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Shi Kun Wang

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Kai-Wen Xu

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SUPERCONFORMAL ALGEBRA OF MEROMORPHIC VECTOR FIELDS WITH THREE POLES ON SUPER-RIEMANN SPHERE *

Shi-Kun Wang
Institute of Applied Mathematics, Academia Sinica, Beijing,
People's Republic of China

and

Kai-Wen Xu **
International Centre for Theoretical Physics, Trieste, Italy
and
Physics Department, Zhejiang University, Hangzhou,
People's Republic of China.

ABSTRACT

Based upon the Riemann-Roch theorem, we construct superconformal algebra of meromorphic vector fields with three poles and the relevant abelian differential of the third kind on super Riemann sphere. The algebra includes two Ramond sectors as subalgebra, and implies a picture of interaction of three superstrings.

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** Permanent address: Kai-Wen Xu, Physics Department, Zhejiang University, Hangzhou, People's Republic of China.

It is well known that the Virasoro algebra and the super Virasoro algebra play very important roles in conformal field theory, the string theory, the superconformal field theory and the superstring theory [1]. A recent generalization of the Virasoro algebra was given by Krichever-Novikov [2] for higher genus Riemann surface, and its super extension. The applications in physics were studied by many papers [3-10]. Another generalization of the Virasoro algebra for multi-poles on Riemann surface was given by [11,12], the algebra of meromorphic vector fields with multi-poles on Riemann surface has been constructed in the papers. In this paper, we will give the supersymmetric version of the algebra. We will concentrate in the case of super version of the algebra of meromorphic vector fields with three poles on the Riemann sphere which could play important roles in the conformal field theory on higher genus Riemann surface in the sense of sewing procedure [13]. We first recall some results from [11]. For the sake of convenience, we take a basis which is slightly different from (but isomorphic to) the one given by [11]. Then we construct superconformal algebra of meromorphic vector fields with three poles on super Riemann sphere. Finally we discuss the abelian differential of the third kind with three poles on super Riemann sphere, and use it to define the Euclidean time and its equal level loop on super Riemann sphere. The procedure of a set of splitting or joining contours propagating from $t = -\infty$ to $t = +\infty$ implies a picture of the interaction of the three superstrings.

Let P_i $(i = 1,2,3)$ denote three different points on Riemann sphere S^2. Let $D = \sum_{i=1}^{3} m_i P_i$ be an arbitary divisor on sphere. It has a pole of order m_i at P_i if $m_i < 0$, and a zero point of order m_i at P_i if $m_i > 0$. Let $H^{\lambda}_{(D)}$ be the space of meromorphic λ differentials.

$$H^{\lambda}_{(D)} = \{\lambda\text{-differential} \quad \eta_{(\lambda)} | (\eta_{(\lambda)}) + D \geq 0\} \tag{1}$$

According to Riemann-Roch theorem, we have

$$dim H^{\lambda}_{(D)} = dim H^{1-\lambda}_{(-D)} + (1 - 2\lambda) + deg D \tag{2}$$

From (2), it follows that $H^{\lambda}_{(D)}$ can exist non-trivally only if

$$deg D = m_1 + m_2 + m_3 \geq -2\lambda \tag{3}$$

For the sake of simplicity, we fix the three points as $Z_{(P_1)} = 0, Z_{(P_2)} = 1, Z_{(P_3)} = \infty$. Then the λ-differential can be uniquely determined (up to multiplication by a constant) as

$$\begin{aligned} L^{\lambda}_{[-2\lambda-m_1-m_2,m_1,m_2]} &= (-1)^{m_1}(z_1 - 1)^{m_2} z_1^{-2\lambda-m_1-m_2}(dz_1)^{\lambda} \\ &= (z_2 + 1)^{m_2} z_2^{m_1}(dz_2)^{\lambda} = (z_3 - 1)^{m_1} z_3^{m_2}(dz_3)^{\lambda} \end{aligned} \tag{4}$$

The local coordinates Z_i surrounding P_i are related in the overlap regions via

$$Z_1 = -\frac{1}{Z_2} = \frac{1}{1 - Z_3} \tag{5}$$

In the case of $\lambda = -1$, we get the meromorphic vector fields with three poles on sphere as

$$L^{-1}_{[2-m_1-m_2,m_1,m_2]} = (-1)^{m_1}(z_1 - 1)^{m_2} z_1^{2-m_1-m_2} \frac{\partial}{\partial z_1} \tag{6}$$

and a basis is provided by the set of vectors

$$L^1_m = L^{-1}_{[2-m,m,0]} = (-1)^m z_1^{2-m}\frac{\partial}{\partial z_1} \quad ; \quad (m \in Z)$$
$$L^2_n = L^{-1}_{[2-n,0,n]} = (z_1-1)^n z_1^{2-n}\frac{\partial}{\partial z_1} \quad ; \quad (n<0) \tag{7}$$

L^1_m and L^2_n are the generators of the algebra of the meromorphic vector fields, the algebraic relations are

$$[L^1_m, L^1_n] = (n-m)L^1_{n+m-1}$$
$$[L^2_m, L^2_n] = (n-m)L^2_{n+m-1}$$
$$[L^1_m, L^2_n] = nL_{[3-m-n,m,n-1]} - mL[3-m-n, m-1, n]$$

$$= \begin{cases} \sum_{k=-1}^{m-1}(k+1-2m)C^m_{k-m+1}L^1_k \\ \quad + \sum_{k=-1}^{n-1}(-1)^{m+n-k-1}(2n-k-1)C^m_{k-n+1}L^2_k \quad ; \quad (m<1) \\ \sum_{k=n-1}^{m+n-1}(-1)^{m+n-k-1}(2n-k-1)C^m_{k-n+1}L^2_k \quad ; \quad (1 \le m < -n+1) \\ \sum_{k=-1}^{n-1}(-1)^{m+n-k-1}(2n-k-1)C^m_{k-n+1}L^2_k \\ \quad + \sum_{p=0}^{m+n-1}(-1)^{m+n-p-1}(2n-p-1)C^m_{p-n+1}\sum_{k=0}^{p}C^k_pL^1_k; \quad (m \ge -n+1) \end{cases} \tag{8}$$

where $C^n_m = \frac{n!}{n!(n-m)!}$.

We can give an abelian differential of the third kind ω, which has the simple poles at $Z=0, Z=\infty, Z=1$, as follows:

$$\omega = \frac{z-2}{z(z-1)}dz. \tag{9}$$

According to Krichever and Novikov[3], we can define the Euclidean time on sphere by means of ω

$$\tau = Re\int_{p_0}^{p}\omega$$
$$= Re[2\ln\frac{z(p)}{z(p_0)} - \ln\frac{z(p)-1}{z(p_0)-1}] \tag{10}$$

and define a one-parameter family C of contours as follows:

$$C_\tau = \{P \in S^2, \tau_{(p)} = \tau\} \tag{11}$$

For $\tau \to -\infty$ the contours C_τ are small circles surrounding Z=0, $\tau \to +\infty$ the contours C_τ are small circles surrounding $Z=1, Z=\infty$. This implies an interacting picture of closed strings . Imagine a small circle surrounding P_i be a closed string, then following the evolution of the circle from $\tau \to -\infty$ to $\tau \to +\infty$ is equivalent to one closed string splitting into two.

Now let us consider the superconformal case.

As the super version of Riemann Roch theorem has not been well understood, we will use $\lambda = -1$, and $\lambda = -\frac{1}{2}$ differentials to construct the superconformal algebra of the meromorphic vector fields with three poles on super Riemann sphere. According to Giddings and Nelson[14], super Riemann surface is a 1|1 complex supermanifold with a superconformal structure. And the class of superconformal structures is the class of supercomplex structures in addition to the requirement that the transition functions be analytic and satisfy extra condition

$$D_{\theta_\alpha}Z_\beta = \theta_\beta D_{\theta_\alpha}\theta_\beta, \tag{12}$$

where $Z_\alpha = f_{\alpha\beta}(Z_\beta,\theta_\beta), \theta_\alpha = \phi_{\alpha\beta}(Z_\beta,\theta_\beta), D_\theta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$.

In our case, we chose

$$Z_1 = -\frac{1}{Z_2} = \frac{1}{1-Z_3} \quad , \quad \theta_1 = -\frac{1}{z_2}\theta_2 = \frac{1}{1-z_3}\theta_3. \tag{13}$$

This gluing conditions obviously satisfy (12).

Using $\lambda = -1$ and $\lambda = -\frac{1}{2}$ differentials in addition to the requirement (13), the superfields can be given as

$$L_{[2-m_1-m_2,m_1,m_2]} = (-1)^{m_1}(z_1-1)^{m_2}z_1^{2-m_1-m_2}\frac{\partial}{\partial z_1}$$
$$+ (-1)^{m_1}\frac{1}{2}[m_2(z_1-1)^{m_2-1}z_1^{2-m_1-m_2} + (2-m_1-m_2)(z_1-1)^{m_2}z^{1-m_1-m_2}]\theta_1\frac{\partial}{\partial\theta_1}$$
$$= (z_2+1)^{m_2}z_2^{m_1}\frac{\partial}{\partial z_2} + \frac{1}{2}[m_2(z_2+1)^{m_2-1}z_2^{m_1} + (m_1-2)(z_2+1)^{m_2}z_2^{m_1-1}]\theta_2\frac{\partial}{\partial\theta_2}$$
$$= (z_3-1)^{m_1}z_3^{m_2}\frac{\partial}{\partial z_3} + \frac{1}{2}[m_2z_3^{m_2-1}(z_3-1)^{m_1} + (m_1-2)z_3^{m_2}(z_3-1)^{m_1-1}]\theta_3\frac{\partial}{\partial\theta_3} \tag{14}$$

$$G_{[1-r-s,r,s]} = (-1)^r(z_1-1)^s z_1^{1-r-s}(\frac{\partial}{\partial\theta_1} - \theta_1\frac{\partial}{\partial z_1})$$
$$= (z_2+1)^s z_2^r(\frac{\partial}{\partial\theta_2} - \theta_2\frac{\partial}{\partial z_2}) \tag{15}$$
$$= z_3^s(z_3-1)^r(\frac{\partial}{\partial\theta_3} - \theta_3\frac{\partial}{\partial z_3}).$$

$L_{[2-m-n,m,n]}$ and $G_{[1-r-s,r,s]}$ can be decomposed into L^1_m, L^2_n and G^1_r, G^2_s respectively:

$$\begin{aligned}
L^1_m &= L_{[2-m,m,0]} = (-1)^m z_1^{2-m}\frac{\partial}{\partial z_1} + (-1)^m \frac{1}{2}(2-m) z_1^{1-m}\theta_1 \frac{\partial}{\partial \theta_1} \quad ; \quad (m \in Z),\\
L^2_m &= L_{[2-m,0,m]} = (z_1-1)^m z_1^{2-m}\frac{\partial}{\partial z_1} + \frac{1}{2}[m(z_1-1)^{m-1} z_1^{2-m}\\
&\quad + (2-m)(z_1-1)^m z_1^{1-m}]\theta_1 \frac{\partial}{\partial \theta_1} \quad ; \quad (m<0)\\
G^1_r &= G_{[1-r,r,0]} = (-1)^r z_1^{1-r}\left(\frac{\partial}{\partial \theta_1} - \theta_1 \frac{\partial}{\partial z_1}\right) \quad ; \quad (r \in Z)\\
G^2_r &= G_{[1-r,0,r]} = (z_1-1)^r z_1^{1-r}\left(\frac{\partial}{\partial \theta_1} - \theta_1 \frac{\partial}{\partial z_1}\right) \quad ; \quad (r<0)
\end{aligned} \tag{16}$$

$L^1_m, L^2_m, G^1_r, G^2_r$ are generators of the superconformal algebra of meromorphic vector fields with three poles on super Riemann sphere. Using (16), we can get the following algebra relations:

$$\begin{aligned}
[L^i_m, L^i_n] &= (n-m) L^i_{n+m-1},\\
[L^i_m, G^i_r] &= (r - \frac{1}{2}n) G^i_{m+r-1},\\
\{G^i_r, G^i_s\} &= -2 L^i_{r+s} \quad ; \quad (i = 1,2);
\end{aligned} \tag{17}$$

$$[L^1_m, L^2_n] = nL_{[3-m-n,m,n-1]} - mL[3-m-n, m-1, n]$$

$$= \begin{cases}
\sum_{k=-1}^{m-1}(k+1-2m)C^n_{k-m+1}L^1_k \\
\quad + \sum_{k=-1}^{n-1}(-1)^{m+n-k-1}(2n-k-1)C^m_{k-n+1}L^2_k \quad ; \quad (m<1)\\
\sum_{k=n-1}^{m+n-1}(-1)^{m+n-k-1}(2n-k-1)C^m_{k-n+1}L^2_k \quad ; \quad (1 \le m < -n+1)\\
\sum_{k=-1}^{n-1}(-1)^{m+n-k-1}(2n-k-1)C^m_{k-n+1}L^2_k \\
\quad + \sum_{p=0}^{m+n-1}(-1)^{m+n-p-1}(2n-p-1)C^m_{p-n+1}\sum_{k=0}^{p}C^k_p L^1_k; \quad (m \ge -n+1)
\end{cases} \tag{18}$$

$$\{G^1_r, G^2_s\} = -2 L_{[2-r-s,r,s]}$$

$$= \begin{cases}
-2\sum_{n=-1}^{r}C^s_{n-r}L^1_n - 2\sum_{n=-1}^{s}(-1)^{r+s-n}C^r_{n-s}L^2_n \quad ; \quad (r<0)\\
-2\sum_{n=s}^{r+s}(-1)^{r+s-n}C^r_{n-s}L^2_n \quad ; \quad (0 \le r < -s)\\
-2\sum_{p=0}^{r+s}(-1)^{r+s-p}C^r_{n-s}\sum_{n=0}^{p}C^p_n L^1_n - 2\sum_{n=-1}^{s}(-1)^{r+s-n}C^r_{n-s}L^2_n \quad ; \quad (r \ge -s)
\end{cases} \tag{19}$$

$$[L^1_n, G^2_r] = -rG_{[3-r-n,n-1,r-1]} + (r - \frac{1}{2}n)G_{[2-r-n,n-1,r]}$$

$$= \begin{cases}
(r - \frac{1}{2}n - 1)\sum_{s=-1}^{n-1}C^r_{s-n+1}G^1_S \\
\quad + \sum_{s=-1}^{r-1}[r + (r - \frac{1}{2}n)(s-r+1)](-1)^{n+r-s-1}C^{n-1}_{s-r+1}G^2_s \quad ; \quad (n<1)\\
\sum_{s=r-1}^{n+r-2}[r + (r - \frac{1}{2}n)(s-r+1)](-1)^{n+r-s-1}C^{n-1}_{s-r+1}G^2_s \quad ; \quad (1 \le n < -r+2)\\
\sum_{s=-1}^{r-1}[r + (r - \frac{1}{2}n)(s-r+1)](-1)^{n+r-s-1}C^{n-1}_{s-r+1}G^2_s \\
\quad + \sum_{p=0}^{n+r-2}[r + (r - \frac{1}{2}n)(p-r+1)](-1)^{n+r-p-1}C^{n-1}_{p-r+1}\sum_{s=0}^{p}C^p_s G^1_s \quad ; \; (n \ge -r+2)
\end{cases} \tag{20}$$

$$
\begin{aligned}
[L_n^2, G_r^1] &= \frac{1}{2} n G_{[3-r-n, r-1, n-1]} + (r - \frac{1}{2} n) G_{[2-r-n, r-1, n]} \\
&= \begin{cases}
(r - \frac{1}{2}n + \frac{1}{2}) \sum_{s=-1}^{r-1} C^n_{s-r+1} G^1_s \\
\quad + \sum_{s=-1}^{n-1} [(r - \frac{1}{2}n)(s - n + 1) - \frac{1}{2}n](-1)^{n+r-s-1} C^{r-1}_{s-n+1} G^2_s \quad ; \quad (r < 1) \\
\sum_{s=n-1}^{n+r-2} [(r - \frac{1}{2}n)(s - n + 1) - \frac{1}{2}n](-1)^{n+r-s-1} C^{r-1}_{s-n+1} G^2_s \quad ; \quad (1 \le r < -n + 2) \\
\sum_{s=-1}^{n-1} [(r - \frac{1}{2}n)(s - n + 1) - \frac{1}{2}n](-1)^{n+r-s-1} C^{r-1}_{s-n+1} G^2_s \\
\quad + \sum_{p=0}^{n+r-2} [(r - \frac{1}{2}n)(p - n + 1) - \frac{1}{2}n](-1)^{n+r-p-1} C^{r-1}_{p-n+1} \sum_{s=0}^{p} C^p_s G^1_s \quad ; \quad (r \ge -n + 2)
\end{cases}
\end{aligned}
\tag{21}
$$

According to the paper [5], a $(\frac{p}{2}, 0)$ holomorphic superdifferential can be defined as

$$
d\theta^p \phi_{(z,\theta)} + d\theta^{p-1} \eta D_\theta \phi_{(z,\theta)} \tag{22}
$$

where $\eta = dz + \theta d\theta$.

So we define an abelian differential of the third kind with three poles (at $z = 0, 1, \infty$) on super Riemann sphere as *)

$$
\omega = \frac{z-2}{z(z-1)} dz + \frac{\partial f_{(z)}}{\partial z} \theta dz + f_{(z)} d\theta \tag{23}
$$

where $f_{(z)}$ is a holomorphic function.

We define the "Euclidean time" on super Riemann sphere as

$$
\begin{aligned}
\tau &= \mathrm{Re} \int_{p_0}^{p} \omega \\
&= \mathrm{Re}[\, 2\, ln \frac{z_{(p)}}{z_{(p_0)}} - ln \frac{z_{(p)} - 1}{z_{(p_0)} - 1} + \theta (f_{(z_p)} - f_{(z_{p_0})})\,]
\end{aligned}
\tag{24}
$$

Let us give this time an explanation, if $\psi_{(\tau,\sigma)}$ is a field defined on super Riemann sphere, then $\psi_{(t,\theta,\sigma)}$ (where $\tau = t + \theta$) satisfies the canonical equation

$$
\begin{aligned}
\frac{\partial \psi_{(t,\theta,\sigma)}}{\partial t} &= [\, H, \psi_{(t,\theta,\sigma)}\,] \\
\frac{\partial \psi_{(t,\theta,\sigma)}}{\partial \theta} &= [\, Q, \psi_{(t,\theta,\sigma)}\,]
\end{aligned}
\tag{25}
$$

*) Here, we would like to thank A.B. Zamolodchikov and C. Vafa for valuable discussions.

where H is Hamiltonian, Q is supersymmetry charge.

We define a one-parameter family C_τ of contours as follows

$$
C_\tau = \{P \in S^2, \tau_{(p)} = \tau\} \tag{26}
$$

For $t \to -\infty$ the contour C_τ is a small circle surrounding Z=0, $t \to +\infty$ the contours C_τ are small circles surrounding Z=1, $Z = \infty$. This implies an interacting picture of closed superstrings. Imagine a small circle surrounding P_i be a closed superstring, then the following evolution of the circle from $t \to -\infty$ to $t \to +\infty$ is equivalent to one closed superstring splitting into two closed superstrings.

Above, we have extended the results [1] to one kind of super case, Ramond sector. Unfortunately, the Neveu-Schwarz sector has not been found, since it is very difficult to expand the half integer binomial of NS sector. But it is clear that the background of our theory concerns with the interaction of three closed superstrings, so it is very interesting to study the relations between our algebra and the chiral algebra. The BRST cohomology and operator formalism of this algebra is under investigation.

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After completion of this work, we were told by K. Wu and L. Bonora that R. Dick and M. Schlichenmaier also obtained some results in the conformal case.

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