SOLUTIONS OF THE CLASSICAL SU(2) YANG–MILLS THEORY IN 2+1 DIMENSIONS WITH THE CHERN–SIMONS TERM: ANSATZ BUILDING *

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ABSTRACT

Here I would like to show a general way of writing the gauge potentials $A^a_\mu$ for which the SU(2) Yang–Mills equations of motion can be simplified and become solvable. A number of exact solutions can be obtained from these simplified equations of motion.

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1. INTRODUCTION

The Chern–Simons term was first proposed by Deser et al. 11, Schonfeld 21, and Jackiw and Templeton 31. The inclusion of the Chern–Simons term to the 2+1 dimensional theory exhibit a number of interesting phenomena. With the presence of the Chern–Simons term, both electrically charged and neutral vortices acquire finite energy both in Abelian and non-Abelian gauge theories 41–71. However only numerical solutions are discussed and no exact solutions have yet been found. Also 2+1 dimensional theories are the high temperature limit of 4-dimensional theories 81–91 and hence have physical applications at high temperatures 101.

In this paper, I would like to report on a general way of writing the gauge potentials $A^a_\mu$, so that the SU(2) Yang–Mills equation of motion can be simplified and made solvable. Some of the solutions of these simplified equations are also being discussed.

In Sec. II of this paper, I will briefly review the SU(2) Yang–Mills theory with the Chern–Simons term with the necessary notations. Sec. III will be divided into three parts. In this section I will show the general way of writing the ansatz after which I will discuss the ansatz in spherical coordinates in part A. Cylindrical coordinates will be dealt with in part B. Here some new exact solutions both in the Minkowski and Euclidean space will be discussed. These solutions are complex, possessed zero energy and have finite action. Part C will be on rectilinear coordinate and both Euclidean and Minkowski space solutions can be obtained from the simplified equations of motion. New exact Minkowski space solutions are also obtained. These complex solutions are wave-like in nature. I will end with some comments in Sec. IV.

2. NOTATIONS

The SU(2) Yang–Mills equations of motion with the Chern–Simons term are

$$\partial_\mu F^a_{\mu\nu} + \epsilon^{abc} A^b_\mu A^c_{\mu\nu} + \frac{1}{2} \epsilon_{\mu\nu\rho} F^{a\mu\nu} = 0$$ (1a)

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu$$ (1b)

The Chern–Simons constant $\xi$ is real in Minkowski space and it is replaced by $-i\xi$ in Euclidean space. The group indices $a, b, c = 1, 2$ and 3. The space indices $\mu, \nu, \alpha = 0, 1, 2$ and 3 in Euclidean space and $\mu, \nu, \alpha = 0, 1, 2$ in Minkowski space. The gauge coupling constant $g = 1$. The nonabelian electric and magnetic field are given respectively by

$$E^a_\mu = F^a_{\mu\nu}$$ (2a)

and

$$B^a_\mu = -\frac{1}{2} \epsilon_{\mu\nu} F^{a\nu}\,; \, i, j = 1, 2$$ (2b)
The energy-momentum tensor is written as

\[ T^{\mu \nu} = F^{\mu \sigma} F_{\sigma}^{\nu} + g^{\mu \nu} L_{YM} \]  

where the Yang–Mills Lagrangian density is

\[ L_{YM} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \]  

The Chern–Simons Lagrangian density is

\[ L_{CS} = -\frac{1}{2} \xi \epsilon^{\mu \nu \rho} (A^a_{\mu} \delta_{\nu} A^a_{\rho} + \frac{1}{3} \epsilon^{abc} A^a_{\mu} A^b_{\nu} A^c_{\rho}) \]  

and the Yang–Mills action with the Chern–Simons term is

\[ S = \int d^3x (L_{YM} + L_{CS}) \]  

Hence the energy density \( T^{00} \) is given by

\[ T^{00} = \frac{1}{2} (E^a E^a + B^a B^a) \]  

3. THE GENERAL ANSATZ

By writing the Euclidean space gauge potentials as

\[ A^a_{\mu} = (\hat{\sigma}^a \psi_1 + \hat{\tau}^a \psi_2) \hat{\partial}_\mu + (\hat{\sigma}^a A_1 + \hat{\tau}^a A_2) \eta_\mu + \hat{\beta}^a \hat{\beta}_\mu \chi \]  

where \( \hat{\sigma}^a, \hat{\tau}^a \) and \( \hat{\beta}^a \) are the same as the orthogonal unit vectors \( \hat{\sigma}_\mu, \hat{\tau}_\mu \) and \( \hat{\beta}_\mu \), the equations of motion (1) can be greatly simplified. In the Minkowski space, the gauge potentials take the form,

\[ A^a_{\mu} = (\hat{\sigma}^a \psi_1 + \hat{\tau}^a \psi_2) \hat{\partial}_\mu + \hat{\beta}^a \hat{\beta}_\mu \chi \]  

(9a)

\[ A^a_{\mu} = (\hat{\sigma}^a A_1 + \hat{\tau}^a A_2), \quad \hat{\beta}^a = \hat{\partial}^a = 0 \]  

(9b)

Here \( \hat{\sigma}^a, \hat{\tau}^a \) and \( \hat{\beta}_\mu \) are similar orthogonal unit vectors and \( i = 1, 2 \).

One restriction on the ansatz (8) is that the corresponding variable of the unit vector \( \hat{\beta}_\mu \) must have the dimension of length.

(A) Spherical Coordinate

In spherical coordinate, the Euclidean space gauge potentials take the form,

\[ A^a_{\mu} = (\hat{\sigma}^a \psi_1 + \hat{\tau}^a \psi_2) \hat{\partial}_\mu + (\hat{\sigma}^a A_1 + \hat{\tau}^a A_2) \hat{\partial}_\mu + \hat{\beta}^a \hat{\beta}_\mu \chi \]  

(10)

where

\[ \hat{\beta}_\mu = \frac{x^a \hat{\beta}_\mu}{r}, \quad \chi = \frac{(z_1^2 + z_2^2)^{1/2}}{r} \]  

(11a)

\[ \hat{\beta}_\mu = \frac{x^a \delta_\mu^a - r^2 \hat{\beta}_\mu}{r^3}, \quad r = (z_1^2 + z_2^2 + z_3^2)^{1/2} \]  

(11b)

and

\[ \hat{\beta}_\mu = \frac{x^a \hat{\beta}_\mu}{r} \]  

(11c)

by writing

\[ \psi_1 = \frac{1}{r} \Phi_1 (r), \quad \psi_2 = \frac{1}{r} \Phi_2 (r) \]  

(12a)

\[ A_1 = -\frac{1}{r} (\Phi_1 (r) - 1), \quad A_2 = \frac{1}{r} \Phi_2 (r) \]  

(12b)

and

\[ \chi = \frac{1}{r} \Phi_2 (r) \]  

(12c)

the Yang–Mills equations of motion (1) reduce to

\[ ( - \Phi_1' - A \Phi_2)' - A (\Phi_2' - A \Phi_1) - \frac{1}{r^2} \Phi_1 (1 - \Phi_1^2 - \Phi_2^2 - \frac{1}{2} \xi (1 - \Phi_1^2 - \Phi_2^2) \]  

(13a)

\[ ( - \Phi_2' + A \Phi_1)' + A (\Phi_1' + A \Phi_2) - \frac{1}{r^2} \Phi_2 (1 - \Phi_1^2 - \Phi_2^2) - \frac{1}{2} \xi (1 - \Phi_1^2 - \Phi_2^2) = 0 \]  

(13b)

\[ ( - \Phi_1' + A \Phi_2)' + A (\Phi_2' - A \Phi_1) - \frac{1}{r^2} \Phi_1 (1 - \Phi_1^2 - \Phi_2^2) + \frac{1}{2} \xi (1 - \Phi_1^2 - \Phi_2^2) = 0 \]  

(13c)

Putting \( \Phi_1 = \Phi \) and \( \Phi_2 = 0 \) \( \dagger \), Eqs. (13) reduce to

\[ A = \frac{-r}{2} (1 - \frac{1}{\xi r^2}) \]  

(14a)

\[ \Phi_1 + \frac{1}{4} \xi^2 (1 - \Phi_1^2) + \frac{\Phi_1}{r^2} (1 - \Phi_2^2) = 0 \]  

(14b)

Here prime means \( \hat{\partial}_r \).

D'Hoker and Vine \( \dagger \) had solved for Eqs. (14). However only numerical solutions are being discussed. This numerical solution is complex. So far no exact solution has been found for the differential equations of (14).

(B) Cylindrical Coordinate

Here the Euclidean space gauge potentials become

\[ A^a_{\mu} = (\hat{\sigma}^a \psi_1 + \hat{\tau}^a \psi_2) \hat{\partial}_\mu + (\hat{\sigma}^a A_1 + \hat{\tau}^a A_2) \hat{\partial}_\mu + \hat{\beta}^a \hat{\beta}_\mu \chi \]  

(15)
where \( \dot{\rho}_1 = \frac{a(t)}{a} \) and \( \psi_1, \psi_2, A_1 \) and \( A_2 \) are functions of \( \rho \) and \( z_3 \). \( \chi \) is a function of \( \rho \) only.

Substituting Eq.(15) into Eq.(1), the Yang–Mills equations of motion becomes, after setting,

\[
\psi_1 = \psi, \quad A_1 = A \quad (16a)
\]

\[
\psi_2 = i\psi + \frac{1}{\rho}, \quad A_2 = iA, \quad (16b)
\]

and replacing \( \chi \) by \( i\chi \),

\[
\nabla^2 A - 2 \chi A' - A \chi' - \frac{1}{\rho} \chi A + \chi^2 A + i\xi(\psi' + \frac{1}{\rho} \psi - \chi \psi) = 0 \quad (17a)
\]

\[
\nabla^2 \psi + \partial_1^2 \psi - \frac{1}{\rho^2} \psi - 2 \chi \psi' - \psi' \chi' - \frac{1}{\rho} \psi + \chi^2 \psi - i\xi(A' - \chi A) = 0 \quad (17b)
\]

\[
\partial_3 (\chi' - \chi A) = -i\xi \partial_3 \psi. \quad (17c)
\]

Here prime means \( \frac{d}{d \rho} \) and \( \nabla^2 \) is \( \frac{1}{\rho^2} \frac{d}{d \rho} (\rho \frac{d}{d \rho}) \). By letting the integration constant be zero, Eq.(17c) becomes,

\[
A' = \chi A - i\xi \psi \quad (18)
\]

Eq.(18) solves Eq.(17a) exactly. To simplify Eq.(17b), I write

\[
\psi(\rho, z_3) = e^{-\xi_3 \rho} \psi(\rho) \quad (19)
\]

and Eq.(17b) reduced to,

\[
\psi'' + \psi' - \frac{1}{\rho^2} \psi - 2 \chi \psi' - \psi' \chi' - \frac{1}{\rho} \psi + \chi^2 \psi = 0 \quad (20)
\]

As a result of Eq.(19), the function \( A \) becomes,

\[
A(\rho, z_3) = e^{-\xi_3 \rho} R(\rho) \quad (21)
\]

and Eq.(18) then reduced to,

\[
R' = \chi R - i\xi \psi \quad (22a)
\]

At this point, it is clear that the function \( \chi(\rho) \) is a completely arbitrary function and Eq.(20) can be solved in a linear fashion. One simple solution is

\[
\psi' = (\chi + \frac{1}{\rho}) \psi \quad (22b)
\]

Hence together with Eq.(22a), Eq.(19) and Eq.(21), the explicit solution is

\[
\psi = i\xi_3 \rho \exp \left( \int \chi d\rho - \xi_3 \right) \quad (23a)
\]

\[
\]

With the solutions of Eq.(23) the Yang–Mills action \( S_0 = \int d^4x \mathcal{L}_{YM} \) is zero. Hence the action \( S \) is just the Chern–Simons action,

\[
S = S_0 + \int d^4x \mathcal{L}_{CS}
\]

\[
= \frac{-1}{2} \xi_3 \chi \exp \left( \int \chi d\rho - \xi_3 \right) \quad (24)
\]

When \( \chi \) is just a negative non–zero constant, that is \( \chi = -b \), the solution of Eq.(23) has finite action,

\[
S = \frac{-\xi_3 \chi}{b^2} \pi \quad (25)
\]

when \( \chi_3 > 0 \). The solution has zero energy and momentum density.

In Minkowski space, the gauge potentials take the form

\[
A^a = (\phi^a_1 \psi_1 + \xi_3 \psi_2 + \xi_3 \phi^a_2) \phi_1 + \xi_3 \phi^a_1 \phi_2, \quad (26a)
\]

\[
A^a = \phi^a_1 \phi_1 + \xi_3 \phi^a_2 \phi_2 \quad (26b)
\]

From Eq.(1) the simplified equations of motion can be solved in a similar manner and the solution obtained is

\[
\psi = a_3 \rho \sin(\xi_3 \rho) \exp(\int \chi d\rho) \quad (27a)
\]

\[
A = (a_3 + \frac{1}{2} \xi_3 a_3^2) \sin(\xi_3 \rho) \exp(\int \chi d\rho) \quad (27b)
\]

where \( \psi \) and \( A \) are related to \( \psi_1, \psi_2, A_1 \) and \( A_2 \) by Eq.(16) and \( \chi \) being replaced by \( i\chi \).

A second cylindrical ansatz can also be written,

\[
A^a = (\phi^a_1 \psi_1 + \xi_3 \psi_2) \phi_1 + (\phi^a_2 \phi_1 + \xi_3 \phi^a_1) \phi_2 + \xi_3 \phi_1 \quad (28)
\]

By writing,

\[
A_1 = i\psi, \quad A_2 = -\psi, \quad (29a)
\]

\[
\psi_1 = \psi, \quad \psi_2 = i\psi, \quad (29b)
\]

\[
\chi = i\xi(n-1) \text{ or } i\xi n \quad (29c)
\]

\( n \) being a constant and solving the equations of motion (1), this Euclidean space solution is found to be

\[
\psi = \frac{a}{\rho^2} e^{-i\xi \phi}, \quad a = \text{constant} \quad (30)
\]

The Minkowski space version of this solution (30) can be similarly found by using the same substitution as Eq.(29a,b) and \( \chi = -n^2 \text{ or } \xi(-n+i) \). \( \psi \) is again given by Eq.(30).
The rectilinear coordinate ansatz first appeared in Refs. 12 and 13. As in Ref. 12,

\[ A_\mu = (\delta_\mu^1 \psi_1 + \delta_\mu^2 \psi_2) \delta_{\alpha_2} + (\delta_\mu^3 A_1 + \delta_\mu^4 A_2) \delta_{\alpha_3} + \delta_\mu^6 b_1 \chi \]  

(31)

and by setting,

\[ \psi = \psi_2 = -\psi_3 = -\frac{i\sqrt{2}}{4} \chi \]  

(32 a)

\[ A(x_2) = A_2 = A_3 \]  

(32 b)

\[ \chi = \sqrt{2} A(x_2) \]  

(32 c)

the Euclidean space Yang-Mills equation of motion (1) is just

\[ \partial_j^2 A(x_2) - \frac{1}{4} \epsilon^2 A(x_2) - 2 A^3 (x_2) = 0 \]  

(33)

Eq. (33) is just the differential equations of the Jacobi elliptic functions. The detailed analysis of the Minkowski as well as the Euclidean space solutions have already been done in Ref. 12.

Time dependent Minkowski space solution can also be obtained when,

\[ \psi_2 = A_2 = \psi(x_2 - x_0) \]  

(34 a)

\[ \psi_3 = A_3 = -\psi(x_2 - x_0) \]  

(34 b)

\[ \chi = i \xi \]  

(34 c)

The gauge potentials then become,

\[ A_\mu = (\delta_\mu^1 - i\delta_\mu^2) (\delta_\mu^3 + \delta_\mu^4) \psi(x_2 - x_0) + i\delta_\mu^6 b_1 \chi \]  

(35)

Here \( \psi \) is a completely arbitrary function of \( x_2 - x_0 \). The electric and magnetic field are given by,

\[ E_\mu = \xi (\delta_\mu^1 - i\delta_\mu^2) \delta_{\alpha_1} \psi(x_2 - x_0) \]  

(36 a)

\[ B_\mu = \xi (\delta_\mu^3 - i\delta_\mu^4) \psi(x_2 - x_0) \]  

(36 b)

The energy density, momentum density and action all vanish.

4. COMMENTS

(1) The Bessel functions solutions of Ref. 12 can be obtained from the cylindrical coordinate ansatz of Eqs. (15) and (26) in the Euclidean and Minkowski space respectively when the functions \( A_2 = \chi = 0 \) and \( \psi_3 = \frac{1}{\rho} \).

(2) In arbitrary \( \chi \) function solutions, the equations of motion (17) are non-linear although they can be solved in a linear way. As a result, a number of explicit solutions can be obtained by using the ansatz of Eqs. (15) and (26), other than the solutions given by Eqs. (23) and (27) in the Euclidean and Minkowski space respectively.

Examples of the other solutions are when Eq. (20) can be solved by writing,

\[ \Psi'(\rho) = \left( \frac{1}{\rho} \right) \Psi(\rho) + i\xi \rho H(\rho) \]  

(37)

When \( m = 0 \), the solution is given by

\[ \psi = \frac{i\xi_1}{\rho} \exp \left( \int \chi d\rho - \xi z_3 \right) \]  

(38 a)

\[ A = (\xi_1 + \xi_1 \rho \theta) \exp \left( \int \chi d\rho - \xi z_3 \right) \]  

(38 b)

When \( m = 1 \), the solution becomes more complicated, as the solution now is

\[ A = \frac{\psi}{(1 + y^2)^{1/2} \exp \left( \int \chi d\rho - \xi z_3 \right) \sqrt{2} \xi z_3} \]  

(39 a)

\[ \psi = \frac{1}{(1 + y^2)^{1/2} \exp \left( \int \chi d\rho - \xi z_3 \right) \sqrt{2} \xi z_3} \]  

(39 b)

where \( y = \frac{a}{a + b} \chi \), \( a, b \) = constants and \( K_0, k_1, L_0 \) and \( L_1 \) are the modified Bessel functions. When \( m \) takes other values, different solutions will be obtained.

The Minkowski space version of solutions (38) and (39) can be obtained in a similar manner with some slight modification.

(3) In contrast to the ansatz of Eq. (15) for which many solutions can be obtained, the second cylindrical ansatz of Eq. (28) has limited solution.

(4) Since the electric and magnetic fields of the ansatz (15) and (16) point in the direction of the null vector \( (\hat{\rho}^0 + \hat{\psi}^1) \) in group space, the energy and momentum density have to be zero. Although the Yang-Mills action is zero, the total action can be a finite quantity by suitable choice of \( \chi \) and for \( x_3 > 0 \). Hence the solutions obtained do not correspond to any physical states. At most they may represent processes when \( x_3 \) runs from zero to infinity.

(5) Similarly, the electric and magnetic fields of the gauge potentials of Eq. (35) are in the direction of null vector \( (\hat{\rho}^0 - \hat{\psi}^1) \) in group space and the energy, momentum density, and action are zero as of result of this group space null vector.
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