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Shui Wang

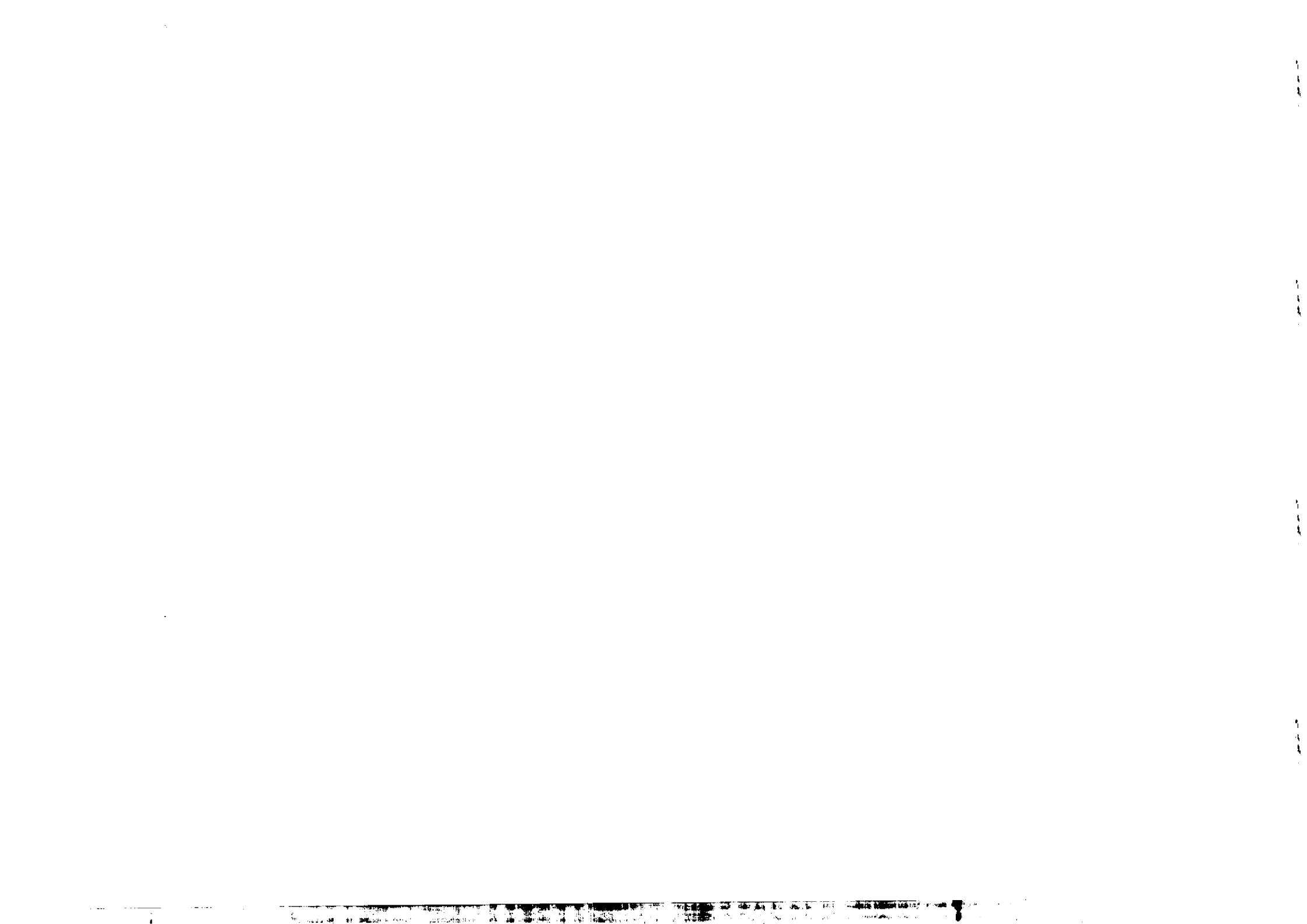


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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

STREAMING GRAVITY MODE INSTABILITY \*

Shui Wang \*\*

International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

In this paper, we study the stability of a current sheet with a sheared flow in a gravitational field which is perpendicular to the magnetic field and plasma flow. This mixing mode caused by a combined role of the sheared flow and gravity is named the streaming gravity mode instability. The conditions of this mode instability are discussed for an ideal four-layer model in the incompressible limit.

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\*\* Permanent address: University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China.

In most applications of space physics, solar physics and laboratory plasma physics, plasma instabilities of various kinds play important roles. Some examples include the Rayleigh-Taylor instability, the Kelvin-Helmholtz instability, the tearing mode instability [1],[2]. Recently, Lee *et al.* [3] and Wang *et al.* [4],[5] have investigated the streaming sausage, kink and tearing instabilities excited in a current sheet with a super-Alfvenic plasma flow.

The purpose of this paper is to study the effects of gravity field on the Kelvin-Helmholtz instability and the streaming sausage and kink instabilities. In the presence of a gravitational field which is perpendicular to the magnetic field and plasma flow, the physical characteristics and growth rates of the instabilities are modified by the gravity field. On the other hand, the magnetic field lines are usually curved in the space and laboratory plasmas. For a plasma on a curved field line, the centrifugal force on particles moving along the field line acts like a gravitational force. This effective gravity may also modify the features of the Kelvin-Helmholtz instability and streaming sausage and kink instabilities. This mixing mode caused by a combined role of the sheared flow and gravity or effective gravity is named the streaming gravity mode instability.

We start from the following ideal incompressible MHD equations

$$\nabla \cdot \vec{v} = 0 \quad (1)$$

$$\frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p = 0 \quad (2)$$

$$\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\nabla p + \frac{1}{4\pi} (\nabla \times \vec{B}) \times \vec{B} + \rho \vec{g} \quad (3)$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) \quad (4)$$

The initial one-dimensional equilibrium configuration in a slab system is described by

$$\begin{aligned} \vec{B}_0 &= B_{0x}(z) \vec{i}_x + B_{0y}(z) \vec{i}_y, \\ \vec{v}_0 &= v_0(z) \vec{i}_x, \quad \rho_0 = \rho_0(z), \\ \rho_c &= \rho_c(z), \quad \vec{g} = -g \vec{i}_z, \end{aligned} \quad (5)$$

where  $g = \text{const.}$  The condition for the transverse pressure balance can be written as

$$\frac{d}{dz} \left( p_0 + \frac{B_{0x}^2 + B_{0y}^2}{8\pi} \right) = -\rho_0 g. \quad (6)$$

The perturbed quantity is expressed by  $q_1(z) \exp[i(k_x x + k_y y - \omega t)]$ . From the linearized form of perturbed MHD equations, we can express all perturbed quantities in terms of the total perturbed pressure  $\Phi$

$$s_1 = -\frac{1}{\alpha^* \omega^*} \frac{d\rho_0}{dz} \frac{d\Phi}{dz}, \quad (7)$$

$$u = \frac{R_x}{\alpha} \Phi - \frac{1}{\alpha^* \omega^*} \frac{dV_0}{dz} \frac{d\Phi}{dz}, \quad (8)$$

$$v = \frac{R_y}{\alpha} \Phi, \quad (9)$$

$$w = \frac{1}{i\alpha^*} \frac{d\Phi}{dz}, \quad (10)$$

$$b_x = -\frac{R_x}{\alpha \omega^*} (\vec{R} \cdot \vec{B}_0) \Phi - \frac{1}{\alpha^* \omega^*} \frac{dB_{0x}}{dz} \frac{d\Phi}{dz}, \quad (11)$$

$$b_y = -\frac{R_y}{\alpha \omega^*} (\vec{R} \cdot \vec{B}_0) \Phi - \frac{1}{\alpha^* \omega^*} \frac{dB_{0y}}{dz} \frac{d\Phi}{dz}, \quad (12)$$

$$b_z = \frac{i}{\alpha^* \omega^*} (\vec{R} \cdot \vec{B}_0) \frac{d\Phi}{dz}, \quad (13)$$

where  $u, v, w$  and  $b_x, b_y, b_z$  are the components of the perturbed plasma velocity  $\vec{v}_1$  and perturbed magnetic field  $\vec{B}_1$ ,  $\rho_1$  is the perturbed plasma density, the Doppler-shifted frequency  $\omega^*$ , the total perturbed pressure  $\Phi$  and  $\alpha, \alpha^*$  are given by

$$\begin{aligned} \omega^* &= \omega - R_x V_0, & \Phi &= p_1 + \frac{1}{4\pi} (\vec{B}_0 \cdot \vec{B}_1), \\ \alpha &= \omega^* \rho_0 - \frac{(\vec{R} \cdot \vec{B}_0)^2}{4\pi \omega^*}, & \alpha^* &= \alpha + \frac{g}{\omega^*} \frac{d\rho_0}{dz}. \end{aligned} \quad (14)$$

The total perturbed pressure  $\Phi$  obeys the following second-order differential equation

$$\frac{d}{dz} \left( \frac{1}{\alpha^* \omega^*} \frac{d\Phi}{dz} \right) - \frac{R^2}{\alpha \omega^*} \Phi = 0. \quad (15)$$

We introduce the perturbed displacement vector  $\xi$

$$\frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} + (\vec{V}_0 \cdot \nabla) \xi = \vec{V}_1. \quad (16)$$

Note that  $\xi_z = iw/\omega^*$ . Using Eqs.(10) and (15), the following differential equation for  $\xi_z$  can be obtained

$$\frac{d}{dz} \left( \alpha \omega^* \frac{d\xi_z}{dz} \right) - R^2 \alpha^* \omega^* \xi_z = 0. \quad (17)$$

As an example, we discuss the stability for the equilibrium state with a four-layer model. Assuming that  $\rho_0, V_0$  and  $B_{0x}$  are  $\rho_I, \rho_{II}, \rho_{III}, \rho_{IV}, V_I, V_{II}, V_{III}, V_{IV}$  and  $B_I, B_{II}, B_{III}, B_{IV}$ , constants, and that  $B_{0y}$  is equal to zero, then  $\omega^*, \alpha$  and  $\alpha^*$  are constants in each region, respectively. Eq.(17) can be reduced to

$$\frac{d^2 \xi_z}{dz^2} - R^2 \xi_z = 0, \quad (18)$$

in each region.

The first match condition at the interface between two different layers can be written as

$$[\xi_z] = 0, \quad (19)$$

where the bracket represents the jump across the interface in slab geometry. On the other hand, the plasma density and its gradient are discontinuous at the interface, we must integrate Eq.(17), across the interface, over an infinitesimal element of  $z$  including the points of discontinuity [1]. Therefore, the second match condition may be represented by

$$\left[ \alpha \omega^* \frac{d \xi_z}{dz} \right] - g k^2 \xi_z [S_0] = 0. \quad (20)$$

Using the solutions of Eq.(18) and the match conditions (19) and (20), we obtain the dispersion equation of the streaming gravity mode for a four-layer model as

$$\begin{aligned} & [\alpha_I \omega_I^* + \alpha_{II} \omega_{II}^* + g k (S_I - S_{II})] \{ [\alpha_{II} \omega_{II}^* - \alpha_{III} \omega_{III}^* + g k (S_{II} - S_{III})] \cdot \\ & \cdot [\alpha_{III} \omega_{III}^* - \alpha_{IV} \omega_{IV}^* - g k (S_{III} - S_{IV})] + e^{-k \delta} [\alpha_I \omega_I^* + \alpha_{II} \omega_{II}^* + \\ & + g k (S_I - S_{II})] [\alpha_{III} \omega_{III}^* + \alpha_{IV} \omega_{IV}^* + g k (S_{III} - S_{IV})] \} + \\ & + [\alpha_I \omega_I^* - \alpha_{II} \omega_{II}^* + g k (S_I - S_{II})] \{ [\alpha_{II} \omega_{II}^* - \alpha_{III} \omega_{III}^* - g k (S_{II} - S_{III})] \cdot \\ & \cdot [\alpha_{III} \omega_{III}^* + \alpha_{IV} \omega_{IV}^* + g k (S_{III} - S_{IV})] + e^{-2k \delta} [\alpha_{II} \omega_{II}^* + \alpha_{III} \omega_{III}^* - \\ & - g k (S_{II} - S_{III})] [\alpha_{III} \omega_{III}^* - \alpha_{IV} \omega_{IV}^* - g k (S_{III} - S_{IV})] \} = 0. \quad (21) \end{aligned}$$

Here we discuss only some special cases.

(1) If the plasma sheet is very thin ( $k \delta \rightarrow 0$ ) and there are no shear flows, i.e.

$$V_0 = 0, \quad B_I = B_{II}, \quad B_{III} = B_{IV}, \quad \rho_I = \rho_{II}, \quad \rho_{III} = \rho_{IV}, \quad (22)$$

then, from Eq.(14) we have

$$\omega^* = \omega, \quad \alpha_I = \alpha_{II}, \quad \alpha_{III} = \alpha_{IV}. \quad (23)$$

Substituting Eqs.(22) and (23) into Eq.(21), the dispersion equation reduces to

$$\omega(\alpha_{II} + \alpha_{III}) + gk(\rho_{II} - \rho_{III}) = 0 \quad (24)$$

From Eq.(24) the dispersion relation for the Rayleigh-Taylor instability in a two-layer homogeneous model [1],[2] can be obtained

$$\omega = \pm \left[ \frac{(\vec{k} \cdot \vec{B}_{II})^2 + (\vec{k} \cdot \vec{B}_{III})^2}{4\pi(\rho_{II} + \rho_{III})} - gk \frac{\rho_{II} - \rho_{III}}{\rho_{II} + \rho_{III}} \right]^{\frac{1}{2}}. \quad (25)$$

The critical condition for the Rayleigh-Taylor instability can be written as

$$gk(S_{II} - S_{III}) > \frac{1}{4\pi} [(\vec{k} \cdot \vec{B}_{II})^2 + (\vec{k} \cdot \vec{B}_{III})^2]. \quad (26)$$

(2) If the plasma sheet is very thin ( $k \delta \rightarrow 0$ ) and the effect of gravity is neglected, i.e.

$$\begin{aligned} g &= 0, \quad V_I = V_{II}, \quad V_{III} = V_{IV}, \\ B_I &= B_{II}, \quad B_{III} = B_{IV}, \quad S_I = S_{II}, \quad S_{III} = S_{IV}, \end{aligned} \quad (27)$$

then, from Eq.(14) we have

$$\omega_I^* = \omega_{II}^*, \quad \omega_{III}^* = \omega_{IV}^*, \quad \alpha_I = \alpha_{II}, \quad \alpha_{III} = \alpha_{IV}. \quad (28)$$

Substituting Eqs.(27) and (28) into Eq.(21), the dispersion equation reduces to

$$\alpha_{II} \omega_{II}^* + \alpha_{III} \omega_{III}^* = 0 \quad (29)$$

From Eq.(29), the dispersion relation for the Kelvin-Helmholtz instability in a two-layer homogeneous model [1],[2] can be obtained

$$\frac{\omega}{k_x} = \frac{S_{II} V_{II} + S_{III} V_{III}}{S_{II} + S_{III}} \pm \left\{ \frac{(\vec{k} \cdot \vec{B}_{II})^2 + (\vec{k} \cdot \vec{B}_{III})^2}{4\pi k_x^2 (S_{II} + S_{III})} - \frac{S_{II} S_{III} (V_{II} - V_{III})^2}{(S_{II} + S_{III})^2} \right\}^{\frac{1}{2}} \quad (30)$$

The critical condition for the Kelvin-Helmholtz instability can be written as

$$(V_{II} - V_{III})^2 > \frac{1}{4\pi k_x^2} \left( \frac{1}{S_{II}} + \frac{1}{S_{III}} \right) [(\vec{k} \cdot \vec{B}_{II})^2 + (\vec{k} \cdot \vec{B}_{III})^2]. \quad (31)$$

(3) If the effect of the gravity field is neglected, and the magnetic field on the two sides of plasma sheet have opposite polarity, i.e.

$$\begin{aligned} g &= 0, \quad V_I = V_{IV}, \quad V_{II} = V_{III} = V_I + V_P, \\ B_I &= -B_{IV}, \quad B_{II} = -B_{III}, \quad S_I = S_{IV}, \quad S_{II} = S_{III}, \end{aligned} \quad (32)$$

then, from Eq.(14) we have

$$\omega_I^* = \omega_{IV}^*, \quad \omega_{II}^* = \omega_{III}^*, \quad \alpha_I = \alpha_{IV}, \quad \alpha_{II} = \alpha_{III} \quad (33)$$

Substituting Eqs.(32) and (33) into Eq.(21), the dispersion equation reduces to

$$\alpha_I \omega_I^* + \alpha_{II} \omega_{II}^* = \pm (\alpha_I \omega_I^* - \alpha_{II} \omega_{II}^*) e^{-2k\delta} \quad (34)$$

From Eq.(34), the dispersion relation for the streaming sausage and kink instabilities in a four-layer homogeneous model of the current sheet, respectively [3], can be obtained

$$\left(\frac{\omega}{k_x}\right)_{1,2} = V_I + \frac{1}{S_{II}(1-e^{-2k\delta}) + S_I(1+e^{-2k\delta})} \left\{ S_{II} V_p(1-e^{-2k\delta}) \pm \left[ S_{II}^2 V_p^2(1-e^{-2k\delta})^2 - [(S_{II} + S_I) - (S_{II} - S_I)e^{-2k\delta}] \cdot [S_{II}(V_p^2 - V_{AII}^2)(1-e^{-2k\delta}) - S_I V_{AI}^2(1+e^{-2k\delta})] \right]^{1/2} \right\} \quad (35)$$

and

$$\left(\frac{\omega}{k_x}\right)_{3,4} = V_I + \frac{1}{S_{II}(1+e^{-2k\delta}) + S_I(1-e^{-2k\delta})} \left\{ S_{II} V_p(1+e^{-2k\delta}) \pm \left[ S_{II}^2 V_p^2(1+e^{-2k\delta})^2 - [(S_{II} + S_I) + (S_{II} - S_I)e^{-2k\delta}] \cdot [S_{II}(V_p^2 - V_{AII}^2)(1+e^{-2k\delta}) - S_I V_{AI}^2(1-e^{-2k\delta})] \right]^{1/2} \right\} \quad (36)$$

where  $V_{AI} = B_I/(4\pi\rho_I)^{1/2}$  and  $V_{AII} = B_{II}/(4\pi\rho_{II})^{1/2}$ . The critical condition for the streaming sausage and streaming kink instabilities can be obtained, respectively, as

$$V_p^2 > \frac{1}{S_I S_{II}(1-e^{-4k\delta})} [S_I(1+e^{-2k\delta}) + S_{II}(1-e^{-2k\delta})] \cdot [S_I V_{AI}^2(1+e^{-2k\delta}) + S_{II} V_{AII}^2(1-e^{-2k\delta})] \quad (37)$$

and

$$V_p^2 > \frac{1}{S_I S_{II}(1-e^{-4k\delta})} [S_I(1-e^{-2k\delta}) + S_{II}(1+e^{-2k\delta})] \cdot [S_I V_{AI}^2(1-e^{-2k\delta}) + S_{II} V_{AII}^2(1+e^{-2k\delta})] \quad (38)$$

(4) If the plasma sheet is very thin ( $k\delta \rightarrow 0$ ), but the effects of both the sheared flow and gravity are considered, i.e.

$$g \neq 0, \quad V_I = V_{II}, \quad V_{III} = V_{IV}, \quad (39)$$

$$B_I = B_{II}, \quad B_{III} = B_{IV}, \quad S_I = S_{II}, \quad S_{III} = S_{IV},$$

then, from Eq.(14) we also have

$$\omega_I^* = \omega_{II}^*, \quad \omega_{III}^* = \omega_{IV}^*, \quad \alpha_I = \alpha_{II}, \quad \alpha_{III} = \alpha_{IV} \quad (40)$$

Substituting Eqs.(39) and (40) into Eq.(21), the dispersion equation reduces to

$$\alpha_{II} \omega_{II}^* + \alpha_{III} \omega_{III}^* + g k (S_{II} - S_{III}) = 0 \quad (41)$$

From Eq.(41), we obtain the dispersion relation for the Kelvin-Helmholtz instability in the presence of the gravitational field which is perpendicular to the magnetic field and plasma flow

$$\frac{\omega}{k_x} = \frac{S_{II} V_{II} + S_{III} V_{III}}{S_{II} + S_{III}} \pm \left\{ \frac{(\vec{k} \cdot \vec{B}_{II})^2 + (\vec{k} \cdot \vec{B}_{III})^2}{4\pi k_x^2 (S_{II} + S_{III})} - \frac{S_{II} S_{III} (V_{II} - V_{III})^2}{(S_{II} + S_{III})^2} - g k \frac{S_{II} - S_{III}}{S_{II} + S_{III}} \right\}^{1/2} \quad (42)$$

The critical condition for the Kelvin-Helmholtz instability with the effect of the gravity field can be written as

$$(V_{II} - V_{III})^2 > \frac{1}{4\pi k_x^2} \left( \frac{1}{S_{II}} + \frac{1}{S_{III}} \right) [(\vec{k} \cdot \vec{B}_{II})^2 + (\vec{k} \cdot \vec{B}_{III})^2] - \frac{g k}{k_x^2} \left( \frac{S_{II}}{S_{III}} - \frac{S_{III}}{S_{II}} \right) \quad (43)$$

From Eqs.(42) and (43), it can be shown that the critical value for the Kelvin-Helmholtz becomes high, when the direction of the gravity field is the same as that of the plasma density gradient ( $\rho_{II} < \rho_{III}$ ). When the direction of the gravity is opposite to the density gradient ( $\rho_{II} > \rho_{III}$ ), the Kelvin-Helmholtz instability in the presence of the gravitational field has a higher growth rate.

(5) For the streaming gravity mode instability, we only consider the initial equilibrium configuration of the current sheet which is described by

$$g \neq 0, \quad v_{II} = v_{III}, \quad B_{II} = -B_{III}, \quad S_{II} = S_{III}, \quad (44)$$

then, from Eq.(14) we have

$$\omega_{II}^* = \omega_{III}^*, \quad \alpha_{II} = \alpha_{III}. \quad (45)$$

Substituting Eqs.(44) and (45) into Eq.(21), the dispersion equation reduces to

$$\begin{aligned} & [\alpha_I \omega_I^* + \alpha_{II} \omega_{II}^* + gk(S_I - S_{II})] [\alpha_{II} \omega_{II}^* + \alpha_{IV} \omega_{IV}^* + \\ & + gk(S_{II} - S_{IV})] + [\alpha_I \omega_I^* - \alpha_{II} \omega_{II}^* + gk(S_I - S_{II})] \cdot \\ & \cdot [\alpha_{II} \omega_{II}^* - \alpha_{IV} \omega_{IV}^* - gk(S_{II} - S_{IV})] e^{-4k\delta} = 0. \end{aligned} \quad (46)$$

To discuss the effects of the gravity field on the streaming sausage and kink instabilities by an analytical method, we restrict ourselves to the study of the case in which the gravitational field is weak. Then the following approximation may be used

$$\alpha_i \omega_i^* = (\alpha_i \omega_i^*)_0 + \omega_i^2 g_i \quad (47)$$

where the subscript  $i$  represents I, II and III for the three regions, respectively.  $\omega_i$  is a small modification of the complex frequency  $\omega$  due to the presence of the gravitational field. Substituting Eq.(47) into Eq.(46), the zero-order equation which neglects the effect of the gravity may be written as

$$\begin{aligned} & [(\alpha_I \omega_I^*)_0 + (\alpha_{II} \omega_{II}^*)_0] [(\alpha_{II} \omega_{II}^*)_0 + (\alpha_{IV} \omega_{IV}^*)_0] + \\ & + [(\alpha_I \omega_I^*)_0 - (\alpha_{II} \omega_{II}^*)_0] [(\alpha_{II} \omega_{II}^*)_0 - (\alpha_{IV} \omega_{IV}^*)_0] e^{-4k\delta} = 0. \end{aligned} \quad (48)$$

The first-order equation which considers the effect of the gravity may be obtained

$$\begin{aligned} & [(\alpha_I \omega_I^*)_0 + (\alpha_{II} \omega_{II}^*)_0] [\omega_1^2 (S_{II} + S_{IV}) + gk(S_{II} - S_{IV})] + \\ & + [(\alpha_{II} \omega_{II}^*)_0 + (\alpha_{IV} \omega_{IV}^*)_0] [\omega_1^2 (S_I + S_{II}) + gk(S_I - S_{II})] + \\ & + [(\alpha_I \omega_I^*)_0 - (\alpha_{II} \omega_{II}^*)_0] [\omega_1^2 (S_{II} - S_{IV}) - gk(S_{II} - S_{IV})] + \\ & + [(\alpha_{II} \omega_{II}^*)_0 - (\alpha_{IV} \omega_{IV}^*)_0] [\omega_1^2 (S_I - S_{II}) + gk(S_I - S_{II})] e^{-4k\delta} = 0. \end{aligned} \quad (49)$$

For a perturbation with long wave length ( $k\delta \ll 1$ ), from Eq.(49) it may be obtained

$$\omega_1^2 = \frac{-gk(\alpha_{II} \omega_{II}^*)_0 (S_I - S_{IV})}{(\alpha_{II} \omega_{II}^*)_0 (S_I + S_{IV}) + S_{II} [(\alpha_I \omega_I^*)_0 + (\alpha_{IV} \omega_{IV}^*)_0]} \quad (50)$$

From Eq.(50), it can be shown that if  $(\alpha_I \omega_I^*)_0$ ,  $(\alpha_{II} \omega_{II}^*)_0$  and  $(\alpha_{IV} \omega_{IV}^*)_0$  are positive, then  $\omega_1^2 \sim -gk(\rho_I - \rho_{IV})$ . Therefore, the system is unstable when the direction of the gravity is opposite to the plasma density gradient ( $\rho_I > \rho_{IV}$ ). It reflects the effect of the gravity field on the streaming sausage and kink instabilities.

From the discussion above, it can be shown that the features of the Kelvin-Helmholtz instability and the streaming sausage and kink instabilities are modified in the presence of a gravitational field which is perpendicular to the magnetic field and the sheared plasma flow. They may evolve into a mixed Kelvin-Helmholtz gravity instability and mixed streaming gravity sausage and kink instabilities, which is simply called the streaming gravity mode instability. A detailed discussion of the growth rates and eigenmode structures of the streaming gravity mode for a continuously varying compressible plasma with a sheared flow in a gravitational field will be published in a separate paper.

#### ACKNOWLEDGMENTS

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