

$$f(\lambda u) = \Big(\frac{c\lambda u}{\lambda^2 - u^2}\Big)^2 [1 + 2\pi i\delta(\lambda^2 - u^2)(\lambda^2 - u^2 + i0)]$$
$$g(\lambda u) = \Big(\frac{c\lambda u}{\lambda^2 - u^2}\Big)^2 [1 - 2\pi i\delta(\lambda^2 - u^2)(\lambda^2 - u^2 - i0)]$$
$$h(\lambda u) = -\Big(\frac{2c\lambda u}{\lambda^2 - u^2}\Big)\Big[1 - \Big(\frac{c\lambda u}{\lambda^2 - u^2}\Big)^2\Big][1 - i\pi\delta(\lambda^2 - u^2)].$$

One can now substitute $R_1(\lambda u), R_2(\lambda u)$ in Eq.(30) to calculate the refined commutation rules of the scattering data and construct the eigenstates.

F. Construction of the discrete Hamiltonian

Since the method of Quantum Inverse Transform effectively works on the discretized $C_n(-L, L)$, it is necessary to formulate a discrete form of the Hamiltonian from the trace of the matrix $T(\lambda)$, (actually the logarithmic derivative of the $Tr\, T(\lambda)$ with respect to λ), i.e.

$$H = \frac{\partial}{\partial\lambda} \ell n\, Tr T(\lambda). \tag{32}$$

When the determinant vanishes, the L operator is transformed into the one dimensional projection operator, i.e. det $|L_n| = 0$ at

$$\lambda = \pm\left(c\,\psi_{1n}\psi_{2n} + \frac{2i}{\Delta}\right)^{1/2} \tag{33}$$

So at such a value of the spectral parameter, L_n is degenerate and one can write L_n as the direct product of two vectors in the form;

$$(L_n)_{ik} = \alpha_i(\Delta)\bar{\beta}_k(\Delta) \tag{34}$$

with

$$\alpha_i(\Delta) = \begin{pmatrix} 2\left(1 - \frac{ic\Delta}{2}\,\psi_{1n}\psi_{2n}\right) \\ -\sqrt{2i\Delta c}\left(1 - \frac{ic\Delta}{2}\,\psi_{1n}\psi_{2n}\right)^{1/2}\psi_{2n} \end{pmatrix}$$

$$\bar{\beta}_k(\Delta) = \left(2\left(1 - \frac{ic\Delta}{2}\psi_{1n}\psi_{2n}\right), -\sqrt{2i\Delta c}\left(1 - \frac{ic\Delta}{2}\psi_{1n}\psi_{2n}\right)^{1/2}\psi_{1n}\right)$$

Constructing the monodromy matrix in a standard manner and using the expression for $(L_n)_{ik}$ in Eq.(34) we get

$$H = \frac{\partial}{\partial\lambda} In\, Tr\, T(\lambda)$$
$$= (-i\Delta)^{1/2} \sum_{n=1}^{N} 2\alpha_1(n-1)^{1/2} + \sum_{n=1}^{N} \frac{\alpha_1(n-1)^{1/2}\alpha_1(n)^{1/2}\left(\alpha_1(n-1)^{1/2} + \alpha_1(n)^{1/2}\right)}{(\alpha(n-1)\cdot\alpha(n))}) \tag{35}$$

where $\alpha(n-1)\cdot\alpha(n)$ denotes the dot product of two dimensional vectors. Eq.(35) actually represents a generalization of the nearest neighbour interaction usually adapted in lattice systems.

G. Exact form of $L_n(\lambda)$ operator [10]

Until now we were working with the form of L_n which is correct up to the order of Δ. But it is possible to construct a form of $L_n(\lambda)$ which is valid for all values of Δ. A procedure to do so was suggested by Sklyanin, Korepin and others. We assume the corrected version of L_n written as

$$\tilde{L}_n(\lambda) = \begin{pmatrix} 1 - \frac{i\Delta}{2}\left(c\,\psi_{1n}\psi_{2n} + \lambda^2\right) & -\Delta\sqrt{c}\,\lambda\psi_{1n}\, f(\Delta, \psi_1\psi_2) \\ -\Delta\sqrt{c}\,\lambda\psi_{2n}\, f(\Delta, \psi_1\psi_2) & 1 + \frac{i\Delta}{2}\left(c\,\psi_{1n}\psi_{2n} + \Delta^2\right) \end{pmatrix} \tag{36}$$

where f is a function of the arguments Δ, ψ_1, ψ_2, such that as $\Delta \to 0$, it goes to one. We now assume that this corrected form of $L_n(\lambda)$ will lead to the same classical r-matrix as in Eq.(13). So that if we again compute $\{\tilde{L}_n(x) \underset{,}{\otimes} \tilde{L}_n(u)\}$ we will obtain some equations determining the function f. In the present case we have

$$i\Delta\{\psi_{1n}\psi_{2n}, \psi_{1n} f\} = -\psi_{1n} f$$
$$i\Delta\{\psi_{1n}\psi_{2n}, \psi_{2n} f\} = \psi_{2n} f \tag{37}$$

an easy solution to these equations is seen to be

$$f = \left(1 + \frac{c\Delta^2}{4}\,\psi_{1n}\psi_{2n}\right)^{1/2} \quad \text{which} \quad \to 1 \quad \text{as} \quad \Delta \to 0. \tag{38}$$

H. Derivation of the properties of the scattering data

In component form we rewrite the space part of the Lan equation as

$$v_{1x} = \frac{i\lambda^2}{2}\, v_1 - \frac{ic}{2}\,\psi_1\, v_1\, \psi_2 + \lambda\sqrt{c}\,\psi_1\, v_2$$
$$v_{2x} = \frac{i\lambda^2}{2}\, v_2 - \frac{ic}{2}\,\psi_1\, v_2\, \psi_2 + \lambda\sqrt{c}v_1\;\; \psi_2 \tag{39}$$

where the terms are all written in specific normal ordered forms.

We use the asymptotic behaviour of the jost solutions to convert these into integral equations which are

$$X_1(x,\lambda)e^{i\lambda^2 x/2} = -\int_{-\infty}^{\infty} d\bar{x}\,\theta(\bar{x}-x)\frac{ic}{2}\,\psi_1(\bar{x})X_1(\bar{x},\lambda)e^{i\lambda^2\bar{x}/2}\psi_2(\bar{x})$$
$$+ \int_{-\infty}^{\infty} d\bar{y}\,\theta(\bar{y}-x)\lambda\sqrt{c}\,e^{i\lambda^2\bar{y}}\psi(\bar{y})X_2(\bar{y},\lambda)e^{-i\lambda^2\bar{y}/2}$$

along with

$$X_2(x,\lambda)e^{-i\lambda^2 x/2} = 1 - \int_{-\infty}^{\infty} d\bar{y}\,\theta(\bar{y}-x)\frac{ic}{2}\,\psi_1(\bar{y})X_2(\bar{y})e^{-i\lambda^2\bar{y}/2}\psi_2(\bar{y})$$
$$+ \int_{-\infty}^{\infty} d\bar{x}\,\theta(\bar{x}-x)e^{i\lambda^2\bar{x}}\lambda\sqrt{c}\,X_1(\bar{x},\lambda)e^{i\lambda^2\bar{x}/2}\psi_2)\bar{x}) \tag{40}$$

We now adopt the procedure of iterating these integral equations and obtain (we display here the result obtained after a few iterations);

$$X_1(x,\lambda)e^{i\lambda^2 x/2} = \int\int\int dx_3\, dx_2\, dy_1\, \theta(y_1 > x_2 > x_3 > x)\left(\frac{ic}{2}\right)^2 \lambda\sqrt{c}\, e^{i\lambda^2 y_1} :$$
$$\psi_1(x_3)\psi_1(x_2)\psi_1(y_1)\psi_2(x_2)\psi_2(x_3) :$$

$$
\begin{aligned}
&+\int\int\int dx_3 dy_2 dy_1\ \theta(y_1 > y_2 > x_3 > x)\left(\frac{ic}{2}\right)^2 \lambda\sqrt{c}\, e^{i\lambda^3 y_1} : \\
&\quad \psi_1(x_3)\psi_1(y_2)\psi_1(y_1)\psi_2(y_1)\psi_2(x_3) : \\
&+\int\int\int dy_3 dy_2 dy_1\ \theta(y_1 > y_2 > y_3 > x)\left(\frac{ic}{2}\right)^{1/2} \lambda\sqrt{c}\, e^{i\lambda^3 y_1} : \\
&\quad \psi_1(y_3)\psi_1(y_2)\psi_1(y_1)\psi_2(y_1)\psi_2(y_2); \\
&+\int\int\int dy_3 dx_2 dy_1\ \theta(y_1 > x_2 > y_3 > x)(\lambda\sqrt{c})^3 e^{i\lambda^3 x_2} e^{i\lambda^3 y_3} e^{i\lambda^3 y_1} : \\
&\quad \psi_1(y_3)\psi_1(y_1)\psi_2(x_2) : \\
&-\int\int dx_3 dy_2\ \theta(y_2 > x_3 > x)\left(\frac{ic}{2}\right)\lambda\sqrt{c}\, e^{i\lambda^3 y_2} : \\
&\quad \psi_1(x_3)\psi_1(y_2)\psi_2(x_3) : \\
&+\int dy_3\ \theta(y_3 > x)\lambda\sqrt{c}\, e^{i\lambda^3 y_3}\psi_1(y_3) + \ldots \qquad (41)
\end{aligned}
$$

The important part to observe is that each term on the right-hand side of Eq.(41) has one operator ψ_1 excess and so the interpretation as a creation operator.

Similarly it is seen that $X_2(\lambda, \lambda)$ contains at every stage the same number of ψ_1 and ψ_2 so is equivalent to the number operator.

3. CONCLUSION

In the above analysis we have discussed the problem of quantization of a new integrable system. The new structure of $r(\lambda, u)$ and $R(\lambda, u)$ are important and distinctive feature of the problem. It is also shown that one can construct an exact form of the classical L_n–operator for all Δ. Bethe eigenstates are easy to construct along with discrete Hamiltonians describing the nearest or next to nearest neighbour interactions, suitable for the description of the model on the lattice. Lastly, the operator interpretation of the scattering data and the exact form of the commutation rules for the scattering data are deduced by proceeding to the $N \to \infty$ limit.

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REFERENCES

[1] Lecture Note on Chaotic Alfven Waves by B. Buti, Spring College on Plasma Physics, 15 May–9 June, 1989.

[2] M.C. Gutzwiller, J. Math. Phys. 8 (1967) 1979; 11 (1970) 1791; 12 (1971) 343.

[3] L.D. Faddeev, Sov. Sci. Rev. Math. Phys. C1 (1987) 107.

[4] V.O. Tarasov, Theor. Math. Phys. 61 (1984) 1211.

[5] E.K. Sklyanin, Dok. Akad. Nauk SSSR 244 (1978) 1337.

[6] E. Gutkin, Phys. Rep. 167 (1988) 1–131.

[7] V.E. Korepin, Comm. Math. Phys. 86 (1982) 391.

[8] M. Wadah, A. Kuniba and T. Konishi, Jap. Phys. Soc. (Japan) 54 (1988) 1710.

[9] M. Wadah and K. Sogo, Prog. Thoer. Phys. 69 (1983) 431.

[10] N.M. Bogolyubov and V.E. Korepin, Theor. Math. Phys. 66 (1986) 455.

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