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REPRESENTATIONS OF BRAID GROUP OBTAINED FROM QUANTUM sl(3) ENVELOPING ALGEBRA *

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ABSTRACT

The quantum Clebsch–Gordan coefficients for the coproduct 6×6 of the quantum sl(3) enveloping algebra are computed. Based on the representation 6, the representation of the braid group and the corresponding link polynomial are obtained. The link polynomials based on the representations of the quantum sl(3) enveloping algebra with one row Young tableau are discussed.

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1. Introduction

Two years ago, Akutsu and Wadati [1] "found an unexpected close connection between physics and mathematics", that is, they found out the similarity between the quantum Yang-Baxter equation (QYBE) and the multiplication rules of the braid group [2]. Taking the spectral parameter u in QYBE to be infinity, they obtained a set of representations of the braid group from the Boltzmann weights of the N-state models, which are the solutions of QYBE, in terms of normalizing, the symmetry breaking transformation and the limit process of $u \to \infty$. Then, they successfully found out a set of link polynomials. This method was discussed independently by Jones.

On the other hand, from the trigonometric solution [3] of the classical Yang-Baxter equation (CYBE)

$$r = - C_0 - 2 C_-$$

$$= - \sum_j H_j \times H_j - 2 \sum_{\alpha \in \Delta_+} E_{-\alpha} \times E_\alpha \qquad (1)$$

where H_j and E_α are the cartan bases, and Δ_+ is the set of positive roots, we computed, based on the quantum sl(2) enveloping algebra (q-sl(2)), the explicit forms of $\check{R}_\ell$ matrix [4,5], satisfying QYBE without the spectral parameter, which is the same as the multiplication rule of the braid group. We found out that the solutions $\check{R}_\ell$ we got are just the same, up to an unimportant factor, as R_{AW} matrix of Akutsu and Wadati. Thus, we have obtained the explicit form of R_{AW} with any j (or N= 2j+1).

In our computation for q-sl(2) [4], we used the same notations as those in the theory of angular momentum, and computed the analogys of CG coefficients, Racah coefficients, 3j-symbol, and 6j-symbol, step by step. Unfortunately, we are not able to compute quantum CG coefficients (q-CG) for the quantum sl(3) enveloping algebra (q-sl(3)) generally, because even in su(3) Lie algebra, C-G coefficients were computed only for some definite representations[6].

Encouraged by the success of computation on q-sl(2), we are interested in generalizing this method to q-sl(3), although we are able to compute only for some definite representations. $\check{R}_q$ matrices based on the fundamental representations of q-sl(n) were calculated[7] and proved to satisfy the Hecke algebra. The fusion of the fundamental representations was discussed[8]. Recently, the operator form of $\check{R}_q$ matrix for q-sl(n) was given[9]. But, in order to construct link polynomials, more explicit forms of $\check{R}_q$ matrices are needed.

In this paper, we compute the q-CG coefficients for the coproduct 6×6, then obtain the $\check{R}_q$ matrix and construct the corresponding link polynomial. Throughout this paper, we assume that q is not a root of one so that all the finite irreducible representations (IR) are nonsingular. In Sec. II, we review q-sl(3) briefly. In Sec. III, we make some conventions on enumerating the states of the irreducible representation, and their relative phases. For the convenience for q-sl(3), we have to change some conventions used in su(3) Lie algebra[6]. The q-CG coefficients and relavent representation matrices are computed in Sec. IV, and then, the $\check{R}_q$ matrix in Sec. V. In terms of the general methods[1,10,11], we construct the link polynomials in Sec. VI. In Sec. VII, some discussions are given for the representations of q-sl(3) with one row Young tableau. We left the computation on the coproduct 8×8 of q-sl(3) in the next paper.

II. Quantum sl(3) Enveloping Algebra

Deform the generators λ_a of su(3) algebra to those of q-sl(3)

$$2\lambda_3 \to h_1, \quad \sqrt{3}\lambda_8 - \lambda_3 \to h_2, \quad \lambda_1 + i\lambda_2 \to e_1$$
$$\lambda_1 - i\lambda_2 \to f_1, \quad \lambda_6 + i\lambda_7 \to e_2, \quad \lambda_6 - i\lambda_7 \to f_2 \tag{2}$$

which satisfy the following multiplication rules

$$k_a = q^{h_a/2}$$
$$k_a e_a = q e_a k_a, \qquad k_a e_b = q^{-1/2} e_b k_a$$
$$k_a f_a = q^{-1} f_a k_a, \qquad k_a f_b = q^{1/2} f_b k_a$$
$$[k_1, k_2] = [e_1, f_2] = [e_2, f_1] = 0$$
$$[e_a, f_a] = \frac{k_a^2 - k_a^{-2}}{q - q^{-1}}$$
$$e_a^2 e_b - (q + q^{-1}) e_a e_b e_a + e_b e_a^2 = 0$$
$$f_a^2 f_b - (q + q^{-1}) f_a f_b f_a + f_b f_a^2 = 0 \tag{3}$$

where $a, b = 1, 2$, and $a \neq b$.

The explicit forms of generators in the fundamental representation of q-sl(3) are given as follows:

$$h_1 = \mathrm{diag}\,(1, -1, 0), \qquad h_2 = \mathrm{diag}\,(0, 1, -1)$$
$$e_1 = \tilde{f}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_2 = \tilde{f}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

where and in the rest of this paper the tidle denotes transpose.

Following the nomenclature in the SU(3) theory, we call the subalgebra q-sl(2) spanned by h_1, e_1 and f_1 q-isospin and that spanned by h_2, e_2 and f_2 q-U-spin, and use q-isospin and q-superspin Y to assign the states in IR:

$$I_3 = h_1/2, \qquad Y = (h_1 + 2h_2)/3$$

$$I^2 \quad \frac{(k_1 - k_1^{-1})(q\, k_1 - q^{-1} k_1^{-1})}{(q - q^{-1})^2} + f_1 e_1 \qquad (5)$$

III, Conventions for Irreducible Representations

We choose the bases of an IR of q-sl(3) so that h_1, h_2 and I^2 are diagonal, namely, each state is the common eigenstate of I^2, I_3 and Y:

$$I^2 \,|I,I_3,Y\rangle = [I][I+1]\, |I,I_3,Y\rangle$$

$$h_1 \,|I,I_3,Y\rangle = 2I_3\, |I,I_3,Y\rangle \qquad (6)$$

$$h_2 \,|I,I_3,Y\rangle = (3Y/2 - I_3)\, |I,I_3,Y\rangle$$

where and throughout this paper

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}} \qquad (7)$$

Because q is not a root of one, we have

$$[m] \neq 0 \quad \text{if} \quad m \neq 0$$

Now, we enumerate the states of an IR from 1 to N as the following order [6] :

a) The state with the highest weight, i.e., with the highest eigenvalue of Y, and the highest eigenvalue of I_3 among the states with the same highest eigenvalue of Y, is enumerated by one.

b) The states with the same Y and I_3 are ordered so that I decreases.

c) The group of states for the same Y but different I_3 are ordered such that I_3 decreases.

d) The group of states for different Y are ordered such that Y decreases.

Sometimes, we denote an IR $[\lambda_1, \lambda_2]$ of q-sl(3) by its dimension N. For example, the fundamental representation [1,0] is

denoted by 3, [1,1] by 8, [2,0] by 6, [2,2] by 6', [4,0] by 15, and [3,1] by 15'. Note not to confuse the notation of an IR $[\lambda_1, \lambda_2]$ with [m] in (5). The enumerations of the states of the relevant representations are listed in Fig. 1.

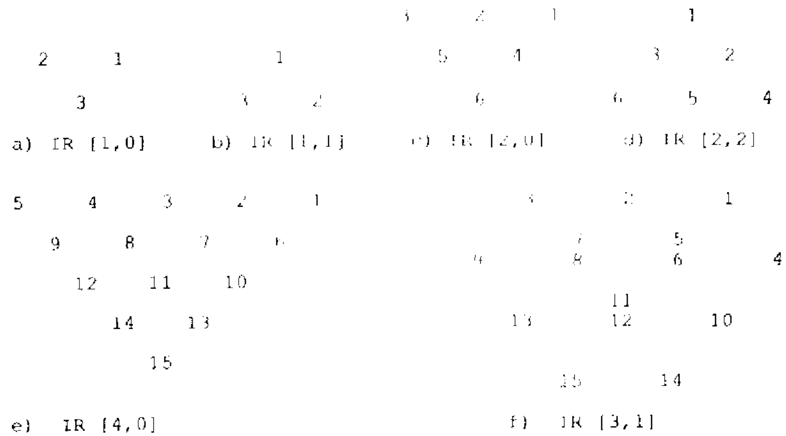

Fig. 1. The enumerations of the eigenstates of IR of q-sl(3).

In this paper we use some different notations to denote the states of IR. When we emphasize the eigenvalues (weight), for example in (6), we use the notation $|I,I_3,Y\rangle$, but in the usual case, in order to reduce the notation we use the enumerated number m to denote a state, for example, $|[2,0]\, m\rangle$ or $|6,m\rangle$ to denote a state in IR [2,0].

To be able to define later uniquely q-CG coefficients of q-sl(3), it is necessary first of all to define precisely the relative phases of the states within an IR. For convenience we adopt a little different conventions as those adopted in su(3) [6] .

i) The relative phases within a definite q-isomultiplet are determined by q-sl(2) phase convention [4] :

$$e_1 \,|I,I_3,Y\rangle = \Gamma^{1}_{-I_3}(q)\, |I,I_3+1,Y\rangle$$

$$f_1 |I, I_3, Y\rangle = \Gamma^{I}_{I_3}(q)\, |I, I_3 - 1, Y\rangle$$

$$\Gamma^{j}_{m}(q) = ([j+m][j-m+1])^{1/2} \qquad (8)$$

ii) The relative phases between the different q-isomultiplets are determined so that the matrix elements of e_2 and f_2 are non-negative when q is positive real.

iii) The coproduct of two IR's is defined as

$$k_a (|N_1, m_1\rangle |N_2, m_2\rangle) = (k_a |N_1, m_1\rangle)(k_a |N_2, m_2\rangle)$$

$$e_a (|N_1, m_1\rangle |N_2, m_2\rangle) = (e_a |N_1, m_1\rangle)(k_a^{-1} |N_2, m_2\rangle) + (k_a |N_1, m_1\rangle)(e_a |N_2, m_2\rangle) \qquad (9)$$

and that replaced e_a by f_a.

iv) The q-CG matrix is real orthogonal when q is positive real, and orthogonal for any q.

As an example, we calculate representation matrices of IR [2,0] of q-sl(3) and the q-CG coefficients for the decomposition of coproduct [1,0] x [1,0] = [2,0] + [1,1] (3 x 3 = 6 + 3*). The states in IR's [1,0], [2,0] and [1,1] are denoted by |m>, |6, m> and |3*, m>, respectively.

The state with the lowest weight of IR [2,0] is simple and can be expressed as a direct product of |3> of IR [1,0]:

$$|6, 6\rangle = |3\rangle|3\rangle \qquad (10a)$$

According to the definition of coproduct (9) and the representation matrices (6) of q-sl(2), we have

$$|6, 5\rangle = [2]^{-1/2} e_2 |6, 6\rangle = [2]^{-1/2} \{q^{1/2} |2\rangle|3\rangle + q^{-1/2} |3\rangle|2\rangle\}$$

$$|6, 4\rangle = e_1 |6, 5\rangle = [2]^{-1/2} \{q^{1/2} |1\rangle|3\rangle + q^{-1/2} |3\rangle|1\rangle\}$$

$$|6, 3\rangle = [2]^{-1/2} e_2 |6, 5\rangle = |2\rangle|2\rangle$$

$$|6, 2\rangle = [2]^{-1/2} e_1 |6, 3\rangle = [2]^{-1/2} \{q^{1/2} |1\rangle|2\rangle + q^{-1/2} |2\rangle|1\rangle\}$$

$$|6, 1\rangle = [2]^{-1/2} e_1 |6, 2\rangle = |1\rangle|1\rangle \qquad (10b)$$

From the requirement of orthogonality of states, we get the state with the lowest weight of IR [1,1]

$$|3^*, 3\rangle = [2]^{-1/2} \{q^{-1/2} |2\rangle|3\rangle - q^{1/2} |3\rangle|2\rangle\} \qquad (11a)$$

then,

$$|3^*, 2\rangle = e_1 |3^*, 3\rangle = [2]^{-1/2} \{q^{-1/2} |1\rangle|3\rangle - q^{1/2} |3\rangle|1\rangle\}$$

$$|3^*, 1\rangle = e_2 |3^*, 2\rangle = [2]^{-1/2} \{q^{-1/2} |1\rangle|2\rangle - q^{1/2} |2\rangle|1\rangle\} \qquad (11b)$$

(10) and (11) gives both representation matrices of IR [2,0] and q-CG coefficients:

$$D^6_q(k_1) = \mathrm{diag}\,(q, 1, q^{-1}, q^{1/2}, q^{-1/2}, 1)$$

$$D^6_q(k_2) = \mathrm{diag}\,(1, q^{1/2}, q, q^{-1/2}, 1, q^{-1})$$

$$D^6_q(e_1)_{12} = D^6_q(e_1)_{23} = D^6_q(e_2)_{35} = D^6_q(e_2)_{56} = [2]^{1/2}$$

$$D^6_q(e_1)_{45} = D^6_q(e_2)_{24} = 1 \qquad (12)$$

the rest of matrix elements are vanishing, and the matrices of f_a are the transpose of those of e_a.

$$
(C_q^{33})_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A & 0 \\ 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (C_q^{33})_{3^*} = \begin{pmatrix} 0 & 0 & 0 \\ B & 0 & 0 \\ 0 & B & 0 \\ -A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B \\ 0 & -A & 0 \\ 0 & 0 & -A \\ 0 & 0 & 0 \end{pmatrix} \qquad (13)
$$

where $A = \{q/[2]\}^{1/2}$ and $B = \{q[2]\}^{-1/2}$. The rows of q-CG are ordered by $m_1 m_2$ of two fundamental representations.

IV. Decomposition of Coproduct 6 x 6 in q-sl(3)

The calculation for the decomposition of coproduct 6 x 6 in q-sl(3) is straightforward but tedious. The key is to determine the states with multiple weights, i.e., the states with the same Y and I_3, by the requirement of orthogonality. Most of the representation matrices can be obtained directly from the enumerations of the states and the representation matrices of q-sl(2). The matrices of f_a are the transpose of those of e_a. What is only needed to list is the matrix of e_2 in IR [3,1], because there are multiple weights in this IR. The nonvanishing matrix elements of e_2 in IR [3,1] are listed in Table 1.

Table 1. The nonvanishing matrix elements of e_2 in IR [3,1]

row	1	2 5	2 6	3	3	7
column	4	5 10	6 10	7	8	11
$D_q(e_2)$	1	$\left\{\frac{[2]}{[3]}\right\}^{\frac{1}{2}}$	$\left\{\frac{[4]}{[3]}\right\}^{\frac{1}{2}}$	$\frac{1}{[3]^{1/2}}$	$\left\{\frac{[4][2]}{[3]}\right\}^{\frac{1}{2}}$	$\frac{[2]}{[3]^{1/2}}$

row	8	8	11	12	9	13
column	11	12	14	14	13	15
$D_q(e_2)$	$\left\{\frac{[4]}{[3][2]}\right\}^{\frac{1}{2}}$	$[3]^{\frac{1}{2}}$	1	$\left\{\frac{[4]}{[2]}\right\}^{\frac{1}{2}}$	$[2]^{\frac{1}{2}}$	$[2]^{\frac{1}{2}}$

The rows of the q-CG matrix for 6 x 6 in q-sl(3) are ordered by m_1 and m_2 which both go from 1 to 6, and the columns are denoted by (15,m), (15′,m) and (6*,m) for the different IR's, respectively. The q-CG matrix is a block matrix, and some submatrices are equal to each other. The nonvanishing matrix elements are listed in Table 2. The equal submatrices are listed in the same table and distingushed by a), b) and so on.

Table 2. Nonvanishing matrix elements of q-CG for 6 x 6 in q-sl(3)

row \ column	a) (15,1) b) (15,5) c) (15,15)
a) 1 1 b) 3 3 c) 6 6	1

row \ column	a) (15,2) b) (15,6) c) (15,4) d) (15,9) e) (15,13) f) (15,14)	a) (15′,1) b) (15′,4) c) (15′,3) d) (15′,9) e) (15′,14) f) (15′,15)
a) 12 b) 14 c) 23 d) 35 e) 46 f) 56	$q\left\{\frac{[2]}{[4]}\right\}^{1/2}$	$q^{-1}\left\{\frac{[2]}{[4]}\right\}^{1/2}$
a) 21 b) 41 c) 32 d) 53 e) 64 f) 65	$q^{-1}\left\{\frac{[2]}{[4]}\right\}^{1/2}$	$-q\left\{\frac{[2]}{[4]}\right\}^{1/2}$

row \ column	a) (15,3) b) (15,10) c) (15,12)	a) (15′,2) b) (15′,10) c) (15′,13)	a) (6*,1) b) (6*,4) c) (6*,6)
a) 13 b) 16 c) 36	$q^2\left\{\frac{[2]}{[4][3]}\right\}^{1/2}$	$\left\{\frac{[2]}{[4]}\right\}^{1/2}$	$q^{-1}[3]^{-1/2}$
a) 22 b) 44 c) 55	$[2]\left\{\frac{[2]}{[4][3]}\right\}^{1/2}$	$(q^{-1}-q)\left\{\frac{[2]}{[4]}\right\}^{\frac{1}{2}}$	$-[3]^{-1/2}$
a) 31 b) 61 c) 63	$q^{-2}\left\{\frac{[2]}{[4][3]}\right\}^{1/2}$	$-\left\{\frac{[2]}{[4]}\right\}^{1/2}$	$q[3]^{-1/2}$

column / row	(15,7)	(15',5)	(15',6)	(6*,2)
15	$q^{2}\left\{\frac{[2]}{[4][3]}\right\}^{\frac{1}{2}}$	$\left\{\frac{[2]}{[4][3]}\right\}^{\frac{1}{2}}$	$[3]^{-1/2}$	$q^{-1}[3]^{-1/2}$
24	$\frac{q^{\frac{1}{2}}[2]}{([4][3])^{1/2}}$	$\frac{q^{-3/2}[2]}{([4][3])^{1/2}}$	$\frac{-q^{3/2}}{([3][2])^{1/2}}$	$\frac{-q^{1/2}}{([3][2])^{1/2}}$
42	$\frac{q^{-1/2}[2]}{([4][3])^{1/2}}$	$\frac{-q^{3/2}[2]}{([4][3])^{1/2}}$	$\frac{q^{-3/2}}{([3][2])^{1/2}}$	$\frac{-q^{-1/2}}{([3][2])^{1/2}}$
51	$q^{-2}\left\{\frac{[2]}{[4][3]}\right\}^{\frac{1}{2}}$	$-\left\{\frac{[2]}{[4][3]}\right\}^{\frac{1}{2}}$	$-[3]^{-1/2}$	$q[3]^{-1/2}$

column / row	(15,8)	(15',7)	(15',8)	(6*,3)
25	$\frac{q^{3/2}[2]}{([4][3])^{1/2}}$	$\frac{q^{-\frac{1}{2}}[2]}{([4][3])^{1/2}}$	$\frac{q^{-\frac{1}{2}}}{([3][2])^{1/2}}$	$\frac{q^{-3/2}}{([3][2])^{1/2}}$
34	$\left\{\frac{[2]}{[4][3]}\right\}^{\frac{1}{2}}$	$q^{-1}\left\{\frac{[2]}{[4][3]}\right\}^{\frac{1}{2}}$	$-q[3]^{-1/2}$	$-[3]^{-1/2}$
43	$\left\{\frac{[2]}{[4][3]}\right\}^{\frac{1}{2}}$	$-q^{1}\left\{\frac{[2]}{[4][3]}\right\}^{\frac{1}{2}}$	$q^{-1}[3]^{-1/2}$	$-[3]^{-1/2}$
52	$\frac{q^{-3/2}[2]}{([4][3])^{1/2}}$	$\frac{-q^{1/2}[2]}{([4][3])^{1/2}}$	$\frac{-q^{\frac{1}{2}}}{([3][2])^{1/2}}$	$\frac{q^{3/2}}{([3][2])^{1/2}}$

column / row	(15,11)	(15',11)	(15',12)	(6*,5)
26	$q^{2}\left\{\frac{[2]}{[4][3]}\right\}^{\frac{1}{2}}$	$\left\{\frac{[2]}{[4]}\right\}^{\frac{1}{2}}$	0	$q^{-1}[3]^{-1/2}$
45	$\frac{q^{\frac{1}{2}}[2]}{([4][3])^{\frac{1}{2}}}$	$\frac{q^{-1/2}-q^{3/2}}{[4]^{1/2}}$	$(q[2])^{-1/2}$	$\frac{-q^{\frac{1}{2}}}{([3][2])^{1/2}}$
54	$\frac{q^{-\frac{1}{2}}[2]}{([4][3])^{1/2}}$	$\frac{q^{-3/2}-q^{1/2}}{[4]^{1/2}}$	$-\left\{\frac{q}{[2]}\right\}^{1/2}$	$\frac{-q^{-1/2}}{([3][2])^{1/2}}$
62	$q^{-2}\left\{\frac{[2]}{[4][3]}\right\}^{\frac{1}{2}}$	$-\left\{\frac{[2]}{[4]}\right\}^{\frac{1}{2}}$	0	$q[3]^{-1/2}$

V. Representations of Braid Group

Now, we are going to compute the representation matrix of the generator b_i of the braid group

$$D(b_i) = \mathbb{1} \times \ldots \times \mathbb{1} \times \check{R}_q \times \mathbb{1} \ldots \times \mathbb{1} \tag{14}$$

where $\check{R}_q$ is located in the ith and (i+1)th positions in the direct product. $\check{R}_q$ matrix is defined as follows[4]:

a) $\check{R}_q$ has r as its classical limit

$$\check{R}_q \sim (\mathbb{1} + (q-1)\, r)\, P$$

where P is the transpose, and r is the solution of classical YBE

$$r = -(h_1 \times h_1 + 3\, Y \times Y)/2 -$$

$$- 2\,(f_1 \times e_1 + f_2 \times e_2 + [f_2, f_1] \times [e_1, e_2]) \tag{15}$$

where the quantum parameter $q \to 1$.

b) $\check{R}_q^{N_1 N_2} D_q^{N_1 N_2} = D_q^{N_2 N_1} \check{R}_q^{N_1 N_2}$

where $D_q^{N_1 N_2}$ is the coproduct of IR's N_1 and N_2, in our case $N_1 = N_2 = 6$, and usually, we will omit the superscripts of $\check{R}_q$.

c) $(\check{R}_q)^{-1} = P\, \check{R}_{q^{-1}}\, P$

From the definition, it can be proved that $\check{R}_q$ satisfies the QYBE without the spectral parameter, and can be expressed as

$$\check{R}_q = \sum_N \xi_N\, q^{\eta(N_1, N_2, N)}\, (C_q^{N_2 N_1})_N\, (\widetilde{C_q^{N_1 N_2}})_N \tag{16}$$

where N =15, 15' and 6* when $N_1 = N_2 = 6$, C_q is the q-CG matrix given in the last section, ξ_N is the symmetry of the CG coefficients of su(3) for exchanging N_1 and N_2 to each other, and given as

$$\xi_{15} = \xi_{6*} = 1, \qquad \xi_{15'} = -1 \tag{17}$$

and $\eta(N_1, N_2, N)$ is calculated in the classical level[4,11]

$$\eta(N_1,N_2,N) = (\widetilde{C^{N_2 N_1}})_N \; (C^{N_2 N_1})_N$$

$$= C_2(N_1) + C_2(N_2) - C_2(N) \qquad (18)$$

where C_2 is the casimir operator and calculated for su(3) in the appendix

$$C_2([\lambda_1,\lambda_2]) = (\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2 + 3\lambda_1)/3 \qquad (19)$$

In our case,

$C_2(3) = C_2(3^*) = 4/3$, $\quad C_2(6) = C_2(6^*) = 10/3$,

$C_2(15) = 28/3$, $\quad C_2(15') = 16/3$

To make the expression more simple, we remove a factor of q from the definition of $\check{R}_q$ such that the term related to the IR, the first row of whose Young tableau is the longest and whose dimension is denoted by N_0, has factor one, i.e., define η' instead of η:

$$\eta'(N) = C_2(N_0) - C_2(N) \qquad (18')$$

Therefore, after removing a factor $q^{-2/3}$ for the fundamental representation, we have

$$\check{R}_q^{33} = (C_q)_6 (\widetilde{C_q})_6 - q^2 (C_q)_{3^*} (\widetilde{C_q})_{3^*} \qquad (20)$$

and after removing a factor $q^{-8/3}$ for the IR [2,0], we have

$$\check{R}_q = (C_q)_{15} (\widetilde{C_q})_{15} - q^4 (C_q)_{15'} (\widetilde{C_q})_{15'} + q^6 (C_q)_{6^*} (\widetilde{C_q})_{6^*} \qquad (21)$$

Obviously, $\check{R}_q^{33}$ satisfies the Hecke algebra

$$(\check{R}_q^{33} - 1)(\check{R}_q^{33} + q^2) = 0 \qquad (22)$$

and $\check{R}_q$ satisfies

$$(\check{R}_q - 1)(\check{R}_q + q^4)(\check{R}_q - q^6) = 0 \qquad (23)$$

Through the straightforward calculation, we obtain

$$\check{R}_q^{33} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & q & 0 & A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & q & 0 & 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & A & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (24)$$

where $A = 1 - q^2$. Obviously, this $\check{R}_q^{33}$ matrix coincides with the previous result [7,8].

The $\check{R}_q$ matrix based on IR [2,0] of q-sl(3) is also a block matrix and some submatrices are equal to each other. The calculation results are listed in Table 3. The equal submatrices are listed in the same table, and distinguished by a), b), and so on.

Table 3. $\check{R}_q$ matrix based on IR [2,0] of q-sl(3).

row \ column	a) 11 b) 33 c) 66
a) 11 b) 33 c) 66	1

row \ column	a) 12 b) 14 c) 23 d) 35 e) 46 f) 56	a) 21 b) 41 C) 32 d) 53 e) 64 f) 65
a) 12 b) 14 c) 23 d) 35 e) 46 f) 56	0	q^2
a) 21 b) 41 c) 32 d) 53 e) 64 f) 65	q^2	$1 - q^4$

row \ column	a) 13 b) 16 c) 36	a) 22 b) 44 c) 55	a) 31 b) 61 c) 63
a) 13 b) 16 c) 36	0	0	q^4
a) 22 b) 44 c) 55	0	q^2	$q(1 - q^4)$
a) 31 b) 61 c) 63	q^4	$q(1 - q^4)$	$1-q^2-q^4+q^6$

column / row	a) 15 a) 26	a) 24 b) 45	a) 42 b) 54	a) 51 b) 62
a) 15 b) 26	0	0	0	q^4
a) 24 b) 45	0	0	q^3	$q^2(1-q^2)(1+q^2)^{\frac{1}{2}}$
a) 42 b) 54	0	q^3	$q^2(1-q^2)$	$q(1-q^2)(1+q^2)^{\frac{1}{2}}$
a) 51 b) 62	q^4	$q^2(1-q^2)(1+q^2)^{\frac{1}{2}}$	$q(1-q^2)(1+q^2)^{\frac{1}{2}}$	$1-q^2-q^4+q^6$

column / row	25	34	43	52
25	0	0	0	q^3
34	0	0	q^4	$q^2(1-q^2)(1+q^2)^{\frac{1}{2}}$
43	0	q^4	0	$q^2(1-q^2)(1+q^2)^{\frac{1}{2}}$
52	q^3	$q^2(1-q^2)(1+q^2)^{\frac{1}{2}}$	$q^2(1-q^2)(1+q^2)^{\frac{1}{2}}$	$1-2q^4+q^6$

VI. Link Polynomials

Substituting the $\check{R}_q$ matrix into (14), we obtain the representation of generators of the braid group, then the representation $D(B,n)$ of any element B of the braid group B_n with n strands. Define a direct product matrix V of n matrices v

$$V = v \times v \times \ldots \times v \tag{25}$$

For the fundamental representation [1,0], v is a 3x3 diagonal matrix

$$v = \mathrm{diag}\,(q^{-2},\ 1,\ q^{2})/(q^{-2}+1+q^{2}) \tag{26a}$$

and for IR [2,0], v is a 6x6 diagonal matrix

$$v = \mathrm{diag}\,(q^{-4},\ q^{-2},\ 1,\ 1,\ q^{2},\ q^{4})/(q^{-4}+q^{-2}+2+q^{2}+q^{4}) \tag{26b}$$

It is easy to check that

$$\sum_n ((1 \times v)\, \check{R}_q)_{mn,m'n} = \delta_{mm'}\, \tau$$
$$\sum_n ((1 \times v)\, \check{R}_q^{-1})_{mn,m'n} = \delta_{mm'}\, \bar{\tau} \tag{27}$$

where for the fundamental representation

$$\tau = (1 + q^{2} + q^{4})^{-1}, \qquad \bar{\tau} = q^{4}\, \tau \tag{28}$$

and for IR [2,0]

$$\tau = (1 + q^{2} + 2q^{4} + q^{6} + q^{8})^{-1}, \qquad \bar{\tau} = q^{8}\, \tau \tag{29}$$

Thus, the link polynomials defined as follows are invariant under the Markov moves

$$\alpha(B,n) = (\tau\bar{\tau})^{-(n-1)/2}\, (\bar{\tau}/\tau)^{e(B)/2}\, \mathrm{Tr}(V\, D(B,n)) \tag{30}$$

where $e(B)$ is the exponential sum of the generators in B [1].

For the fundamental representation [1,0], the Alexander-Conway relation (or called the Skein relation) is

$$\alpha(Ab^{2}B,n) = q^{2}(1-q^{2})\, \alpha(AbB,n) + q^{6}\, \alpha(AB,n), \quad A,B \in B_n \tag{31a}$$

and

$$\alpha(E,2) = q^{-2} + 1 + q^{2} \tag{31b}$$

This link polynomial is a little different from the Jones polynomial. For comparison we give the anology of the Jones polynomial:

$$\alpha_J(Ab^{2}B,n) = q^{2}(1-q^{2})\, \alpha_J(AbB,n) + q^{4}\, \alpha_J(AB,n), \quad A,B \in B_n \tag{32a}$$

$$\alpha_J(E,2) = q^{-1} + q \tag{32b}$$

For the IR [2,0], the Alexander-Conway relation is

$$\alpha(Ab^{2}B,n) = q^{4}(1-q^{4}+q^{6})\, \alpha(Ab^{2}B,n) + q^{12}(1-q^{2}+q^{6})\, \alpha(AbB,n)$$
$$-q^{22}\, \alpha(AB,n), \qquad A,B \in B_n \tag{33a}$$

$$\alpha(E,2) = q^{-4} + q^{-2} + 2 + q^{2} + q^{4} \tag{33b}$$

This link polynomial is different from the Akutsu-Wadati polynomial [1].

VII. Discussions

We have computed q-CG coefficients and the $\check{R}_q$ matrix for IR [2,0] of q-sl(3). Now, we are going to discuss the general properties of those for IR $[\lambda,0]$ of q-sl(3) with one row Young tableau.

The decomposition of the coproduct $[\lambda,0]\times[\lambda,0]$ in q-sl(3) is

$$[2\lambda,0]+[2\lambda-1,1]+[2\lambda-2,2]+\ldots+[\lambda,\lambda] \qquad (34)$$

The symmetry of the CG coefficients of SU(3) is

$$\xi_{[2\lambda-\mu,\mu]}=(-1)^{\mu}$$

and the difference of the casimir operators of SU(3) is

$$\eta'([2\lambda-\mu,\mu])=C_2([2\lambda,0])-C_2([2\lambda-\mu,\mu])$$
$$=2\lambda\mu-\mu^2+\mu \qquad (35)$$

Both ξ and η' happen to coincide with those in the decomposition of the coproduct $[\lambda,0]\times[\lambda,0]$ in q-sl(2), namely, their $\check{R}_q$ matrices have the same eigenvalues and satisfy the same conditions:

$$\prod_N(\check{R}_q-\xi_N q^{\eta'(N)})=0 \qquad (36)$$

Following the standard method to build the link polynomials [1,10,11], we define

$$v=\mathrm{diag}(q^{-2\lambda},q^{-2\lambda+2},q^{-2\lambda+4},\ldots,q^{0},q^{-2\lambda+4},$$
$$q^{-2\lambda+6},\ldots,q^{2},\ldots,q^{2\lambda}) \qquad (37)$$

then, we obtain

$$\tau=\left(\sum_{m=0}^{\lambda}\sum_{n=0}^{\lambda-m}q^{2n+4m}\right)^{-1}$$
$$\bar{\tau}=q^{4\lambda}\tau \qquad (38)$$

Therefore, the link polynomials (30) based on IR $[\lambda,0]$ of q-sl(3) are differrent from those on IR $[\lambda,0]$ of q-sl(2).

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Appendix. Casimir Operator of SU(3)

Define

$$T_2([\lambda_1,\lambda_2])=\mathrm{Tr}\,(I_3^2)$$

where $[\lambda_1,\lambda_2]$ is an IR of SU(3). As a representation of SU(2) it can be decomposed as a direct sum of IR's $[\mu_1,\mu_2]$ of SU(2), for which we have

$$T_2^{(0)}([\mu_1,\mu_2])=\frac{1}{12}(\mu_1-\mu_2)(\mu_1-\mu_2+1)(\mu_1-\mu_2+2)$$

Thus,

$$T_2([\lambda_1,\lambda_2])=\sum_{\mu_2=0}^{\lambda_2}\sum_{\mu_1=\lambda_2}^{\lambda_1}T_2^{(0)}([\mu_1,\mu_2])$$
$$=\frac{1}{48}(\lambda_1+2)(\lambda_2+1)(\lambda_1-\lambda_2+1)(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2+3\lambda_1)$$

Since the dimension of IR $[\lambda_1,\lambda_2]$ of SU(3) is $(\lambda_1+2)(\lambda_2+1)(\lambda_1-\lambda_2+1)/2$, we obtain (19).

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