1. INTRODUCTION

Recently, people pay great attention to study the phenomena of seismicity constrained by self-similar structure of medium. Fukao and Furumoto (1985) proposed a two-dimensional model of faulting, in which there are some characteristic lengths between a largest scale (e.g. a size of great earthquake) and a smallest scale (e.g. a grain size), and the ratio between any successive two characteristic lengths is a constant $r$. A growth cycle of faulting begins with rupturing of unit area at a smaller characteristic size, and stops with that of unit area at a larger characteristic size. This paper expands this theory to three-dimensional situation by using simpler and more direct mathematical treatment, and explains some fundamental statistical laws, such as the energy ratio of precursor to main shock, magnitude difference between the magnitudes of main shock and its largest aftershock, and $b$ value in magnitude-frequency relationship. These results were compared with the observations and the results of Fukao and Furumoto.

2. SELF-SIMILAR MODEL OF THREE-DIMENSIONAL FAULT GROWTH

Fig. 1 illustrates a spread cycle of fault rupture. The initial figure is a cuboid with length $a$, width $b$ and thick $c$. According to the observations on strong earthquakes occurred in China, usually the relations $a > b$ and $a > c$ are tenable. In Fig. 1, we take $r = 3$, i.e. each block is separated into $r^3 = 27$ small blocks, and each small block is separated into 27 smaller blocks, ... continually separating these formations with this way. Finally we establish a three-dimensional and self-similar structure.

We agree on that the sizes or orders of blocks are arranged from large to small while the ordinal numbers are pointed from small to large. In other words, the ordinal number is smaller, the larger or higher the order or size is. Within an order $i$, the $r^3$ blocks have the same size $a_i$, $b_i$, and $c_i$, and the same volume $a_i b_i c_i$, within the successive higher order $i - 1$, the blocks have the same size $a_{i-1}$, $b_{i-1}$ and $c_{i-1}$, and the same volume $a_{i-1} b_{i-1} c_{i-1}$; and this expression holds:

$$\frac{a_{i-1}}{a_i} = \frac{b_{i-1}}{b_i} = \frac{c_{i-1}}{c_i} = r,$$

or

$$\frac{a_{i-1} b_{i-1} c_{i-1}}{a_i b_i c_i} = r^3.$$  \hfill (2)

We consider a rupture volume $V$, which satisfied the condition

$$a_i b_i c_i \leq V < a_{i-1} b_{i-1} c_{i-1} = r^3 a_i b_i c_i.$$  \hfill (3)

It is clear that if $V$ is larger, both of the number $n$ of the block involved by volume $V$ and the probability of continuing to rupture are larger. When $n$ reaches a critical number $n_c$ ($n_c < r^3$),
the rupture will not stop and burst out to form a ruptured block with a higher size or order. Let \( p_1 \) be the rupture probability of a single block, then \( p_1 \) is

\[
p_1 = \frac{1}{n_0},
\]

and the \( p_n \), probability of continuing to rupture under the condition that \( n \) blocks have ruptured, can be considered to be proportional to the number \( n \), that is

\[
p_n = np_1, \quad (n < n_0).
\]

When \( n = n_0, p_n = 1 \), we have the expression as follows:

\[
p_n = \begin{cases} n_0 \quad &n < n_0 \\ 1 \quad &n \geq n_0 \end{cases}
\]

Thus, the probability \( p \), with which the rupture develops from one block into \( n \) blocks, is this:

\[
p = \prod_{i=1}^{n} p_n = \frac{n!}{n_0^n} \quad (n < n_0).
\]

The number \( N \) of this kind of rupture, which involves blocks more than \( n_0 \), can be written as

\[
N = r^3 p = r^3 \frac{n!}{n_0^n} \quad (n < n_0).
\]

According to (6), the probability with which the rupture develops from one block in order \( i \) into one block in order \( i - 1 \), is that:

\[
p_i = \prod_{n=1}^{n_0} p_n = \frac{n_0!}{n_0^{n_0}}
\]

The formula (9) is the same as expression (8) of Fukao and Furumoto in form.

Fig. 2 shows level-segments ① and ② within the three-dimensional self-similar structure presented above. The order-difference between the initial and final block in level-segment ① equals \( m \), as the same as that in level-segment ②. The order-difference between the final block of level-segment ② and the initial block of ① is equal to \( R \).

In level-segment ①, the volume \( V_i \) of final block contains \( N_{01} \) initial blocks, each with volume \( V_j \). \( N_{01} \) is

\[
N_{01} = r^3_{10}.
\]

The number \( N_1 \), of blocks with ruptured volume larger than \( V_j \) can be shown as:

\[
N_1 = N_{01} \frac{m}{3}
\]

In the level-segment ②, the corresponding number respectively is

\[
N_{02} = r^3_{20},
\]

and

\[
N_2 = N_{02} \frac{m}{3}.
\]

In (11) and (13), the subscript ① and ② respectively indicates the level-segment ① and ②.

Because of the self-similar structure, it must be true:

\[
N_1 = N_2.
\]

According to that the order-difference between the final block \( V_i \) of level-segment ② and the initial block \( V_j \) of level-segment ① is \( R \), we can write down two relationships as follows:

\[
N_{01} = r^3_{10} N_{02},
\]

and

\[
(p_{10})_0 = (p_{20})_0 (p_1)^R.
\]

By using these formulas, it is obtained:

\[
N_1 = r^3_{10} N_{02} \frac{m}{3} = N_2 = N_{02} \frac{m}{3}.
\]

Combining (17) with (9), we finally obtain that

\[
r^3 = \frac{n_0!}{n_0^{n_0}}.
\]

Formula (18) is important to describe the numerical relation between \( r \) and \( n_0 \) for the three-dimensional self-similar model.

Table 1 shows the values of \( n_0 \) and the corresponding values of \( r \). We prefer to take the pair of \( n_0 \) and \( r \) as

\[
\begin{cases}
   n_0 = 6, \\
r = 3.6564
\end{cases}
\]

since if then it will be seen from the discussion as below that some fundamental relationships of seismology can be explained very well.
3. PRECURSOR AND MAIN SHOCK

If the rupture stops at an order of block, we call this size of rupture as main rupture or main shock, and call previous rupture at relatively small sizes as precursor or foreshock. Let $E_p$ and $E_m$ be the energy of precursor and main shock respectively, and the ratio $\alpha$ is

$$\alpha = \frac{E_p}{E_m}. \quad (20)$$

The observations on seismicity give $\alpha = 0.08$ (Fukao and Furumoto, 1985). In the present model, assuming that the released energy by earthquake is direct proportional to the ruptured volume, the ratio of ruptured volume $V_p$ of precursor to the $V_m$ of main shock is also $\alpha$:

$$\alpha = \frac{V_p}{V_m}. \quad (21)$$

and $\frac{V_p}{V_m}$ can be provided from the model.

As the analysis above, the maximum number of block contained by $V_p$ is $n_k - 1$. The main shock occurs at higher order and may include $n$ blocks, each of them contains $r^3$ smaller blocks at order of precursor. Thus, the ratio of volume is

$$\frac{V_p}{V_m} = \frac{n_k - 1}{n r^3}. \quad (22)$$

But the probability of main shock stops at one block ruptured is largest, so we have

$$\alpha = \frac{n_k - 1}{r^3}. \quad (23)$$

Substituting $n_k = 6$ and $r = 4$ in (23), $\alpha = 0.077 \approx 0.08$ is given. This result is very consistent with the observed data mentioned by Fukao and Furumoto.

4. MAIN SHOCK AND LARGEST AFTERSHOCK

The occurrence of main shock decreases the homogeneous of stress at the order of block of main shock, but increases that of lower order of block and causes many small earthquakes – aftershocks. Our model predicts that the difference $\Delta M$ between the magnitude $M_m$ of main shock and $M_a$ of largest aftershock is about 1.21.

In formula (25), the subscript $m$ and $a$ respectively indicates main shock and largest aftershock.

Since the energy is proportional to the ruptured volume as analyzed previously, it is obtained:

$$\Delta M = M_m - M_a = \frac{1}{C} \ell \frac{g}{r^3}. \quad (26)$$

Substituting $r = 4$ and $C = 1.5$ from observed data, $\Delta M$ is calculated as

$$\Delta M = 1.21. \quad (27)$$

This conclusion is consistent with many observations (Gutenberg and Richter, 1949; Zhou Huilan et al., 1980) pretty well.

5. MAGNITUDE–FREQUENCY RELATION

Following Fukao and Furumoto, we consider a fault which ruptures and causes earthquakes. The rupture involving total fault is called zero order, its magnitude is pointed as $M_{ma} = 0$, its rupture volume is marked $V_{max}$, and its frequency must be one. We mark the magnitude and volume respectively as $M$ and $V$ for the smaller rupture. Similar to get (25) and (26), we can have

$$M - M_{max} = -\frac{1}{1.5} \ell \frac{g}{r^3} \frac{V_{max}}{V}. \quad (28)$$

and further obtain

$$M = \frac{2}{3} \ell \frac{g}{r^3} \frac{V}{V_{max}}. \quad (29)$$

In the same order of rupture, if one block, with volume $V_0$, is broken, then rupture volume $V = V_0$; if $n$ blocks are broken, then $V = nV_0$; when $n = n_k$, rupture volume reaches its maximum, that is, $V_{max} = r^3 V_0$. Thus, (29) can be written as that:

$$M_{1n} = \frac{2}{3} \ell \frac{g}{r^3} \frac{n}{V_{max}}. \quad (30)$$

Similar to (8) the number $N_{1n}$ of ruptures with volume larger than $V$ is

$$N_{1n} = \frac{r^3 (n - 1)!}{n_k - 1}. \quad (n = 1, 2, \ldots, n_k). \quad (31)$$

In formulas (30) and (31), the subscripts 1 for $M$ and $N$ mean the first order. By using these two formulas, the magnitude $M_{1n}$ and its frequency $N_{1n}$ can be calculated. The results of $M_{1n}$ and $N_{1n}$ for $n_k = 6$ and $r = 4$ or $r > 6$ are listed in Table 2.

For the rupture at order $k$ ($k > 1$), combining (30) and (31) with (27), the formulas for calculating magnitude $M_{kn}$ and its frequency $N_{kn}$ are these:
\[ M_{k1} = M_{11} - 21(k - 1), \quad (k > 1) \]
\[ N_{k1} = 65^{k-1} N_{11}. \]

Here, \( k > 1 \), and \( \gamma = 65 \) has been used.

In the coordinate of \( \log N \) versus \( \log M \), the magnitude–frequency relation is a broken-line very much alike to stairs. When the magnitude minus 1 21, the same shape of broken-line repeat again. The slope of this broken-line is \( -1.5 \). Thus, our model gives the \( b \) value in the relationship of magnitude to frequency is \( 1.5 \). This result is close, but with slightly high, to the statistical value from data of seismicity in many various areas in the world.

6. DISCUSSIONS

In this paper, we use a different mathematical treatment from Fukao and Furumoto and more forthrightly provide a three-dimensional self-similar model for the fault growth. This model gives \( \alpha = 0.08 \), \( \Delta M = 1.21 \) and \( b = 1.5 \). Fukao and Furumoto's two-dimensional model gives \( \alpha = 0.2, \Delta M = 1.4 \) and \( b = 1.0 \). Comparing our model and theirs with observations, both of two models are satisfactory to explain the values of \( \Delta M \) and \( b \), but our model is more successful in discussing the ratio \( \alpha \) of energy between the precursor and main shock.

Acknowledgments

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REFERENCES


Table 1
Values of $n_c$ and $r^3$ calculated from formula (18).

<table>
<thead>
<tr>
<th>$n_c$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^3$</td>
<td>2.0</td>
<td>4.5</td>
<td>10.7</td>
<td>26.0</td>
<td>64.8</td>
<td>163.4</td>
<td>416.1</td>
</tr>
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<td>$n_c/r^3$</td>
<td>1.0</td>
<td>0.67</td>
<td>0.37</td>
<td>0.19</td>
<td>0.09</td>
<td>0.04</td>
<td>0.02</td>
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Table 2
The magnitude $M_{1n}$ and frequency $N_{1n}$ calculated for the first order of rupture.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$V/V_{max}$</th>
<th>$M_{1n}$</th>
<th>$N_{1n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>6/65</td>
<td>-0.690</td>
<td>$65 \times \frac{6}{65} = 1.00$</td>
</tr>
<tr>
<td>5</td>
<td>5/65</td>
<td>-0.743</td>
<td>$65 \times \frac{5}{65} = 1.20$</td>
</tr>
<tr>
<td>4</td>
<td>4/65</td>
<td>-0.807</td>
<td>$65 \times \frac{4}{65} = 1.81$</td>
</tr>
<tr>
<td>3</td>
<td>3/65</td>
<td>-0.891</td>
<td>$65 \times \frac{3}{65} = 3.61$</td>
</tr>
<tr>
<td>2</td>
<td>2/65</td>
<td>-1.01</td>
<td>$65 \times \frac{2}{65} = 10.8$</td>
</tr>
<tr>
<td>1</td>
<td>1/65</td>
<td>-1.21</td>
<td>$65 \times 1 = 65.0$</td>
</tr>
</tbody>
</table>
FIGURE CAPTIONS

Fig. 1 A spread cycle of fault growth. See the text for detail.

Fig. 2 The illustration for proving formula $r^{-3} = \frac{h^3}{V}$, $m$ is the order-difference between initial and final size in level-segment 1 as the same as in level-segment 2. $R$ is the order-difference between final size of level-segment 2 and initial size of segment 1. $V$ is the volume of block.
Fig. 2

Direction of increasing ordinal number
or decreasing order.