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The Extended Local Gauge Invariance and the BRS Symmetry in Stochastic Quantization of Gauge Fields

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Abstract

We investigate the BRS invariance of the first-class constrained systems in the context of the stochastic quantization. For the first-class constrained systems, we construct the nilpotent BRS transformation and the BRS invariant stochastic effective action based on the D+1 dimensional field theoretical formulation of stochastic quantization. By eliminating the multiplier field of the gauge fixing condition and an auxiliary field, it is shown that there exists a truncated BRS transformation which satisfies the nilpotency condition. The truncated BRS invariant stochastic action is also derived. As the examples of the general formulation, we investigate the BRS invariant structure in the massless and massive Yang-Mills fields in stochastic quantization.

1. Introduction

Stochastic Quantization (in short, SQ)[1] was first introduced by Parisi and Wu as an interesting alternative quantization method. It has two advantages. One is the manifest Lorentz covariance and the other is that the explicit gauge fixing procedure is not necessary for the quantization of gauge fields. The manifest Lorentz covariance in SQ is expected to provide a new approach to the quantization method of the second-class constrained systems. On the other hand, for the first-class constrained systems, SQ of gauge fields has been investigated in connection with the ordinary quantization method in which the well-known Faddeev-Popov determinant factor[2] appears.

SQ is originally formulated by the Langevin equation coupled with the white noise which is appropriate for the numerical calculation, while not so suitable for the investigation of the symmetry properties of the system (for example, renormalizability, unitarity, spontaneous symmetry breakdown, etc)[3] . To investigate these symmetry properties, SQ is reformulated by an action principle, that is, an effective action is introduced[4] which is directly connected with the Fokker-Planck equation for the probability distribution functional. Based on the stochastic action, SQ is interpreted as a D+1 dimensional field theory which also enables us to introduce a canonical formulation in SQ by regarding the fctitious time as the time coordinate in the canonical formulation. Furthermore, recently the BRS invariant formulation is introduced for SQ of Yang-Mills field and the Ward-Takahashi identity in SQ is investigated[5][6] . The main advantage of this approach is that the field theoretical tools developed in the ordinary quantization method are easily applied for SQ.

The purpose of this paper is to construct the BRS invariant formulation of SQ, that is, we formulate the BRS transformation which satisfies the nilpotency condition and the BRS invariant stochastic action for the general first-class constrained systems. We clarify the BRS structure in SQ based on the canonical formulation of SQ which is powerful to investigate the algebraic structure of the first-class constrained systems.

The paper is organized as follows. In section 2, we first briefly review the construction of the stochastic action and the canonical formulation of SQ. Secondly, we show that there exist a conserved charge in the sense of the fictitious time development provided that the classical action is invariant under the fictitious time independent local gauge transformation. We also clarifies the condition of the first-class constraint by using the conserved charge. In section 3, we show that there exists an extended local gauge symmetry which is dependent on the fictitious time by constructing the invariant stochastic action. In section 4, we formulate the nilpotent BRS transformation and the BRS invariant stochastic action. We show that there are two type of the nilpotent BRS transformations. One is the extended BRS transformation which is a generalization of the well-known BRS transformation[7][8] in the ordinary quantization method. The other is the truncated BRS transformation. By eliminating the auxiliary field which is introduced as a multiplier field for the constraint and the multiplier field for the gauge fixing condition in the extended BRS transformation, we show that there remains the truncated BRS transformation which also satisfies the nilpotency condition. The section 5 and 6 are devoted to the application of the general formulation of the first-class constrained systems. As the examples, we consider the massless Yang-Mills field in section 5 and the massive Yang-Mills field in section 6. The section 7 is devoted to discussions. In the appendix, we show that the ordinary distribution functional in the path-integral formulation including the Faddeev-Popov determinant factor is obtained as the equilibrium distribution functional of the Fokker-Planck equation.

2. The Basic Concepts

We first give a general description of the first-class constrained systems by introducing a canonical formulation in SQ.

We start from a Langevin equation,

$$\begin{aligned} \dot{q}^A &= -\frac{\partial S_{cl}}{\partial q^A} + \eta^A, \\ \left\langle \eta^A(t)\eta^B(t') \right\rangle &= 2\delta^{AB}\delta(t-t'). \end{aligned} \tag{2.1}$$

which describes a quantum mechanical system. S_{cl} is the classical Euclidean action of the system. The dot ($\cdot$) denotes the derivative with respect to the fictitious time, t. The capital Roman subscript denotes the whole set of indices (including the spacetime coordinates in the case of a quantum field theoretical system). The expectation value of an observable $O(q)$ is given by

$$\begin{aligned}\langle O(q)\rangle &\equiv \lim_{t\to\infty}\langle O(q_\eta)\rangle, \\ &= \int \mathcal{D}\eta O(q_\eta)e^{-\frac{1}{4}\int dt\eta^A\eta^A},\end{aligned} \tag{2.2}$$

where q_η^A is the solution of the Langevin equation (2.1).

An effective stochastic action is defined from the Langevin equation (2.1) by inserting the unity,

$$1 = \int \mathcal{D}q\delta(\dot{q}^A + \frac{\partial S_{cl}}{\partial q^A} - \eta^A)\det(\delta^{BC}\partial_t + \frac{\partial^2 S_{cl}}{\partial q^B\partial q^C}), \tag{2.3}$$

into the following generating functional of the white noise.

$$\begin{aligned}Z &\equiv \int \mathcal{D}\eta e^{-\frac{1}{4}\int dt\eta^A\eta^A}, \\ &= \int \mathcal{D}\eta\mathcal{D}q\delta(\dot{q}^A + \frac{\partial S_{cl}}{\partial q^A} - \eta^A)\det(\delta^{BC}\partial_t + \frac{\partial^2 S_{cl}}{\partial q^B\partial q^C})e^{-\frac{1}{4}\int dt\eta^A\eta^A}, \\ &= \int \mathcal{D}p\mathcal{D}q\det(\delta^{AB}\partial_t + \frac{\partial^2 S_{cl}}{\partial q^A\partial q^B})e^{\int K_0 dt},\end{aligned} \tag{2.4}$$

where

$$K_0 \equiv -p_A p_A + ip_A(\dot{q}^A + \frac{\partial S_{cl}}{\partial q^A}). \tag{2.5}$$

In the functional Z in (2.4), we have omitted the source term of the white noise. The auxiliary field, p_A, is introduced as an integration variable to represent the δ-functional in (2.3).

In order to investigate the algebraic structure of the constrained systems, it is useful to introduce a canonical formulation in SQ. By regarding (2.5) as a Legendre transformation $(q^A, \dot{q}^A) \rightarrow (q^A, p_A)$, we obtain a hamiltonian formulation[8].

$$
\begin{aligned}
K_0 &\equiv i p_A \dot{q}^A - H_0, \\
H_0 &= p_A p_A - i p_A \frac{\partial S_{cl}}{\partial q^A},
\end{aligned}
\tag{2.6}
$$

with

$$
[p_A, q^B] = -i\delta_A^B. \tag{2.7}
$$

The equations of motion are given by

$$
\begin{aligned}
\dot{q}^A &= [H_0, q^A] = -2ip_A - \frac{\partial S_{cl}}{\partial q^A}, \\
\dot{p}_A &= [H_0, p_A] = \frac{\partial^2 S_{cl}}{\partial q^A \partial q^B} p_B,
\end{aligned}
\tag{2.8}
$$

which are equivalent to the variation of the stochastic action (2.5).

For the case of non-constrained systems, it is well-known that the determinant factor in (2.4) can also be brought into the effective stochastic action by using Grassmannian variables $\bar{C}_A$ and C^A,

$$
K_{eff} = K_0 + \bar{C}_A(\dot{C}^A + \frac{\partial^2 S_{cl}}{\partial q^A \partial q^B} C^B). \tag{2.9}
$$

The effective stochastic action (2.9) has the supersymmetry δ defined by

$$
\begin{aligned}
\delta q^A &= i\epsilon C^A, \\
\delta C^A &= 0, \\
\delta \bar{C}_A &= \epsilon p_A, \\
\delta p_A &= 0.
\end{aligned}
\tag{2.10}
$$

which, first discussed in connection with the mechanism of the dimensional reduction[9], *comes from the fact that the Langevin equation (2.1) is coupled with the white noise*. On the other hand, in the first-class constrained systems, we will show in the following three sections, that there exists a BRS-type supersymmetry instead of (2.10).

Now let us consider constrained systems, that is, the case where the classical action S_{cl} is invariant under the infinitesimal local gauge transformation

$$\delta q^A \equiv E^A_a \alpha^a, \tag{2.11}$$

where α^a is a transformation parameter which is assumed to be independent of the fictitious time. E^A_a is a function of q^B. The invariance of the classical action is expressed as

$$\delta S_{cl} = \frac{\partial S_{cl}}{\partial q^A} E^A_a \alpha^a = 0. \tag{2.12}$$

Here we define the local gauge transformation of p_A

$$\delta p_A = -\frac{\partial E^B_b}{\partial q^A} \alpha^b p_B. \tag{2.13}$$

The transformation property of p_A is defined to be equivalent to that of $\frac{\partial S_{cl}}{\partial q^A}$ from the commutation relation (2.7), that is, $p_A = -i\frac{\partial}{\partial q^A}$. The commutation relation (2.7) is invariant under the transformation (2.11) and (2.13).

The stochastic action K_{inv} corresponding to K_0 in (2.5) is defined so as to be invariant under the transformation (2.11) and (2.13) by introducing a metric G^{AB} as follows ;

$$K_{inv} \equiv i p_A \dot{q}^A - p_A G^{AB} (p_B - i \frac{\partial S_{cl}}{\partial q^B}), \tag{2.14}$$

where the transformation of the metric G^{AB} is defined by

$$\begin{aligned} \delta G^{AB} &\equiv \frac{\partial G^{AB}}{\partial q^C} E^C_c \alpha^c, \\ &= G^{AC} \frac{\partial E^B_a}{\partial q^C} \alpha^a + G^{CB} \frac{\partial E^A_a}{\partial q^C} \alpha^a. \end{aligned} \tag{2.15}$$

(2.15) implies that the metric G^{AB} must satisfy the equation

$$-\frac{\partial G^{AB}}{\partial q^C} E^C_a + G^{AC} \frac{\partial E^B_b}{\partial q^C} + G^{CB} \frac{\partial E^A_a}{\partial q^C} = 0. \tag{2.16}$$

This is interpreted as follows. Let us consider the configuration space, that is a manifold $\{q^A, G_{AB}\}$, which is specified by the coordinates q^A and the metric G_{AB} (

G_{AB} is the inverse matrix of G^{AB}). The equation (2.16) implies that the functional E_a^A is the Killing vectors on the manifold. As we will show in the following section, the metric G^{AB} is trivial in the case of massless Yang-Mills field. On the other hand, for the massive Yang-Mills field, we will construct the metric G^{AB} from the variable q^A explicitly and show its transformation property (2.15).

Since the transformation parameter α^a in (2.11) is independent of the fictitious time, there exists the conserved charge,

$$Q(\alpha) \equiv i(\delta q^A)p_A = iE_a^A p_A \alpha^a, \tag{2.17}$$

which generates the transformations (2.11), (2.13) and (2.15),

$$\begin{aligned} \delta q^A &= [Q(\alpha), q^A], \\ \delta p_A &= [Q(\alpha), p_A], \end{aligned} \tag{2.18}$$

by the canonical commutation relation (2.7). From (2.15) and (2.16), it is clear that the charge consistently generates the transformation of the metric G^{AB}

$$\delta G^{AB} = [Q(\alpha), G^{AB}]. \tag{2.19}$$

It is also easy to check explicitly that the fictitious time development is generated by the hamiltonian which is defined from the invariant stochastic action (2.14),

$$H_{inv} \equiv \frac{1}{\sqrt{G}} p_A \sqrt{G} G^{AB} (p_B - i\frac{\partial S_{cl}}{\partial q^B}), \tag{2.20}$$

where $G \equiv det G_{AB}$. We note that the operator ordering is important (p_A is identified to the differential operator $p_A = -i\frac{\partial}{\partial q^A}$), that is, (2.20) implies that the derivative with respect to q^A is the covariant derivative on the manifold $\{q^A, G_{AB}\}$.

The equations of motion,

$$\begin{aligned} \dot{q}^A &= [H_{inv}, q^A], \\ \dot{p}_A &= [H_{inv}, p_A], \end{aligned} \tag{2.21}$$

which are equivalent to the variation of the stochastic action (2.14) due to the "

Killing vector equation " (2.16.), justifies the conservation of the charge

$$\dot{Q}(\alpha) = [H_{inv}, Q(\alpha)] = 0, \tag{2.22}$$

in the present canonical formulation of SQ.

The situation which we have explained above corresponds to the $A_0 = 0$ gauge case in the ordinary canonical quantization of Yang-Mills field. In that case, there remains a residual local gauge symmetry associated with the transformation parameter which is independent of the (true) time coordinates, and the generators of the residual gauge symmetry have a closed algebra. In the analogy of the $A_0 = 0$ gauge case in the ordinary quantization method, we here restrict ourselves to the case in which the generators (2.17) have the closed algebra

$$[Q_a, Q_b] = u_{ab}^c Q_c. \tag{2.23}$$

where u_{ab}^c is the " structure constant ". We assume that the structure constant u_{ab}^c is independent of q^A and p_A. Note that the little Roman subscript also includes the spacetime coordinates in the case of a quantum field theoretical system. The closed algebra (2.23) is equivalent to the condition

$$-\frac{\partial E_a^A}{\partial q^B} E_b^B + \frac{\partial E_b^A}{\partial q^B} E_a^B = u_{ab}^c E_c^A, \tag{2.24}$$

on the function E_a^A in (2.11). We call this type of the constraint (2.23) or the condition (2.24) as the first-class constraint in SQ. We will see, for the case of the massive Yang-Mills field, that a nonlinear function E_a^A appears satisfying the condition (2.24).

Here we comment on the improvement of the stochastic action (2.14). We introduce the metric G_{AB} and G^{AB} by the invariance principle of the stochastic action based on the canonical formulation of SQ. The Fokker-Planck equation which is obtained from the stochastic hamiltonian (2.20)

$$\dot{\mathcal{P}} = -H_{inv}(p_A \rightarrow -i\frac{\partial}{\partial q^A})\mathcal{P}, \tag{2.25}$$

is manifestly invariant under the fictitious time independent local gauge transformation (2.11). A similar improvement of the Fokker-Planck equation and the corresponding Langevin equation are discussed in connection with the general covariance of the Fokker-Planck equation under the general coordinate transformation $q^{A'} \rightarrow f^{A'}(q)$[10-14]. In these cases, however, the conditions (2.16) and (2.24) which are essential for the first-class constrained systems in SQ have never clarified.

In this section, we define the first-class constrained system in the context of stochastic quantization. In the ordinary quantization method, the constraints arise from the definition of the canonical momentum variables and the consistency condition of the constraints. On the other hand, in SQ, since the canonical momentum p_A is introduced artificially in the canonical formulation , the closure of the constraint algebra (2.23) is assumed as the condition of the first-class constraints. This is the characteristic feature of the constrained systems in SQ.

To summarize, for the constrained system which is invariant under the local gauge transformation (2.11) in which the functional E^A_a satisfies the condition (2.24), we always construct a local gauge invariant stochastic action (2.14) by solving the " Killing vector equation " (2.16) with respect to the metric G^{AB}. In the following section, we show that there exists an extended local gauge symmetry in the first-class constrained systems by adding the conserved charge (2.17) to the stochastic action (2.14).

3. The Extended Local Gauge Symmetry in SQ

Following to the basic formulation of the first-class constrained systems in the previous section, we construct an extended stochastic action which has an extended local gauge symmetry with the fictitious time dependent gauge transformation parameter by introducing a multiplier field ϕ^a of the constraint.

We define an extended stochastic action

$$K \equiv K_{inv} - Q(\phi), \tag{3.1}$$

where K_{inv} is defined in (2.). $Q(\phi)$ is obtained by changing the gauge parameter α^a to the auxiliary field ϕ^a in (2.16). We consider the gauge transformation (2.11), (2.13) and (2.15) with the fictitious time dependent gauge transformation parameter $\alpha^a = \alpha^a(t)$. The variation of the extended stochastic action (3.1) is

$$\begin{aligned} \delta K &= \delta(ip_A\dot{q}^A) - [Q(\alpha), Q(\phi)] - Q(\delta\phi), \\ &= Q(\dot{\alpha}) - Q([\alpha \times \phi]) - Q(\delta\phi), \\ &= Q(\dot{\alpha} - [\alpha \times \phi] - \delta\phi), \end{aligned} \tag{3.2}$$

where

$$[\alpha \times \phi]^a \equiv u^a_{bc}\alpha^b\phi^c. \tag{3.3}$$

From (3.2), we find that the extended stochastic action (3.1) is invariant under the fictitious time dependent local gauge transformation provided that the auxiliary field ϕ^a is transformed as

$$\delta\phi^a = \dot{\alpha}^a - [\alpha \times \phi]^a. \tag{3.4}$$

We note here that the only necessary condition for the existence of this extended local gauge symmetry is the closure of the algebra (2.23) for the conserved charge or equivalently the condition (2.24).

In the appendix, We show that, under the special choice of the auxiliary field ϕ^a, the following Fokker-Planck equation derived from the stochastic action (3.1)

$$\dot{P} = \frac{1}{\sqrt{G}}\frac{\partial}{\partial q^A}\{\sqrt{G}G^{AB}(\frac{\partial}{\partial q^B} + \frac{\partial S_{cl}}{\partial q^B}) - \sqrt{G}E_a^A\phi^a\}P. \tag{3.5}$$

where ∂_A denotes the derivative with respect to the variable q^A, gives the well-known distribution functional including the Faddeev-Popov determinant factor as the equilibrium distribution functional.

4. BRS Invariance in the Extended Stochastic Action and the Consistent Truncation of the Auxiliary Fields

In this section, we formulate an extended BRS invariant stochastic action based on the extended local gauge invariance in the previous section. We also consider a consistent truncation of the auxiliary fields of the BRS transformation.

The extended BRS transformation is constructed from the extended local gauge transformation by replacing

$$\alpha^a \rightarrow i\epsilon C^a, \tag{4.1}$$

where ϵ is a Grassmannian constant. C^a is the Faddeev- Popov ghost field. The extended BRS transformation is given by

$$\begin{aligned}
\tilde{\delta}_{BRS}\, q^A &= i\epsilon E_a^A C^a, \\
\tilde{\delta}_{BRS}\, p_A &= -i\epsilon \frac{\partial E_b^B}{\partial q^A} p_B C^b \\
\tilde{\delta}_{BRS}\, \phi^a &= i\epsilon(\dot{C}^a - [C \times \phi]^a), \\
\tilde{\delta}_{BRS}\, C^a &= -\frac{i\epsilon}{2}[C \times C]^a, \\
\tilde{\delta}_{BRS}\, \bar{C}_a &= \epsilon B_a, \\
\tilde{\delta}_{BRS}\, B_a &= 0.
\end{aligned} \tag{4.2}$$

We note that the BRS transformation of the metric G^{AB}, corresponding to (2.15), is induced by the BRS transformation of q^A since the metric G^{AB} is not a fundamental variable but the solution of the Killing vector equation (2.16) as a function of q^A. It is easy to show the nilpotency of the extended BRS transformation (4.2) by using the relation (2.24) and the Jacobi identity of the generators in (2.17) and (2.23).

Since the extended BRS transformation (4.2) is a D+1 dimensional generalization of the ordinary BRS transformation[7] [8], except that the canonical momentum variable p_A is included, the BRS invariant stochastic action is constructed as follows. Let F^a be a gauge fixing function which is a functional of the variables q^A, p_A, ϕ^a and B_a. The BRS invariant gauge fixing term and Faddeev-Popov ghost term is given by

$$\tilde{K}_{GF+FP} \equiv \tilde{\delta}'_{BRS}(i\bar{C}_a F^a), \tag{4.3}$$

where $\tilde{\delta}'_{BRS}$ is defined by $\tilde{\delta}_{BRS} \equiv \epsilon\tilde{\delta}'_{BRS}$. We obtain the extended BRS invariant stochastic action

$$\tilde{K}_{BRS} = K + \tilde{K}_{GF+FP}. \tag{4.4}$$

From the practical point of view, since (4.2) includes many auxiliary fields, $\tilde{K}_{BRS}$ is rather complex than the BRS invariant action in the ordinary quantization method. Here we consider a consistent truncation of the auxiliary field ϕ^a and the multiplier field B_a in (4.2) to remain a nilpotent BRS transformation. To do this, we consider a special choice of the gauge fixing function

$$F^a = \phi^a - \chi^a. \tag{4.5}$$

χ^a is a function of only q^A. $\tilde{K}_{GF+FP}$ is reduced to

$$\begin{aligned}\tilde{K}_{GF+FP} &\equiv \tilde{\delta}'_{BRS}(i\bar{C}_a(\phi^a - \chi^a)),\\ &= iB_a(\phi^a - \chi^a) + \bar{C}_a(\dot{C}^a - [C \times \phi]^a - \frac{\partial\chi^a}{\partial q^A}E^A_b C^b).\end{aligned} \tag{4.6}$$

We use the equations of motion for ϕ^a and B_a for the consistent truncation of the extended BRS transformation.

$$\begin{aligned} B_a + i\bar{C}_b C^c u^b_{ca} - E^A_a p_A &= 0, \\ \phi^a - \chi^a &= 0. \end{aligned} \tag{4.7}$$

Substituting the equations of motion into the BRS transformation (4.2), we have a truncated BRS transformation

$$\begin{aligned} \delta_{BRS}\, q^A &= i\epsilon E^A_a C^a, \\ \delta_{BRS}\, p_A &= -i\epsilon \frac{\partial E^B_b}{\partial q^A} p_B C^b, \\ \delta_{BRS}\, C^a &= -\frac{i\epsilon}{2}[C \times C]^a, \\ \delta_{BRS}\, \bar{C}_a &= \epsilon(-iu^c_{ab} C^b \bar{C}_c + E^A_a p_A). \end{aligned} \tag{4.8}$$

Note that the nilpotency of the truncated BRS transformation is easily shown <u>without</u> use of the equations of motion. It is also interesting that the truncated BRS transformation is independent of the gauge fixing function χ^a after eliminating the multiplier field B_a.

The truncated BRS invariant stochastic action is obtained by integrating out the auxiliary fields B_a and ϕ^a successively in the extended BRS invariant stochastic action (4.4).

$$\begin{aligned} K_{BRS} &\equiv K_{inv} - Q(\chi) + \bar{C}_a(\dot{C}^a - [C \times \chi]^a - \frac{\partial \chi^a}{\partial q^A} E^A_b C^b), \\ &= K_{inv} + \bar{C}_a \dot{C}^a - \delta'_{BRS}(i\bar{C}_a \chi^a). \end{aligned} \tag{4.9}$$

From (4.9), we find that K_{BRS} is invariant under the truncated BRS transformation (4.8) by noting

$$\delta_{BRS}(ip_A \dot{q}^A + \bar{C}_a \dot{C}^a) = 0. \tag{4.10}$$

Now we derive the corresponding BRS charge. The BRS charges Q_{BRS} for the truncated BRS transformation (4.8) are defined by

$$\begin{aligned} Q_{BRS} &\equiv (\delta'_{BRS}\, q^A)p_A + i(\delta'_{BRS}\, C^a)\bar{C}_a, \\ &= iE^A_a C^a p_A + \frac{1}{2}[C \times C]^a \bar{C}_a. \end{aligned} \tag{4.11}$$

The BRS charge correctly generates the truncated BRS transformation with the commutation relations

$$\begin{aligned} [p_A, q^B] &= -i\delta^B_A, \\ \{i\bar{C}_a, C^b\} &= -i\delta^b_a. \end{aligned} \tag{4.12}$$

As we have shown in (2.17), (2.18), (2.19) and (2.20) that the canonical formulation is consistent with the action principle of SQ in the local gauge invariant formulation if we are careful for the operator ordering. By a similar fashion, in the case of the BRS invariant formulation, it is possible to show explicitly that the charge Q_{BRS} is conserved by regarding the BRS invariant stochastic action (4.9) as the Legendere transformation $\{q^A, \dot{q}^A, C^a, \dot{C}^a\} \rightarrow \{q^A, p_A, C^a, \bar{C}_a\}$. (4.9) implies that the BRS invariant stochastic hamiltonian is given by

$$\begin{aligned} K_{BRS} &\equiv ip_A\dot{q}^A + \bar{C}_a\dot{C}^a - H_{BRS}, \\ H_{BRS} &= H_{inv} + \{Q_{BRS}, i\bar{C}_a\chi^a\}, \end{aligned} \tag{4.13}$$

where H_{inv} is given in (2.20).

The BRS invariant stochastic action K_{BRS} in (4.9) and the stochastic hamiltonian (4.13) are obviously invariant under the scale transformation of the ghost fields.

$$\begin{aligned} \delta_{gh}C^a &= \rho C^a, \\ \delta_{gh}\bar{C}_a &= -\rho\bar{C}_a. \end{aligned} \tag{4.14}$$

The invariance implies that the ghost number charge

$$Q_{gh} \equiv -C^a\bar{C}_a, \tag{4.15}$$

conserves. The BRS charge (4.11) and the ghost number charge (4.16) satisfy the well-known algebra

$$\begin{aligned}\{Q_{BRS}, Q_{BRS}\} &= 0,\\ [Q_{gh}, Q_{gh}] &= 0,\\ [Q_{gh}, Q_{BRS}] &= Q_{BRS}.\end{aligned} \tag{4.16}$$

We comment here the canonical formulation of the extended BRS invariant stochastic action (4.4). If we want to impose a commutation relation between ϕ^a and B_a, we must consider a gauge fixing function which includes $\dot{\phi}^a$; however, in this case we have a Faddeev-Popov term which includes $\dot{\bar{C}}^a$. This implies that, for taking the stochastic action as a Legendere transformation of a corresponding hamiltonian formulation, it is natural to consider the truncated version of the BRS transformation. Although we can construct a canonical formulation of the extended BRS invariant stochastic action in principle, we consider the truncated version of the BRS transformation by taking the minimal principle for the auxiliary fields in the following sections. It is also clear from the truncated BRS transformation (4.8) that the stochastic quantization enlarge the phase space artificially by introducing the canonical momentum fields and then confine the auxiliary fields in the sense of Kugo-Ojima's quartet mechanism[14] which, in the case of the present BRS invariant stochastic quantization, realized in the artificially enlarged phase space $\{q^A, p_A, C^a, \bar{C}_a\}$ based on the nilpotency of the BRS transformation (4.8).

5. BRS Invariance of the Massless Yang-Mills Field in SQ

Based on the general formulation of the BRS symmetry in SQ, as a simple example, we investigate the BRS symmetry in the case of the massless Yang-Mills field. The classical gauge invariant action is given by

$$\begin{aligned}S_{YM} &= \frac{1}{4}\int d^D x F^a_{\mu\nu} F^a_{\mu\nu},\\ F^a_{\mu\nu} &\equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + A_\mu \times A_\nu{}^a.\end{aligned} \tag{5.1}$$

where $A_\mu \times A_\nu{}^a \equiv f^{abc} A^b_\mu A^c_\nu$.

We start from the Langevin equation

$$\dot{A}^a_\mu = D_\lambda F^a_{\lambda\mu} + \eta^a_\mu,$$
$$\left\langle \eta^a_\mu(x,t)\eta^b_\nu(x',t')\right\rangle = 2\delta^{ab}\delta_{\mu\nu}\delta^D(x-x')\delta(t-t'). \tag{5.2}$$

To derive the stochastic action corresponding to (2.5), we note that the determinant factor vanishes in this case, that is

$$\det\left(\delta^{ab}\delta_{\mu\nu}\partial_t + \frac{\delta^2 S_{YM}}{\delta A^a_\mu \delta A^b_\nu}\right) = 0. \tag{5.3}$$

The reason is that there is no local gauge invariant combination which is at most second order with respect to the field variable A^a_μ.

The stochastic action corresponding to (2.5) is given by

$$K_0^{YM} \equiv \int d^D x\Big\{ -\pi^a_\mu\pi^a_\mu + i\pi^a_\mu(\dot{A}^a_\mu - D_\lambda F^a_{\lambda\mu})\Big\}. \tag{5.4}$$

K_0^{YM} is invariant under the fictitious time independent local gauge transformation.

$$\delta A^a_\mu = D_\mu \alpha^a \equiv \partial\alpha^a + A_\mu \times \alpha^a,$$
$$\delta\pi^a_\mu = \eta \times \alpha^a. \tag{5.5}$$

The generator $Q(\alpha)$ of (5.5) is

$$Q(\alpha) \equiv i\int d^D x D_\lambda \alpha^a(x)\pi^a_\lambda(x), \tag{5.6}$$

where the gauge transformation parameter $\alpha^a(x)$ is <u>independent</u> of the fictitious time. By using the commutation relation,

$$\left[\pi^a_\mu(x), A^b_\nu(x')\right] = -i\delta^{ab}\delta_{\mu\nu}\delta^D(x-x'), \tag{5.7}$$

it is easy to see that the generator $Q(\alpha)$ satisfies

$$\left[Q(\alpha_1), Q(\alpha_2)\right] = Q(\alpha_1 \times \alpha_2). \tag{5.8}$$

From (5.8), we find that the structure constant f^{abc} corresponds to the " structure constant " u^c_{ab} in (2.20). Since the metric G^{AB} in (2.14) which satisfies (2.16) is δ^{AB}

in the case of the massless Yang-Mills field, we identify K_0^{YM} as the fictitious time independent stochastic action K_{inv}^{YM} corresponding to (2.14), that is, $K_{inv}^{YM} \equiv K_0^{YM}$

.

To obtain the extended local gauge transformation, we introduce an auxiliary field $\phi^a(x,t)$. From (3.1), we define

$$\begin{aligned} K &\equiv K_{inv}^{YM} - Q(\phi), \\ &= \int d^D x \Big\{ - \pi_\mu^a \pi_\mu^a + i\pi_\mu^a (\dot{A}_\mu^a - D_\lambda F_{\lambda\mu}^a - D_\mu \phi^a) \Big\}, \end{aligned} \tag{5.9}$$

which is invariant under the fictitious time dependent local gauge transformation

$$\begin{aligned} \delta A_\mu^a &= D_\mu \alpha^a \equiv \partial \alpha^a + A_\mu \times \alpha^a, \\ \delta \pi_\mu^a &= \eta \times \alpha^a, \\ \delta \phi^a &= \dot{\alpha}^a + \phi \times \alpha^a. \end{aligned} \tag{5.10}$$

The extended BRS transformation is obtained by replacing $\alpha^a \to i\epsilon c^a$

$$\begin{aligned} \tilde{\delta}_{BRS}\, A_\mu^a &= i\epsilon D_\mu c^a, \\ \tilde{\delta}_{BRS}\, \pi_\mu^a &= i\epsilon \pi_\mu \times c^a, \\ \tilde{\delta}_{BRS}\, \phi^a &= i\epsilon(\dot{c}^a + \phi \times c^a), \\ \tilde{\delta}_{BRS}\, c^a &= -\frac{i\epsilon}{2} c \times c^a, \\ \tilde{\delta}_{BRS}\, \bar{c}^a &= \epsilon B^a, \\ \tilde{\delta}_{BRS}\, B^a &= 0. \end{aligned} \tag{5.11}$$

The extended BRS invariant stochastic action is

$$\tilde{K} = K_{inv}^{YM} + \tilde{\delta}'_{BRS}(i\bar{c}^a F^a), \tag{5.12}$$

where F^a is a gauge fixing function.

By using the field equations which are obtained by the variation of ϕ^a and B^a, we can take a consistent truncation of the auxiliary fields as explained in the previous section. To truncate (5.11), we consider a special choice of the gauge fixing function

$$F^a = \phi^a - \alpha\partial_\mu A^a_\mu, \tag{5.13}$$

where α is introduced as a gauge parameter. By integrating the auxiliary fields B^a and ϕ^a successively, we have

$$\begin{aligned} K_{BRS} = \int d^Dx\Big\{ &- \pi^a_\mu\pi^a_\mu + i\pi^a_\mu(\dot{A}^a_\mu - D_\lambda F^a_{\lambda\mu} - \alpha D_\mu\partial_\lambda A^a_\lambda) \\ &+ \bar{c}^a(\dot{c}^a - \alpha D_\mu\partial_\mu c^a)\Big\}. \end{aligned} \tag{5.14}$$

(5.14) is invariant under the following truncated BRS transformation

$$\begin{aligned} &\delta_{BRS}\, A^a_\mu = i\epsilon D_\mu c^a, \\ &\delta_{BRS}\, \pi^a_\mu = i\epsilon\pi_\mu \times c^a, \\ &\delta_{BRS}\, c^a = -\frac{i\epsilon}{2} c \times c^a, \\ &\delta_{BRS}\, \bar{c}^a = \epsilon(-D_\mu\pi^a_\mu - ic \times \bar{c}^a). \end{aligned} \tag{5.15}$$

The extended BRS transformation (5.11) and the truncated one (5.15) are discussed in ref.[6] and ref.[5], respectively, though the relation between the two BRS transformations are not clarified. We here note that the truncated BRS transformation (5.15) is also obtained if we chose $F^a = \phi^a - \chi^a$ with the arbitrary function $\chi^a = \chi^a(A)$.

The BRS charge for the truncated BRS transformation (5.15) is given by

$$\begin{aligned} Q_{BRS} &\equiv \int d^Dx((\delta'_{BRS}\, A^a_\mu)\pi^a_\mu + i(\delta'_{BRS}\, c^a)\bar{c}^a), \\ &= \int d^Dx(D_\mu A^a_\mu\pi^a_\mu + \frac{1}{2}\int d^Dx c \times c^a\bar{c}^a). \end{aligned} \tag{5.16}$$

The BRS charge generates (5.15) with the commutation relation (5.8) and

$$\{i\bar{c}^a(x), c^b(x')\} = -i\delta^{ab}\delta^D(x - x'). \tag{5.17}$$

In section 2, we introduce the metric G^{AB} for the invariance of the stochastic action under the local gauge transformation. As we have shown in this section, in the case of the massless Yang-Mills field, the metric G^{AB} is trivial, i.e. $G^{AB} = \delta^{AB}$. In the following section, as a nontrivial example, we investigate the massive Yang-Mills field based on the BRS invariant formulation of the stochastic quantization.

6. BRS Invariance of the Massive Yang-Mills Field in SQ

The classical action of the massive Yang-Mills field is given by

$$S_{cl} = S_{YM} + S_{mass}, \qquad S_{mass} = \frac{m^2}{2}(A^a_\mu - K^a_\mu)(A^a_\mu - K^a_\mu). \tag{6.1}$$

where the auxiliary field K_μ is introduced to make the classical action invariant under the local gauge transformation. The simplest way to determine the transformation property of the auxiliary field is to define K_μ as a pure gauge field[17]:

$$K_\mu \equiv U^{-1}\partial_\mu U, \qquad U = e^\theta, \tag{6.2}$$

where we use the notation $\theta = \theta^a T^a$. The anti-hermite matrix, T^a, is the generator of the Lie algebra.

$$[T^a, T^b] = f^{abc}T^c, \qquad tr(T^aT^b) = -2\delta^{ab}. \tag{6.3}$$

The local gauge transformation of the Yang-Mills field $A_\mu = A^a_\mu T^a$ is defined by

$$A^\omega_\mu = \omega^{-1}A_\mu\omega + \omega^{-1}\partial_\mu\omega, \qquad = \partial_\mu\alpha + [A_\mu, \alpha], \tag{6.4}$$

with $\omega = e^\alpha$. If we define the local gauge transformation of the auxiliary field θ in (6.1) by

$$U^\omega = U\omega, \tag{6.5}$$

we have

$$K^\omega_\mu = \omega^{-1}K_\mu\omega + \omega^{-1}\partial_\mu\omega. \tag{6.6}$$

Now it is obvious that the classical action (6.1) is invariant under the local gauge transformations (6.4) and (6.5).

Since, for the field variable A^a_μ, the local gauge transformation is the same as that of the Yang-Mills field case, we concentrate on the investigation of the transformation property of the auxiliary field θ^a in (6.2). The local gauge transformation of θ^a (6.5) is written in the infinitesimal form

$$\delta\theta^a = L^a{}_b(\theta)\alpha^b(x), \tag{6.7}$$

where $L^a{}_b$ is introduced as the inverse of the matrix $K^a{}_b$: $K^a{}_c L^c{}_b = \delta^a_b$

$$K^a{}_b \equiv -\frac{1}{2}\{T^a U^{-1}\partial_b U\}. \tag{6.8}$$

By the definition, K_μ is written by $K^a_\mu = K^a{}_b\partial_\mu\theta^b$. Since K_μ is a pure gauge field, its field strength vanishes giving the condition

$$\partial_b K^a{}_c - \partial_c K^a{}_b + f^{ade}K^d{}_b K^e{}_c = 0, \tag{6.9}$$

where ∂_a denotes $\frac{\partial}{\partial\theta^a}$. The inverse matrix $L^a{}_b$ satisfies a similar equation

$$-(\partial_d L^a{}_b)L^d{}_c + (\partial_d L^a{}_c)L^d{}_b = f^{dbc}L^a{}_d. \tag{6.10}$$

This is just the condition (2.24) which, as we have shown in section 3 and 4, automatically guarantees the existence of the nilpotent BRS transformation in SQ and the BRS invariance of the stochastic action. In the ordinary quantization method, eqs.(6.9) and (6.10) are also the fundamental equations for the existence of the BRS invariance[14][17].

Our next task is to construct the metric tensor G^{AB} (or G_{AB}) in (2.15). To do this, we must obtain the solution of the Killing vector equation (2.16). Fortunately, in the case of the massive Yang-Mills field, it is easy to find the metric G^{AB} which satisfies (2.16). Let us define

$$
\begin{aligned}
G^{ab} &\equiv L^a{}_c L^b{}_c, \\
G_{ab} &\equiv K^c{}_a K^c{}_b,
\end{aligned}
\tag{6.11}
$$

which satisfies $G^{ac}G_{cb} = \delta^a_b$. By the help of the equations (6.9) and (6.10), we have

$$
\begin{aligned}
\delta G^{ab} &\equiv \partial_c G^{ab} L^c{}_d \alpha^d, \\
&= G^{ac}(\partial_c L^b{}_d)\alpha^d + G^{cb}(\partial_c L^a{}_d)\alpha^d, \\
\delta G_{ab} &\equiv \partial_c G_{ab} L^c{}_d \alpha^d, \\
&= -G_{ac}(\partial_b L^c{}_d)\alpha^d - G_{cb}(\partial_a L^c{}_d)\alpha^d.
\end{aligned}
\tag{6.12}
$$

(6.12) implies that the metrices (6.11) satisfy the Killing vector equation.

Before constructing the BRS invariant stochastic action, we confirm the correspondence between the formulation in section 2 and the result in this section. The configuration space $\{q^A\}$ is defined by $\{q^A\} = \{A^a_\mu(x), \theta^a(x)\}$. The metric is given by

$$
\begin{aligned}
G^{AB} &= \begin{pmatrix} \delta^{ab}\delta_{\mu\nu} & 0 \\ 0 & G^{ab} \end{pmatrix}, \\
G_{AB} &= \begin{pmatrix} \delta^{ab}\delta_{\mu\nu} & 0 \\ 0 & G_{ab} \end{pmatrix}.
\end{aligned}
\tag{6.13}
$$

In this manifold, $\{E^A_a\alpha^a\} = \{D_\mu\alpha^a(x), L^a{}_b\alpha^b(x)\}$ are the Killing vectors. Especially, we note that E^A_a satisfies the condition for the closure of the constraints (2.24).

For the canonical formulation, we introduce the field variable, $\pi_a(x)$, as the canonical momentum of θ^a. Then the canonical momentum space is given by $\{p_A\} = \{\pi^a_\mu(x), \pi_a(x)\}$. The conserved charge defined in (2.17) is given by

$$
\begin{aligned}
Q(\alpha) &\equiv i\int d^Dx(\delta A^a_\mu(x)\pi^a_\mu(x) + \delta\theta^a(x)\pi_a(x)), \\
&= i\int d^Dx(D_\mu\alpha^a\pi^a_\mu + L^a{}_b\alpha^b\pi_a).
\end{aligned}
\tag{6.14}
$$

The charge satisfies

$$[Q(\alpha_1), Q(\alpha_2)] = Q(\alpha_1 \times \alpha_2), \tag{6.15}$$

by the help of (6.10).

To give the local gauge invariant stochastic action, we note that the multiplier field ϕ^a in (3.1) is also sufficient in the case of the massive Yang-Mills field. The invariant stochastic action for massive Yang-Mills field which is invariant under the fictitious time <u>independent</u> local gauge transformation is given by

$$\begin{aligned} K_{inv} = \int d^D x \Big\{ & -\pi^a_\mu \pi^a_\mu + i\pi^a_\mu (\dot{A}^a_\mu - D_\lambda F^a_{\lambda\mu} + m^2(A^a_\mu - K^a{}_b \partial_\mu \theta^b)) \\ & - \pi_a G^{ab} \pi_b + i\pi_a (\dot{\theta}^a + m^2 L^a{}_b \tilde{D}_\mu A^b_\mu - m^2(\Box\theta^a + \Gamma^a_{bc} \partial_\mu \theta^b \partial_\mu \theta^c)) \Big\}, \end{aligned} \tag{6.16}$$

where

$$\tilde{D}_\mu A^a_\mu \equiv \partial_\mu A^a_\mu + f^{abc} K^b{}_d \partial_\mu \theta^d A^c_\mu. \tag{6.17}$$

We also obtain the invariant action under the fictitious time <u>dependent</u> local gauge transformation

$$\begin{aligned} K &\equiv K_{inv} - Q(\phi), \\ &= K_{inv} - i \int d^D x (-D_\mu \pi^a_\mu + L^b{}_a \pi_b) \phi^a. \end{aligned} \tag{6.18}$$

Since the structure constant is f^{abc} in the case of the massive Yang-Mills field, the transformation of the auxiliary field ϕ^a is the same as that given in (5.11) for the fictitious time dependent local gauge transformation.

Now we clarify the BRS structure of the massive Yang-Mills field in SQ. The extended BRS transformation is given by (5.11) and

$$\begin{aligned} \delta_{BRS}\, \theta^a &= i\epsilon L^a{}_b c^b, \\ \delta_{BRS}\, \pi_a &= -i\epsilon \partial_a L^b{}_c \pi_b c^c. \end{aligned} \tag{6.19}$$

By using the gauge fixing function $\chi^a = \chi^a(A_\mu, \theta)$, we choose $F^a = \phi^a - \chi^a$ in (5.3) to truncate the extended BRS transformation. The truncated BRS transformation is

$$
\begin{aligned}
&\delta_{BRS}\ A^a_\mu = i\epsilon D_\mu c^a,\\
&\delta_{BRS}\ \pi^a_\mu = i\epsilon \pi_\mu \times c^a,\\
&\delta_{BRS}\ \theta^a = i\epsilon L^a{}_b c^b,\\
&\delta_{BRS}\ \pi_a = -i\epsilon \partial_a L^b{}_c \pi_b c^c,\\
&\delta_{BRS}\ c^a = -\frac{i\epsilon}{2} c \times c^a,\\
&\delta_{BRS}\ \bar{c}^a = \epsilon(-D_\mu \pi^a - ic \times \bar{c}^a + L^b{}_a \pi_b).
\end{aligned}
\tag{6.20}
$$

The BRS transformation is, of course, nilpotent $\delta^2_{BRS} = 0$. The (truncated) BRS invariant stochastic action is given by

$$
K_{BRS} = K_{inv} + \bar{c}^a \dot{c}^a - \delta'_{BRS}(i\bar{c}^a \chi^a), \tag{6.21}
$$

where K_{inv} is defined in (6.16).

In this section, we showed that the metric G^{AB}, which is introduced in the general formulation of the local gauge invariant stochastic action, is really exists satisfying the Killing vector equation (2.16) as a nontrivial function of q^A in the case of the massive Yang-Mills field. Especially, in this case, the function $K^a{}_b$ and $L^a{}_b$ play the role of the " Vielbein " clearly shown in (6.11).

It is also possible to derive the Ward-Takahashi identities based on the truncated BRS transformation (6.20), however, it is obvious from the power counting that the massive Yang-Mills theory is perturbatively unrenormalizable in the sense of the loop expansion in SQ.

7. Discussions

In the paper, we constructed the BRS invariant formulation for the stochastic quantization of the general first-class constrained systems. We showed that there exists the extended BRS transformation (4.2) which is an D+1 dimensional field theoretical analogue of the well-known BRS transformation in the ordinary quantization method. By eliminating the two auxiliary fields, we showed that there remains the truncated BRS transformation (4.8) which is also nilpotent. We also found the systematic method to construct the truncated BRS invariant stochastic action. As the application of the general formulation, we clarified the BRS invariance in the massless and massive Yang-Mills fields in SQ

In the general formulation of the local gauge invariant stochastic action, we introduced the metric G_{AB} in the configuration space $\{q^A\}$. By using the metric, we proposed the geometric interpretation of the first-class constraint systems. That is, from the condition (2.16), the functional E_a^A of the local gauge transformation in (2.11) is just the Killing vector in the manifold of the configuration which is specified by $\{q^A, G_{AB}\}$. From the fact that the local gauge invariant stochastic hamiltonian (2.20) is written in the form of the covariant derivative with respect to q^A in the manifold $\{q^A, G_{AB}\}$, the Fokker-Planck equation derived from the stochastic hamiltonian

$$\dot{\mathcal{P}} = -H_{inv}(p_A \rightarrow -i\frac{\partial}{\partial q^A})\mathcal{P}, \tag{7.1}$$

is almost invariant under the general coordinate transformation $q^A \rightarrow q^{A'} = f^{A'}(q)$ in the manifold $\{q^A, G_{AB}\}$ provided that the distribution functional $\mathcal{P}$ is regarded as a scalar quantity. The invariance, however, is broken by the term $\frac{\partial S_{cl}}{\partial q^A}$ in the stochastic hamiltonian since the classical action is not invariant under the arbitrary transformation $q^A \rightarrow q^{A'} = f^{A'}(q)$. The invariance of the Fokker- Planck equation is only realized in the special case in which the direction of the coordinate transformation is given by the Killing vector, that is, $q^A \rightarrow q^{A'} = q^A + E_a^A(q)\alpha^a$. This is an interesting geometrical interpretation of the first-class constrained systems in the context of the stochastic quantization.

There remains many open questions in SQ. One of the most important problems is how to derive the S-matrix in SQ. Since there is no concept corresponding to the asymptotic state with respect to the (true) time coordinate in spacetime, it is difficult to define the S-matrix in the present BRS invariant formulation of the stochastic quantization. The BRS invariant formulation of SQ is also applicable to the gravitational field. The analysis is to be published in elsewhere.

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APPENDIX

In this appendix, We prove that, with an appropriate choice of the auxiliary field ϕ^a as a functional of q^A, the Fokker-Planck equation which is derived from the stochastic action (3.1) gives the distribution functional including the well-known Faddeev-Popov determinant factor as the equilibrium distribution functional.

We consider the Fokker-Planck equation. As we have noted in (2.20), in order to derive the hamiltonian operator from the stochastic action (3.1), we must be careful to the operator ordering. The Fokker-Planck equation derived from the stochastic action (3.1) is given by

$$\begin{aligned} \dot{\mathcal{P}} &= -\mathcal{H}\mathcal{P}, \\ \mathcal{H} &\equiv -\frac{1}{\sqrt{G}}\frac{\partial}{\partial q^A}\{\sqrt{G}G^{AB}(\frac{\partial}{\partial q^B} + \frac{\partial S_{cl}}{\partial q^B}) - \sqrt{G}E^A_a\phi^a\}. \\ &= -\nabla^A(\partial_A + \partial_A S_{cl}) + \nabla_A E^A_a\phi^a. \end{aligned} \tag{A.1}$$

where ∂_A and ∇_A denotes the derivative and the covariant derivative on the manifold $\{q^A, G_{AB}\}$ with respect to the variable q^A, respectively. We note that the Killing vector equation (2.16) implies

$$\nabla_A E_a^A = 0, \tag{8.2}$$

then the last term in (A.1) is rewritten by $E_a^A \partial_A \phi^a$.

The equilibrium distribution functional satisfies

$$\mathcal{H}\mathcal{P}_{eq} = 0. \tag{8.3}$$

We expect that, under the suitable choice of the auxiliary field ϕ^a as a function of q^A, the equilibrium distribution is given by the well-known functional[2] .

$$\mathcal{P}_{F\text{-}P} \equiv \int e^{-S_{cl}-S_{F\text{-}P}}, \tag{8.4}$$

where $\int$ denotes $\int \mathcal{D}B\mathcal{D}C\mathcal{D}\bar{C}$. $S_{F\text{-}P}$ denotes the gauge fixing term and the Faddeev-Popov determinant term;

$$\begin{aligned} S_{F\text{-}P} &= \hat{\delta}_{BRS} F, \\ \hat{\delta}_{BRS} &\equiv -iB_a \frac{\partial}{\partial \bar{C}_a} - \frac{i}{2}[C \times C]^a \frac{\partial}{\partial C^a} + E_a^A C^a \frac{\partial}{\partial q^A}, \\ F &= \bar{C}_a(\chi^a + \frac{\alpha}{2} B^a). \end{aligned} \tag{8.5}$$

where χ^a, a function of only q^A, denotes the gauge fixing function. The operator $\hat{\delta}_{BRS}$ is the generator of the BRS transformation[7][8] in the ordinary quantization method which satisfies the nilpotency condition.

Now we prove that the equilibrium distribution functional is given by $\mathcal{P}_{F\text{-}P}$ in (A.4) if we chose

$$\phi^a = -\int C^a \nabla^A (\partial_A F e^{-S_{tot}}) \mathcal{P}_{F\text{-}P}^{-1}, \tag{8.6}$$

where $S_{tot} = S_{cl} + S_{F\text{-}P}$. By inserting (A.4) and (A.6) into the equation (A.3), we have

$$
\begin{aligned}
\mathcal{H}\mathcal{P}_{F\text{-}P} &= \nabla^A\Big\{\int(\partial_A\hat{\delta}_{BRS}F)e^{-S_{tot}}\Big\} + \nabla_A(E^A_a\phi^a\mathcal{P}_{F\text{-}P}), \\
&= \nabla^A\Big\{\int\hat{\delta}_{BRS}(\partial_A F)e^{-S_{tot}}\Big\} + \nabla^A\Big\{\int(\partial_A E^B_a)C^a(\partial_B F)e^{-S_{tot}}\Big\} \qquad (8.7) \\
&\quad - \nabla_A\Big\{\int E^A_a C^a(\nabla^B(\partial_B F)e^{-S_{tot}})\Big\}.
\end{aligned}
$$

By integrating by parts with respect to the variables C^a and B^a in (A.7), we finally obtain

$$
\mathcal{H}\mathcal{P}_{F\text{-}P} = \nabla_A\Big\{(G^{AC}\partial_C E^C_a + G^{CB}\partial_C E^A_a - \partial_C G^{AB}E^C_a)\int C^a(\partial_B F)e^{-S_{tot}}\Big\}. \qquad (8.8)
$$

From the Killing vector equation (2.16), we find that the functional $\mathcal{P}_{F\text{-}P}$ is really an equilibrium distribution functional under the special choice of the auxiliary field ϕ^a in (A.1). The above proof is the generalization of the Yang-Mills case[20][21]. In the Yang-Mills case, it is not necessary to introduce the metric G^{AB}.

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