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PROJECTION–ITERATION METHODS FOR SOLVING NONLINEAR OPERATOR EQUATIONS *

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ABSTRACT

In this paper, the authors investigate a nonlinear operator equation in uniformly convex Banach spaces as in metric spaces by using stationary and nonstationary generalized projection–iteration methods. Convergence theorems in the strong and weak sense were established.

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0. Introduction

The projection-Iteration methods for solving linear and nonlinear operator equations have been applied by many authors(see for instance [1] [2],[3]).

In this paper , we generalize some results in [1], [2] , [7] and others by using these methods for solving the operator equation

$$x = T(x,x) \qquad (10)$$

It is obvious that this equation contains the equation $x = Tx$ as special case . For approximation methods , it is well-known that the following successive approximation

$$x_n = T_n(x_n, x_{n-1}) , \quad n = 1,2\ldots$$

generalizes many approximation methods , such as method Picard, method Seidel , method Newton-Kantorovich, and so on (seefor instance [2])

Here we will study the convergence in the strong and weak sense of some generalized projection iteration methods ,stationary as well as nonstationary used to solve equation (1) .

In sections 1 and 2 the strong convergence shall be discussed .

In sections 3 and 4 we will be concerned with the weak convergence.

1. Strong convergence. Stationary case.

Theorem 1.1

Let X be a topological space ,D a compact set of X, and T a continuous mapping of DxD into D , and

$$F = \{x \in D : T(x,x) = x\} . \qquad \text{Let } g ; D\times F \longrightarrow [0,+\infty)$$

Assume that

1.1. $F \neq \emptyset$ and $g(x,p) = 0$, $x \in D$, $p \in F \Leftrightarrow x = p \in F$

1.2 $g(T(x,y),p) < \max\{g(x,p), g(y,p)\}$, $\forall x,y \in D$, $p \in F \in$ such that the equality $x = y = p$ does not hold

I.3 The systems

$$u_n = T(u_n, v_n)$$
$$v_n = T_k(u_n, u_{n-I}) \qquad (2)$$

are solvable for arbitrary $u_o \in D$ and for all $n \geq I$, where

$T_I(x,y) = T(x,y)$, $T_i(x,y) = T(x, T_{i-I}(x,y))$, $i = 2,3\dots,k$.

Moreover

I% If the function $g(x,p)$ is continuous in x , then the sequence (u_n) , (v_n) constructed by (2) converge(in the topology of X) to a unique root of equation (I)

2% If the function $g(x,p)$ satisfies the following condition

(I.4) From $\lim g(u_n,p) = \lim g(v_n,p)$ (for $p \in F$) , and v being any limit point of the sequence (v_n) it follows that either $g(u,p) \leq g(v,p)$ or $g(u,p) \geq g(v,p)$ for all limit point u of the sequence (u_n)

Then the sequences (u_n) , (v_n) constructed by (2) converge to the unique root p of equation (I).

Proof

By (I.2) we have

$g(u_n,p) < \max(g(u_n,p), g(v_n,p))$, i.e. $g(u_n,p) < g(v_n, p)$

However

$g(v_n,p) = g(T_k(u_n,u_{n-I}),p) < \max g(u_n,p), g(T_{k-I}(u_n,u_{n-I}), p)$

$\leq \dots\dots \leq \max(g(u_n,p), g(u_{n-I},p))$

Hence

$g(u_n,p) < g(v_n,p) < g(u_{n-I},p)$

Therefore $\lim g(u_n,p) = \lim g(v_n,p)$

By using the compactness of D we obtain subsequences $(u_{n_j}) \subset (u_n)$, $(v_{n_j}) \subset (v_n)$ such that $u_{n_j} \longrightarrow u$, $v_{n_j} \longrightarrow v$

I% If g is continuous , then $g(u,p) = g(v,p)$. On the other hand we have $T(u_{n_j}, v_{n_j}) \longrightarrow T(u,v)$ and

$g(u,p) = g(T(u,v),p) < \max\{g(u,p), g(v,p)\}$, a contradiction.

Hence we get $u = v = p$.

Remark : F contains only one element. Indeed if $p \neq q$, $p,q \in F$ then

$g(q,p) = g(T(q,q),p) < \max g(q,p)$.

which cannot be held .

Therefore the sequences (u_n) , (v_n) converge to a unique root p of equation (I).

2% It is clear that from the sequences (u_{n_j}), (v_{n_j}) we can choose subsequences $(u_{n_{j_k}-I})$ such that $u_{n_{j_k}-I} \longrightarrow \bar{u}$.

Then we have

$u_{n_{j_k}} = T(u_{n_{j_k}}, v_{n_{j_k}})$, $u_{n_{j_k}}$ $u = T(u,v)$

$v_{n_{j_k}} = T_k(u_{n_j}, u_{n_{j_k}-I})$, $v_{n_{j_k}}$ $v = T_k(u,\bar{u})$

Hence

$g(u,p) < \max(g(u,p),g(v,p))$ implies $g(u,p) < g(v,p)$

$g(v,p) < \max(g(u,p),g(\bar{u},p))$ implies $g(v,p) < g(\bar{u},p)$

if either $u=v=p$ or $u = \bar{u} = p$ does not hold. However the last inequalities are contrary to (I.4). Therefore $u = v = p$ or $u = \bar{u} = p$

that is $v = T_k(u,u) = T_k(p,p) = p$.

Q.E.D.

Theorem I.2

Assume that (I.1), (I.3) hold and instead of (I.2) the following conditions are fulfiled

(I.5)
$$g(T(x,y),p) \begin{cases} < \max(g(x,p),g(y,p)) & \text{if } g(x,p) = g(y,p) \\ \leqslant \max(g(x,p),g(y,p)) & \text{if } g(x,p) = g(y,p) \end{cases}$$
$\forall x,y \in D$, $p \in F$

(I.6) $\forall y \bar{\in} F$, $x \in D$ satisfying $x = T(x,y)$, $\exists q = q(x,y) \in F$ such that $g(T(x,y),q) < \max(g(x,q),g(y,q))$

Moreover if

1° either function g is continuous

2° or function g satisfies assertion (I.4)

then any subsequence (u_{n_j}), (v_{n_j}) must converge at the same time to a solution p of equation (I).

Further in order to the sequences (u_n), (v_n) converge at the same time to a solution $p \in F$, it is sufficient that

(I.7) function g is lower semicontinuous, and X is compact

(I.8) $g(u_n,p)$ uniformly converges on F for a element $u_o \in D$

(I.9) If $u_k \longrightarrow u \in F$, then for every $p \in F$, $p \neq u$,
$\underline{\lim} g(u_k,p) > \underline{\lim} g(u_k,u)$

Proof

As in theorem I.I, we have

$\lim g(u_n,p) = \lim g(v_n,p)$

Now from $(u_n),(v_n)$ we take (u_{n_j}), (v_{n_j}) such that $u_{n_j} \longrightarrow u$, $v_{n_j} \longrightarrow v$

We will show that $v \in F$.

If it is contrary $v \bar{\in} F$, then $\exists q = q(u,v)$ such that $g(u,q) =$

$g(T(u,v),q) < \max(g(u,q),g(v,q))$, i.e. $g(u,q) < g(v,q)$ (4)

If function g is continuous, then by (3) it follows that $g(u,q) =$

$= g(v,q)$, which is in contradiction to (4). Therefore $v \in F$

In the case 2° we obtain

$v_{n_{j_k}} \longrightarrow v$, $v_{n_{j_k}} = T_k(u_{n_{j_k}}, u_{n_{j_k}-I}) \longrightarrow T_k(u,\bar{u}) = v$

It follows that

$g(v,q) = g(T_k(u,\bar{u}),q) \leqslant \max(g(u,q), g(\bar{u},q))$

that is $g(v,q) \leqslant g(\bar{u},q)$, in contradiction to (4) and to (I.4).

Hence $v \in F$.

Furthermore if $g(u,v) \neq g(v,v) = 0$, then

$g(u,v) = g(T(u,v),v) < \max(g(u,v), g(v,v))$, which is impossible.

Whence $g(u,v) = 0$, that is $u = v \in F$

For further discussions we need the following lemma

Lemma I.I.

Let $f_n(p)$ be lower semicontinuous and bounded on a closed set F and $f_n(p)$ uniformly converges to $f(p)$ on F, then $f(p)$ is lower semicontinuous on F too.

It is not difficult to prove this assertion.

Now we continue our proof of Theorem 1.2

Let us consider function $f(p) = \lim g(u_n\ p)$ on F . By the continuity of T we assert that F is closed , and F lies in D which is a compact set , hence F is compact too.

By (I.8) it follows that $g(v_n,p)$ converges uniformly to $f(p)$ on F , and by Lemma I.I , $f(p)$ is lower semicontinuous on F , whence f attains its infimum on F , say, at a point p(*).We shall prove that p is the unique limit point of the sequences (u_n) and (v_n) .

As being shown above , the subsequences (u_{n_j}), (v_{n_j}) where $u_{n_j} \longrightarrow u$ $v_{n_j} \longrightarrow v$, have a unique limit point $u = v \in F$. If $u = v \neq p$ then by (I.9) it follows that $\underline{\lim} g(u_n,p) = \underline{\lim} g(u_{n_j},p) > \underline{\lim} g(u_{n_j},u) =$

$= \underline{\lim} g(u_n,u)$ which leads to the contradiction of(*).

Therefore $u = v = p$, that is $u_n \longrightarrow p$, $v_n \longrightarrow p$

And this completes the proof of Theorem I.2

Remark.In the above stated Theorems we are concerned with topological space X , continuous mapping T , compact set D. In the metric case (X,d) , instead of a compact set D we may consider a closed one in X and a compact mapping $T : D\times D \longrightarrow D$ and a additional condition

1.12 $\exists p \in F$ such that from $d(x,p) \longrightarrow \infty$ it follows that $g(x,p) \to \infty$.

Then keeping mentioned above conditions (1.1),...,(1.9) , the conclusions in our Theorems remain valid and contains Theorems in [1-3] as special cases .

2. Nonstationary Method.Strong convergence .

So far we have discussed stationary projection-iteration method , that is operator T remains fixed.It may be seen that this method is applicable in the nonstationary case , that is , when operator T depends on each step, say, T_n .

Namely we will investigate the following algorithm

$$u_n = T_n(u_n,v_n) , \quad n \geq I$$
$$v_n = T_{n,k}(u_n,u_{n-I})$$

where $T_{n,I}(x,y) = T_n(x,y)$, $T_{n,i}(x,y) = T_n(x,T_{n,i-I}(x,y))$, $i=2,\dots,k$

Let (X,d) be metric space , D a closed subset in X, $g:D\times F \longrightarrow [0,\infty)$ a function satisfying condition

(2.I) $\exists p \in F$ such that from $d(x,p) \longrightarrow \infty$ it follows that $g(x,p) \to \infty$

and $T_n : D\times D \longrightarrow D$, $n \geq 0$ ($T_o = T$) being compact mappings.

Moreover the following inequalities hold

(2.2) $d(T_n(x,y),T_o(x,y)) \leq a_n$, $a_n \longrightarrow 0$, $\forall x,y \in D$

Theorem 2.I.

Suppose the mentioned above assumptions are fulfiled .Moreover the following conditions are satisfied :

(2.3) $F = \{ x \in D : T_o(x,x) = x \} \neq \emptyset$ and

$g(x,p) = 0$, $x \in D$, $p \in F \iff x = p \in F$

(2.4) $g(T_n(x,y),p) < \max(g(x\ p),g(y,p))$, $\forall x,y \in D$, $p \in F$

$n = 0,1,2,\dots$such that the equalities $x = y = p$ does not hold .

(2.5) The systems

$$u_n = T_n(u_n, v_n) \quad , \quad n \geq I$$
$$v_n = T_{n,k}(u_n, u_{n-I})$$

are solvable for all n =I,2,... and for any $u_o \in D$. Then the assertions in Theorem I.I for the cases I%, 2% remains valid .

Theorem 2.2

Assume that conditions (2.3) , (2.5) are satisfied . Moreover the below additional assumptions hold

(2.6)
$$g(T(x,y),p) \begin{cases} < \max(g(x,p),g(y,p)) & \text{if } g(x,p) \neq g(y,p) \\ \leq \max(g(x,p),g(y,p)) & \text{if } g(x,p) = g(y,p) \end{cases}$$

$\forall x,y \in D$, $p \in F$, $n \geq 0$

(2.7) $\forall y \in F$, $x \in D$ such that $x = T(x,y)$, $\exists q = q(x,y)$ such that $g(T_o(x,y)\, q) < \max(gx,q),g(y,q))$

Furthermore if function g is either continuous or satisfies condition (I.4), then any convergent subsequence (u_{n_j}) , (v_{n_j}) converges at the same time to a solution $p \in F$.

If function g additonally satisfies conditions (I.7) , (I.8),I.9), then the sequences (u_n), (v_n) have just one limit point $p \in F$

The proofs of these theorems are analogous to the ones in Theorems I.I and I.2 .It suffices to establish the fact that from the sequences (u_n), (v_n) we can choose convergent subsequences (u_{n_j}), (v_{n_j}) . Neverthertless this assertion may be taken from a Lemma in the book [4] of O.A.Liskovetz stated as follows

Lemma 2.I

Let (X,d) be metric space , M_n , n=O.I,2,...compact subsets in X, and

$M_n \xrightarrow{B} M_o$, i.e. $B(M_n M_o) \to 0$ as $n \to \infty$, or

$$\sup_{x \in M_n} \inf_{y \in M_o} d(x,y) \to 0 \quad \text{as } n \to \infty$$

Then the set $\bigcup_{n=0} M_n$ is compact too .

The proof of this Lemma we omit (see[4])

Now it is clear that all u_n , v_n belong to any sphere B with center p in D. Then $T_n : B \to M_n$, n=0,I,.... are compact , and u_n , $v_n \in M_n$, n=I,2,...

Further

$$\sup_{x \in M_n} \inf_{y \in M_o} d(x,y) = \sup_{x_I, x_2 \in B} \inf_{y_I, y_2 \in B} d(T_n(x_I, x_2), T_o(y_I,y_2)) \leq$$
$$\leq \sup_{x_I, x_2 \in B} d(T_n(x_I,x_2), T_o(x_I,x_2)) \leq a_n \to 0 \text{ as } n \to \infty$$

From this we obtain $M_n \xrightarrow{B} M_o$ and by Lemma I.2 $\bigcup_{n=0} M_n$ is compact

Thus we can take from (u_n), (v_n) convergent subsequences (u_{n_j}), (v_{n_j}) .

3.Stationary Method.Weak convergence.

Weak convergence of iteration process for nonexpancive and quasinonexpansive mappings in Banach spaces was studied by many authors (see[4]-[7]).

In the same way , we will discuss weak convergence of the projection-iteration sequences (u_n), (v_n) investigated in Section I as follows

$$u_n = T(u_n, v_n) \quad , \quad n \geq I$$
$$v_n = T_k(u_n, u_{n-I}) \qquad (2)$$

Lemma 3.I

Let X be a strictly convex Banach space , D a convex subset in X, $T: D\times D \to D$ a continuous mapping satisfying condition

$$\| T(x,y)-p \| \leq \max (\| x-p \| , \| y-p \|)$$

$\forall x,y \in D$, $p \in F = \{ x \in D : T(x,x) = x \}$

If $F \neq \emptyset$, then F is convex .

The proof of this Lemma is similar to the one in [6]

Lemma 3.2 (see [7])

In a Banach space if the functional f is quasiconvex , i.e.

$f(tx-(I-t)y) \leq \max(f(x),f(y))$ and lower semicontinuous , then f is weakly lower semicontinuous.

From on now , a mapping V is said to be quasiconvex if

$$\left\| V\left(\frac{x_I+x_2}{2}\right) \right\| \leq \max (\| V(x_I) \| , \| V(x_2) \|)$$

Theorem 3.I

Let X be a strictly convex and reflexive Banach space , D a convex closed subset in X , T a continuous mapping from $D\times D$ into D and

(3.I) $F \neq \emptyset$

(3.2)

$$\| T(x,y)-p \| \begin{cases} < \max(\| x-p \| , \| y-p \|) & \text{if } \| x-p \| \neq \| y-p \| \\ \leq \max(\| x-p \|, \| y-p \|) & \text{if } \| x-p \| = \| y-p \| \end{cases}$$

(3.3) The systems (2) are solvable for all n , and $\| u_n - v_n \| \to 0$ as $n \to \infty$

(3.4) Either the mapping I_x-T , where $(I_x-T)(x,y) = x-T(x,y)$ is quasiconvex or the mapping I_y-T , where $(I_y-T)(x,y) = y-T(x,y)$ is quasiconvex.

(3.5) The space X has the following property

If $(y_n) \subset X$, $y_n \xrightarrow{w} y_o \in X$, then

$\underline{\lim} \| y_n - y \| > \underline{\lim} \| y_n - y_o \| , \forall y \neq y_o , y \in X$

then the sequences (u_n), (v_n) defined by (2) weakly converge at the same time to a solution $p \in F$.

Theorem 3.2

Let X be a uniformly convex Banach space , D a convex closed subset in X and $T:D\times D \to D$ a mapping satisfying conditions (3.I),(3.2),(3.7).

a) Assume that for a point $c \in [0,I)$ the mapping J_c-T , where $J_c(x,y) = cx + (I-c)y$ is quasiconvex on $D\times D$ .Then for any $t \in (o,I)$ the sequences

$$u_n = ((tJ_c-(I-t)T)(u_n,v_n)$$
$$v_n = ((tJ_c-(I-t)T)_k (u_n,u_{n-I})$$

weakly converges at the same time to a solution $p \in F$

b) When the mapping I_x-T is quasiconvex , we construct a algorithm as follows :

$\tilde{u}_o = u_c$, then by means of the following system

$$\begin{aligned} u_n &= T(u_n,v_n) , \quad n \geq I \\ v_n &= T_k (u_n,\tilde{u}_{n-I}) \end{aligned} \qquad (4)$$

we obtain (u_I,v_I) , then we take $\tilde{u}_I = tu_I + (I-t)v_I$ and again using system (4) we get (u_2,v_2) , and so on .

We claim that such obtained sequences (u_n),(v_n) will weakly converges to the the solution $p \in F$

Proof of Theorem 3.I

By using the same arguments as in Theorem 2.I we get

$$\| u_n - p \| \leq \| v_n - p \| \leq \| u_{n-I} - p \|$$

Hence

$$\lim \| u_n - p \| = \lim \| v_n - p \|$$

It follows that

$(u_n),(v_n) \subset B(p, \|u_o - p\|)$, where B is a sphere with center at a point $p \in F$. Moreover $(u_n),(v_n) \subset D$ and D is a convex closed subset of a reflexive Banach space X. Therefore from $(u_n),(v_n)$ we can choose weakly convergent subsequences (u_{n_j}), (v_{n_j})

$$u_{n_j} \xrightarrow{w} u, \qquad v_{n_j} \xrightarrow{w} v$$

However

$$\|u-v\| \leqslant \underline{\lim} \| u_{n_j} - v_{n_j} \| = \lim \| u_n - v_n \| = 0$$

Whence $u = v$

By (3.4) , if the mapping $I_y - T$ is quasiconvex (and certainly continuous) then it is weakly lower semicontinuous , consequently we have

$$\|v - T(u,v)\| \leqslant \underline{\lim} \| v_{n_j} \; T(u_{n_j}, v_{n_j}) \| = \lim \| v_{n_j} - u_{n_j} \| = 0$$

and we get $v = T(u,v)$, that is $u = v = T(u,u) = T(v,v)$.

If the mapping $I_x - T$ is quasiconvex (and continuous certainly) , then as above we have

$$\| u - T(u,v) \| \leqslant \underline{\lim} \| u_{n_j} - T(u_{n_j} - v_{n_j}) \| = 0$$

Thus $u = T(u,v)$, that is $u = v = T(u,u) = T(v,v)$.

Therefore in both cases the subsequences $(u_{n_j}),(v_{n_j})$ weakly converge

to a solution $p \in F$.

Now letting $f(p) = \lim \| u_n - p \|$ .This function is defined on the set $F_0 = F \cap B(p, \|u_o - p\|)$ which is closed , convex and bounded by the convexity and closedness of F . And hence it is weakly compact. On the other hand , it is easy to see that $f(p)$ is convex and continuous , and whence $f(p)$ is weakly lower semicontinuous . Therefore $f(p)$ attains on F_0 its infinimum , say, at a point p . We will show that p is a unit limit point of both sequences $(u_n),v_n)$.

Indeed if there exists a subsequence (u_{n_k}) weakly converging to $\bar{u} \neq p$ then by (3.5) we obtain

$$\underline{\lim} \| u_{n_k} - p \| > \underline{\lim} \| u_{n_k} - \bar{u} \|$$

However

$$\underline{\lim} \| u_{n_k} - p \| = \lim \| u_n - p \| = f(p)$$

and

$$\underline{\lim} \| u_{n_k} - \bar{u} \| = \lim \| u_n - u \| = f(\bar{u})$$

yielding a contradiction.

The proof of Theorem 3.I thus is complete .

<u>Proof of Theorem 3.2.</u>

It suffices to show that the constructed sequences $(u_n),(v_n)$ satisfy condition(3.3), i.e. $\| u_n - v_n \| \to 0$ as $n \to \infty$

a) Firstly we study the case $c \neq I$

We have

$$\| u_n - p \| = \| (tJ_c + (I-t)T)(u_n, v_n) - p \| \leqslant t \| J_c(u_n, v_n) - p \| +$$

$$+(I-t)\| T(u_n, v_n) - p \| \leqslant t(c \| u_n - p \| + (I-c) \| v_n - p \|) + (I-t) \| T(u_n, v_n) - p \|$$

But

$$(I-t)\| T(u_n,v_n)-p \| \leqslant (I-t)\max(\| u_n-p \|, \| v_n-p \|)$$

if $\| u_n-p \| \neq \| v_n-p \|$

$$\leqslant (I-t)\max(\| u_n-p \|, \| v_n-p \|)$$

if $\| u_n-p \| = \| v_n-p \|$

It follows that

$$\| u_n-p \| \leqslant \| v_n-p \|$$

Analogously we obtain

$$\| v_n-p \| = \|(tJ_c + (I-t)T)_k (u_n,u_{n-I})-p \| \leqslant \| u_{n-I}-p \|$$

Therefore there exists d such that

$$d = \lim \| u_n-p \| = \lim \| v_n-p \|$$

If $d = 0$, then $u_n \longrightarrow p$, $v_n \longrightarrow p$ and certainly

$$u_n \xrightarrow{w} p, \quad v_n \xrightarrow{w} p$$

If $d \neq 0$, then $\left\| \dfrac{u_n-p}{\| v_n-p \|} \right\| = \dfrac{\| u_n-p \|}{\| v_n-p \|} \longrightarrow 1$

Nevertherless

$$\frac{u_n-p}{v_n-p} = \left\| t \left[\frac{c(u_n-p)+(I-c)(v_n-p)}{\| v_n-p \|} \right] + (I-t) \frac{T(u_n,v_n)-p}{\| v_n-p \|} \right\|$$

Putting

$$w_n = \frac{c(u_n-p)+(I-c)(v_n-p)}{\| v_n-p \|} \qquad z_n = \frac{T(u_n,v_n)-p}{\| v_n-p \|}$$

Obviously that

$$\| w_n \| \leqslant I, \quad \| z_n \| \leqslant I$$

Hence, from $\| tw_n+(I-t)z_n \| \longrightarrow 0$ as $n \longrightarrow \infty$ and by the uniform convexity of X, it follows that $\| w_n-z_n \| \longrightarrow 0$ as $n \longrightarrow \infty$, that is

$$\| c(u_n-p)+(I-c)(v_n-p) - (T(u_n,v_n)-p) \| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

Taking into account that

$$\| cu_n+(I-c)v_n-T(u_n,v_n) \| = \frac{I}{I-t} \| (I-t)(cv_n+(I-c)v_n)-(I-t)T(u_n,v_n) \| =$$

$$= \frac{I}{I-t} \| cu_n+(I-c)v_n - [t(cu_n+(I-c)v_n+(I-t)T(u_n,v_n)] \| =$$

$$= \frac{I}{I-t} \| cu_n+(I-c)v_n-u_n \| = \frac{I}{I-t} \| (I-c)(v_n-u_n) \|$$

We claim

$$\| v_n-u_n \| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

At the same time, it is easily to be seen

$$(J_c-T)(u_n-v_n) = cu_n+(I-c)u_n-T(u_n,v_n) = \frac{I-c}{I-t}(v_n-u_n)$$

Now as in Theorem 2.I we have

$$u_{n_j} \xrightarrow{w} u, \quad v_{n_j} \xrightarrow{w} v, \quad u = v$$

Because of the quasiconvexity of the mapping J_c-T we get

$$\| cu+(I-c)v-T(u,v)\| = \|u-T(u,u)\| \leq \underline{\lim} \|(J_c-T)(u_{n_j},v_{n_j})\| =$$

$$= \lim \| v_n-u_n\| \frac{I-c}{I-t} = 0 ,$$

that is $u = v = T(u,u) = T(v,v)$.

It remains to apply the same arguments as in the proof of Theorem 2.I .

b) The case $c = I$.

By (3.I) we obtain

$$\| u_{n+I}-p\| \leq \| v_{n+I}-p \| \leq \| \tilde{u}_n-p \|$$

and $\| u_n-p \| \leq \| v_n-p \|$

But $\tilde{u}_n = tu_n + (I-t)v_n$, we get

$$\| \tilde{u}_n-p \| \leq \|v_n-p \|$$

By the inequalities

$$\| v_{n+I}-p \| \leq \|\tilde{u}_n-p \| \leq \| v_n-p \|$$

We assert that there exists d such that

$$d = \lim \| \tilde{u}_n-p \| = \lim \| v_n-p \|$$

If $d = 0$, then $\| v_n-p \| \longrightarrow 0$ as $n \longrightarrow \infty$

If $d \neq 0$, then $\left\| \frac{\tilde{u}_n-p}{\|v_n-p\|} \right\| \longrightarrow I$

However

$$\left\| \frac{\tilde{u}_n-p}{\|v_n-p\|} \right\| = \left\| \frac{t(u_n-p)}{\|v_n-p\|} + \frac{(I-t)(v_n-p)}{\| v_n-p \|} \right\|$$

Letting

$$w_n = \frac{u_n-p}{\|v_n- p\|} \qquad z_n = \frac{v_n-p}{\| v_n-p\|}$$

We have

$$\| w_n \| \leq I , \quad \| z_n \| = I$$

Consequently

$\| w_n-z_n \| \to 0$ as $n \longrightarrow \infty$, i.e.

$$\left\| \frac{u_n-p}{\| v_n-p\|} - \frac{v_n-p}{\| v_n-p\|} = \frac{u_n-v_n}{\| v_n-p \|} \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

Therefore

$\| u_n-v_n \| \to 0$ as $n \longrightarrow \infty$

And further we discuss as in the proof of Theorem 2.I.

4. Weak convergence . Nonstationary case .

<u>Theorem 4.I</u>

Let X be a Banach space , D a closed subset of X, T_n , $n \geq 0$ continuous, weakly compact mappings from DxD into D. Assume that the following conditions hold

(4.1) $F = \{ x : T(x,x) = x \} \neq \emptyset$

(4.2)
$$\|T_n(x,y)-p\| \begin{cases} < \max(\|x-p\|, \|y-p\|) & , \text{ if } \|x-p\| \neq \|y-p\| \\ \leq \max(\|x-p\|, \|y-p\|), & \text{ if } \|x-p\| = \|y-p\| \end{cases}$$

$\forall x,y \in D$, $p \in F$, $n \geq 0$

(4.3) For $c \in [0,I]$ the mapping $J_c - T_o$ is quasiconvex on DxD .

(4.4) The space X has the property

If $(y_n) \subset X$, $y_n \xrightarrow{w} y_o \in X$ then

$\underline{\lim} \|y_n - y\| > \underline{\lim} \|y_n - y_o\|$, $\forall y \neq y_o$, $y \in X$

(4.5) The systems

$$u_n = T_n(u_n, v_n) \quad , \quad n \geq I$$
$$v_n = T_{n,k}(u_n, u_{n-I}) \qquad (5)$$

are solvable for all $n \geq I$, and $\|u_n - v_n\| \to 0$ as $n \to \infty$

(4.6) $\|T_n(x,y) - T_o(x,y)\| \leq \varepsilon_n$, $\varepsilon_n \to 0$ as $n \to \infty$, $\forall x,y \in D$

Then the sequences $(u_n),(v_n)$ determined by (5) weakly converge at the same time to a solution of equation (I).

Proof

As in the proof of Theorem 2.I , we have

$\lim \|v_n - p\| = \lim \|u_n - p\| \leq \|u_o - p\|$

Hence

$(u_n), (v_n) \subset D_o = B(p, \|u_o - p\|) \cap D$

which is bounded and closed in D .

Let M_n be the image of the restriction of T_n on $D_o x D_o$, $n \geq 0$.

Because of the weak compactness of T_n,M_n are weakly compact . By means of the constructed method , it is clear that $u_n, v_n \in M_n$, $n \geq I$. We will prove that $M = \bigcup_{n \geq 0} M_n$ is weakly compact

Indeed by (4.6) we obtain $M_n \xrightarrow{B} M_o$, which is equivalent in Banach spaces to $M_n \longrightarrow M_o$ in the strong topology of X.Consequently $M_n \xrightarrow{B-w} M_o$. Now by using Lemma 2.I of O.A.Liskovetz we assert that $M = \bigcup_{n \geq 0} M_n$ is weakly compact .

Therefore from (u_n) , (v_n) , we can choose weakly convergent subsequences (u_{n_j}), (v_{n_j}) , $u_{n_j} \xrightarrow{w} u$, $v_{n_j} \xrightarrow{w} v$

But

$\|u - v\| \leq \underline{\lim} \|u_{n_j} - v_{n_j}\| = \lim \|u_n - v_n\| = 0$

that is $u = v$.

On the other hand

$\|(J_c - T_o)(u,v)\| = \|cu + (I-c)v - T_o(u,v)\| \leq$

$$\leq \underline{\lim} \| cu_{n_j} + (I-c)v_{n_j} - T_o(u_{n_j}, v_{n_j}) \| \leq$$

$$\leq \underline{\lim}(\| cu_{n_j - 1} + (I-c)v_n - T_{n_j}(u_{n_j}, v_{n_j}) \| + \mathcal{E}_{n_j})$$

$$\leq \underline{\lim} \| (I-c)(v_{n_j} - u_{n_j}) \| \quad (\text{ because } u_{n_j} = T_{n_j}(u_{n_j}, v_{n_j}))$$

$$= (I-c)\lim \| v_n - u_n \| = 0$$

Hence

$$cu + (I-c)v = T_o(u,v)$$

and $u = v$, whence $u = v \in F$.

We will show that $(u_n),(v_n)$ have just one limit point (in the weak sense) , say , p .

For this aim , we consider the function $f(p) = \underline{\lim} \| v_n - p \|$ defined on $F_o = F \cap M_o$. Since the function f is weakly lower semicontinuous and the set F is weakly compact (as the image of a bouded closed set by a weakly compact mapping) , we claim that the function f attains on F_o its infinimum at a point , say , $\bar{p}$.

Suppose that a subsequence (u_{n_k}) weakly converges to $u \neq \bar{p}$. then by (4.4) we have

$$\underline{\lim} \| u_{n_k} - \bar{p} \| > \underline{\lim} \| u_{n_k} - u \| \quad , \text{ a contradiction.}$$

Therefore $u_n \xrightarrow{w} \bar{p}$, $v_n \xrightarrow{w} \bar{p}$

This completes the proof of Theorem 4.I.

Theorem 4.2.

Let X be a strictly convex and reflexive Banach space , D a subset in X containing convex closed subsets D_n , $n \geq 0$, which are images of continuous mappings T on DxD (i.e. $T_n : DxD \longrightarrow D_n$) . Assume that the conditons (4.I) , (4.3) , (4.6) are fulfiled , where the mapping $J_c - T_o$ is quasiconvex on DxD or D'x D' , $D_o \subset D' \subset D$.

Then the sequences (u_n), (v_n) defined by (5) weakly converge at the same time to a solution of equation (I).

Proof

The proof of this theorem is similar to the one in Theorem 3.I and 3.3 . Using the convexity of the set $F \cap B(p, \| u_o - p \|)$ in a strictly convex space and the weak compactness of the set $\bigcup_{n=0}^{\infty} (B(p, \| u_o - p \|) \cap D_n$

(the union of bounded convex closed subsets in reflexive space) and the weak convergence $D_n \xrightarrow{B-w} D_o$

Theorem 4.3.

Let X be a uniformly convex Banach space , T_n , $n \geq 0$ continuous mappings from DxD to D_n, where D_n , $n \geq 0$, are convex closed subsets in $D \subset X$. Suppose that the conditions (4.I), (4.4), (4.5),(4.6) are fulfiled . Moreover

a) Let for any $c \in [0,I]$ the mapping $J_c - T_o$ be quasiconvex on DxD , or on DxD' in the case D_o strictly contains itself in D , and D' is a convex subset in D and contains itself in D_o

If for any $t \in (0,I)$ th systems

$$\begin{aligned} u_n &= (tJ_c+(I-t)T_n)(u_n,v_n) \quad , \quad n \geqslant I \\ v_n &= (tJ_c + (I-t)T_n)_k (u_n,u_{n-I}) \end{aligned} \qquad (6)$$

are solvable, then the sequences (u_n), (v_n) defined by (6) weakly converge to a solution of equation(I) .

b) Let the mapping I_x-T_o be quasiconvex on $D \times D$, or on $D \times D'$ (see a))

If for any $t \in (0,I)$ the systems

$$\begin{aligned} u_n &= T_n(u_n,v_n) \quad , \quad n \geqslant I \\ v_n &= T_{n,k}(u_n, \tilde{u}_{n-I}) \end{aligned} \qquad (7)$$

where $\tilde{u}_{n-I} = tu_{n-I} +(I-t)v_{n-I}$, $n \geqslant 2$, $\tilde{u}_o = u_o$

then the sequences $(u_n),(v_n)$ constructed by (7) weakly converge to a solution of equation (I) .

To prove this Theorem , applying the arguments used in the proofs of Theorems 3.2 and 4.I .

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