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LOCAL SUPERSYMMETRY IN NON-RELATIVISTIC SYSTEMS *

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ABSTRACT

Classical and quantum non-relativistic interacting systems invariant under local supersymmetry are constructed by the method of taking square roots of the bosonic constraints which generate timelike reparametrization, leaving the action unchanged. In particular, the square root of the Schrödinger constraint is shown to be the non-relativistic limit of the Dirac constraint. Contact is made with the standard models of Supersymmetric Quantum Mechanics through the reformulation of the locally invariant systems in terms of their true degrees of freedom. Contrary to the field theory case, it is shown that the locally invariant systems are completely equivalent to the corresponding globally invariant ones, the latter being the Heisenberg picture description of the former, with respect to some fermionic time.

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1. INTRODUCTION

Supersymmetric quantum mechanical [SSQM] models viewed as one-dimensional field theories, provide explicit realizations of the basic supersymmetry algebra [1]. The interest in such models has been mainly twofold: on the one hand we find the search for realistic applications of them [2], while on the other, they are considered merely as a simplified arena where new ideas in supersymmetry are generated, tested and subsequently generalized. A distinguished example of the latter point of view constitutes the introduction of the Witten index as a characterization for spontaneously broken supersymmetric theories [3].

Generally speaking, the construction of SSQM models has been characterized by the use of supercharges which are linear in the Grassman variables and which generate a global symmetry of the system. That is to say, the parameters of the induced transformation do not depend on time, being just constant Grassmann numbers. Having in mind the richer structure and interesting possibilities exhibited by systems possessing gauge supersymmetry, i.e. those coupled to supergravity [4], the question of how to extend the above mentioned quantum mechanical models to include time-dependent supersymmetry transformations naturally arises. Most of the work along these lines has been done in the realm of the relativistic point particle which then naturally includes the concept of the relativistic spining particle, because the additional Grassmann variables can be interpreted as intrinsic spin degrees of freedom[5]. There are some recent works which deal with the problem of constructing non-relativistic systems having local supersymmetry [6,7,8]. In this paper we present an alternative systematic method for such construction, which is based on the idea that the generators of time dependent supersymmetry, when considered as constraints on the system, are square roots of the constraints which generate the reparametrization invariance of the same system [9]. Within the context of quantum mechanics, this method has been applied mostly to the relativistic point particle where the reparametrization invariance of the original action is manifest [10]. The corresponding constraint is the Klein-Gordon operator while its square root, leading to the fermionic constraint, is the Dirac operator.

One can proceed along similar lines in the non-relativistic case by considering the free particle as a constrained system invariant under time reparametrization. As it is well-known this is achieved, for example, by introducing a new coordinate $t(\tau)$ and by rewriting the free particle action as [11]

$$S = \int_0^1 d\tau \, \frac{1}{2m} \frac{\dot{\vec{x}}^2}{\dot{t}} \tag{1}$$

The condition $\delta S = 0$ reproduces the usual dynamical equations if the coordinates $x(\tau), t(\tau)$ are kept fixed at the end points $\tau = 0, \tau = 1$. The canonical action is given by

$$S = \int_0^1 d\tau \, [p_0 \dot{t} + \vec{p}\cdot\dot{\vec{x}} - N\mathcal{H}] \tag{2}$$

where $p_0, \vec{p}$ are the momenta canonically conjugated to $t, \vec{x}$ respectively; N is a Lagrange multiplier and

$$\mathcal{H} = p_0 + \frac{1}{2m}\vec{p}^{\,2} \tag{3}$$

is a first class constraint. The two extra degrees of freedom, t and p_0, that we have added to phase space are removed by the constraint (3) upon gauge fixing. The action (2) is invariant under the local transformations generated by $\mathcal{H}$

$$\delta V = (V, \varepsilon(\tau)\mathcal{H}), \tag{4}$$

where V is any function of the canonical variables and (,) denotes the usual Poisson bracket. The Lagrange multiplier transformation is given by $\delta N = \dot{\varepsilon}(\tau)$ [12]. The parameter $\varepsilon(\tau)$ is restricted only at the end points by $\varepsilon(0) = \varepsilon(1) = 0$, as required by the action principle. Using the standard language, we refer to the above symmetry of the action as a gauge symmetry, even though one is dealing with a non internal symmetry.

Upon Dirac quantization, $\mathcal{H}$ becomes the Schrödinger operator and the constraint condition becomes the Schrödinger equation

$$\left(\frac{1}{i}\frac{\partial}{\partial t} + \frac{\vec{p}^{\,2}}{2m}\right)\psi = 0. \tag{5}$$

This can be naturally extended to the interacting case by redefining the constraint as

$$\mathcal{H} = p_0 + H \tag{6}$$

where H is now the full physical (gauge invariant) Hamiltonian

$$H = \frac{1}{2m}\vec{p}^{\,2} + V(\vec{x}). \tag{7}$$

The paper is organized as follows. In Section 2 we construct the square root of the Schrödinger operator in the free case, which is obtained by taking the non–relativistic limit of the Dirac operator, as suggested by the fact that the Schrödinger operator is the non–relativistic limit of the Klein–Gordon operator. In Section 3 the interacting case is discussed. It is shown that the gauge field associated to the invariance under local supersymmetry can be completely eliminated. This is in contrast with the case for a field theory, where local supersymmetry requires the introduction of a physical spin 3/2 field: the gravitino. In fact, one can see that the local supersymmetric Witten model reduces to the global one, in the appropriate coordinates through a finite supersymmetry rotation. This transformation can be interpreted as the passage from a Schrödinger to a Heisenberg picture with respect to some fermionic time. Finally, Section 4 contains a short summary and the conclusions.

2. THE SQUARE ROOT OF THE FREE SCHRÖDINGER EQUATION

In the Introduction we briefly reviewed how the Schrödinger equation can be understood as a bosonic constraint restricting the allowed states in a reparametrization invariant description

of a non-relativistic system. This is a useful remark that allows us to construct a locally supersymmetric action for a non-relativistic system. There is a standard procedure for constructing a locally supersymmetric extension of a bosonic system invariant under general coordinate transformations: the generators of the local supersymmetry transformations are the square roots of the bosonic constraints responsible for the invariance under general coordinate transformations [9].

In the case of the relativistic point particle such procedure starts from the Klein-Gordon equation as the original bosonic constraint and leads to the Dirac equation as the resulting fermionic constraint. Both constraints obey a closed graded supersymmetry algebra. Having in mind that the Schrödinger equation is the non-relativistic limit of the Klein-Gordon equation, we look for a fermionic constraint that can be obtained as the corresponding non-relativistic limit of the Dirac operator. We study first the non interacting case in order to determine the basic structure of the constraints.

Let us consider the following form of the Dirac equation

$$(i\gamma_5\gamma^\mu p_\mu + m\gamma_5)\psi = 0, \tag{8}$$

where $p_\mu = \frac{1}{i}\partial_\mu$, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $\eta^{\mu\nu} = diag(-,+,+,+)$, $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$, $\gamma_5^2 = -1$ and $\hbar = 1$. Using a representation for the Dirac matrices given by

$$\gamma^0 = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^i = i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma_5 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

we obtain the non-relativistic limit for equation (8)

$$\begin{pmatrix} \vec{\sigma}\cdot\vec{p} & p_0 \\ 2m & -\vec{\sigma}\cdot\vec{p} \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix} = 0. \tag{9}$$

where $p_0 = \frac{1}{i}\frac{\partial}{\partial t} \ll m$.

Defining

$$\hat{S} = -\frac{i}{2\sqrt{m}}\begin{pmatrix} \vec{\sigma}.\vec{p} & p_0 \\ 2m & -\vec{\sigma}.\vec{p} \end{pmatrix} \tag{10}$$

we easily verify that

$$\{\hat{S}, \hat{S}\} = -\left(\frac{\vec{p}^{\,2}}{2m} + p_0\right)I \equiv -\hat{\mathcal{H}}I, \tag{11a}$$

$$[\hat{\mathcal{H}},\ \hat{S}] = 0, \quad [\hat{\mathcal{H}}, \hat{\mathcal{H}}] = 0. \tag{11b}$$

Here I is the 4 x 4 identity matrix. The normalization of $\hat{S}$ is such that when we consider the classical theory according to the prescription

$$\frac{1}{i}[\hat{A}, \hat{B}]_\pm \rightarrow (A, B), \tag{12}$$

with $\hat{A} \to A$, the first relation in (11) reads $(S, S) = i\mathcal{H}$. In order to understand better the nature of the symmetries generated by $\hat{S}$, we look at the classical limit. Following Ref. [10] we introduce the new variables

$$\hat{\theta}^\mu = \frac{1}{\sqrt{2}} i\gamma_5\gamma^\mu, \qquad \hat{\theta}_5 = \frac{1}{\sqrt{2}}\gamma_5, \tag{13}$$

which allow us to rewrite the fermionic constraint as

$$\hat{S} = \frac{1}{\sqrt{2m}}(\vec{p}\cdot\vec{\hat{\theta}} + p_0(\frac{\hat{\theta}_5 + \hat{\theta}^0}{2}) + 2m(\frac{\hat{\theta}_5 - \hat{\theta}^0}{2})). \tag{14}$$

The classical limit is now obtained by considering the bosonic operators x^μ, p_μ as real numbers with the usual Poisson brackets relations, while the fermionic operators are replaced by real Grassmann variables θ^μ, θ_5. According to the prescription (12), the only non-vanishing Poisson brackets of θ^μ, θ_5 are

$$(\theta^\mu, \theta^\nu) = i\eta^{\mu\nu}, \qquad (\theta_5, \theta_5) = i. \tag{15}$$

The physical system that we are considering at this stage is the free, non-relativistic spinning point particle of mass m. The fact that a constrained description is employed means that the phase space contains extra coordinates, bosonic as well as fermionic, besides the dynamical ones $(\vec{x}, \vec{p}, \vec{\theta})$, and that the action is to be varied with respect to all of them. The classical version of the algebra (11) ensures that both constraints are first class and consequently we need two extra variables of each type: t, p_0 and θ^0, θ_5 respectively. The fermionic contribution to the kinetic part of the action is

$$\int_0^1 d\tau\, \frac{i}{2}(\dot{\vec{\theta}}\cdot\vec{\theta} - \dot{\theta}^0\theta^0 + \dot{\theta}_5\theta_5). \tag{16}$$

Introducing the following combinations of the extra fermionic variables

$$\theta_+ = \frac{1}{2}(\theta_5 + \theta^0), \qquad \theta_- = \theta_5 - \theta^0, \tag{17}$$

the corresponding kinetic term in the Lagrangian is

$$-\dot{\theta}^0\theta^0 + \dot{\theta}_5\theta_5 = 2\dot{\theta}_+\theta_- - \frac{d}{d\tau}(\theta_+\theta_-). \tag{18}$$

The complete action for our system, which generalizes (2), is then

$$\begin{aligned} S = \int_0^1 d\tau\, [&\dot{\vec{x}}\cdot\vec{p} + \dot{t}p_0 + \frac{i}{2}\dot{\vec{\theta}}\cdot\vec{\theta} + i\dot{\theta}_+\theta_- \\ &- N(\tau)(p_0 + \frac{\vec{p}^2}{2m}) - iM(\tau)\frac{1}{\sqrt{2m}}(\vec{\theta}\cdot\vec{p} + \theta_+p_0 + \theta_-m)] \\ &+ \frac{i}{2}[\vec{\theta}(0)\cdot\vec{\theta}(1) + (\theta_+(0) - \theta_+(1))(\theta_-(0) + \theta_-(1))], \end{aligned} \tag{19}$$

where $M(\tau)$ is a fermionic Lagrange multiplier. The variations at the end points are restricted so that the action be stationary under arbitrary changes of the coordinates around a classical trajectory for $0 < \tau < 1$ and they are

$$\delta b(0) = \delta b(1) = 0, \qquad \delta f(0) + \delta f(1) = 0, \tag{20}$$

where b and f denote the bosonic coordinates $\vec{x}, t$ and the fermionic coordinates $\vec{\theta}, \theta_+, \theta_-$ respectively. These conditions are chosen in such a way that they provide a unique solution for the equations of motion [10].

The action (19) possesses two kinds of local symmetries generated by

$$\mathcal{H} = p_0 + \frac{\vec{p}^2}{2m}, \tag{21a}$$

$$S = \frac{1}{\sqrt{2m}}(\vec{\theta}\cdot\vec{p} + \theta_+p_0 + \theta_-m), \tag{21b}$$

respectively. The local supersymmetry transformations of the dynamical variables are induced by S and are given by

$$\begin{aligned} &\delta x^i = \frac{1}{\sqrt{2m}}\eta\theta^i, \qquad \delta p_i = 0, \\ &\delta t = \frac{1}{\sqrt{2m}}\eta\theta_+, \qquad \delta p_0 = 0, \\ &\delta\theta^i = -\frac{i}{\sqrt{2m}}\eta p_i, \\ &\delta\theta_+ = -i\sqrt{\frac{m}{2}}\eta, \\ &\delta\theta_- = -\frac{i}{\sqrt{2m}}p_0\eta, \\ &\delta N = iM\eta, \qquad \delta M = \dot{\eta}, \end{aligned} \tag{22}$$

where $\eta = \eta(\tau)$ is an arbitrary local Grassmanian parameter restricted by $\eta(0) = \eta(1) = 0$ because of the fixed end points conditions.

The action (19) can be rewritten in second order form by eliminating $\vec{p}, p_0, N$ in favour of the velocities, from the following equations of motion

$$N = \dot{t} - \frac{iM\theta_+}{\sqrt{2m}}, \tag{23a}$$

$$p_0 = -\frac{\vec{p}^2}{2m}, \tag{23b}$$

$$\frac{p_i}{m} = \frac{\dot{x}^i}{\dot{t}} + \frac{iM}{\sqrt{2m}}(\frac{\dot{x}^i}{\dot{t}^2}\theta_+ - \frac{\theta^i}{\dot{t}}). \tag{23c}$$

There is no conflict between the already prescribed end point conditions and the defining relations (23). Substituting (23) in (19) we obtain

$$S = \int_0^1 d\tau \, (\frac{m}{2}\frac{\dot{\vec{x}}^2}{\dot{t}} + \frac{i}{2}\dot{\vec{\theta}}\cdot\vec{\theta} + i\dot{\theta}_+\theta_- + iM\sqrt{\frac{m}{2}}(-\frac{\dot{\vec{x}}\cdot\vec{\theta}}{\dot{t}} + \frac{\dot{\vec{x}}^2}{2\dot{t}^2}\theta_+ - \theta_-)) + B.T. \tag{24}$$

This result coincides with the starting point taken if Ref.[8] to discuss the free spining non-relativistic particle.

It is sometimes found in the literature that $e = \dot{t}$ is considered as an independent coordinate [6, 7]. This however is incorrect unless e is also the Lagrange multiplier that goes together with the generator of time reparametrizations in the canonical action.

In closing let us identify the true degrees of freedom of the theory described by the action (19). This can be done by rewriting the action of the system in terms of supersymmetric invariant quantities. In order to achieve this, the variables p_0, N, M are substituted in the action (19) using Eqs. (23a,b) and the one obtained by varying θ_-. The result is

$$S = \int_0^1 d\tau \, [\dot{\vec{x}}\cdot\vec{p} - \frac{\vec{p}^2}{2m}\dot{t} + \frac{i}{2}\dot{\vec{\theta}}\cdot\vec{\theta} - \frac{i}{m}\dot{\theta}_+(\vec{\theta}\cdot\vec{p} - \frac{\vec{p}^2}{2m}\theta_+)] + B.T. \tag{25}$$

We observe that θ_- has dropped out from the action. The only reference to this variable remains in the boundary term through the combination $\alpha_- = \theta_-(0) + \theta_-(1)$. When the action (25) is varied with respect to θ_+, we find the boundary contribution

$$-i\frac{\delta\theta_+}{m}(\vec{\theta}\cdot\vec{p} - \frac{\vec{p}^2}{2m}\theta_+)|_0^1 + \frac{i}{2}(\delta\theta_+(0) - \delta\theta_+(1))\alpha_- \tag{26}$$

which must be set equal to zero. Recalling the end point restriction $\delta\theta_+(0) + \delta\theta_+(1) = 0$ we obtain

$$\delta\theta_+(1)[\theta_-(1) + \theta_-(0) - \alpha_-] = 0. \tag{27}$$

Here θ_- is a shorthand notation for

$$-\frac{1}{m}(\vec{\theta}\cdot\vec{p} - \frac{\vec{p}^2}{2m}\theta_+). \tag{28}$$

It would have been incorrect to consider (28) as a definition of θ_- valid for the whole history of the system because the condition $\delta(\theta_-(1) + \theta_-(0)) = 0$ is not recovered from (28). Nevertheless, impossing the correct boundary condition (27) effectively means that we are using such definition, but only at the end points.

Going back to the action (25) we notice that it is invariant under reparametrization and also under the following local supersymmetry transformations

$$\begin{aligned} \delta\vec{x} &= \frac{i}{m}\eta(\tau)(\vec{\theta} - \vec{p}\frac{\theta_+}{m}), \\ \delta\vec{p} &= 0, \\ \delta t &= 0, \\ \delta\vec{\theta} &= \eta(\tau)\frac{\vec{p}}{m}, \\ \delta\theta_+ &= \eta(\tau). \end{aligned} \tag{29}$$

as can be verified by direct substitution. The above transformations are generated by $s = \Pi_+ + i(\vec{\theta}\cdot\vec{p} - \frac{\vec{p}^2}{2m}\theta_+)$ which arises as a constraint $s \simeq 0$ from the action (25). This results when Π_+, the momentum canonically conjugated to θ_+, is defined to be the left derivative of the Lagrangian in (25) with respect to $\dot{\theta}_+$, as it is usually done. The constraint s must be proportional to S, with $p_0 = -\frac{\vec{p}^2}{2m}$ in the latter. This allows us to identify $\Pi_+ = i\theta_-$ together with the Poisson bracket $(\theta_+, \Pi_+) = -1$. Let us remark that Π_+ is purely imaginary with the conventions adopted.

Let us now introduce the combinations

$$\begin{aligned} \vec{X} &= \vec{x} - \frac{i}{m}\theta_+\vec{\theta}, \\ \vec{\Theta} &= \vec{\theta} - \frac{\vec{p}}{m}\theta_+, \end{aligned} \tag{30}$$

which are invariant under the local transformations (29). In terms of these new variables, the action (25) is rewritten as

$$S \equiv \int_0^1 d\tau \, [\dot{\vec{X}}\cdot\vec{p} - \frac{\vec{p}^2}{2m}\dot{t} + \frac{i}{2}\dot{\vec{\Theta}}\cdot\vec{\Theta}] + B.T. \tag{31}$$

which corresponds to the first order formulation of non-interacting three-dimensional bosonic and fermionic degrees of freedom and where all information concerning local supersymmetry invariance is lost.

3. LOCAL SUPERSYMMETRIC SYSTEMS WITH INTERACTIONS

In this section we extend the previous results in order to include interactions which will produce after quantization, the locally invariant generalization of the standard supersymmetric quantum mechanics [1]. To do this we redefine the fermionic constraint (21b) in the form

$$S = \frac{1}{\sqrt{2m}}(Q + p_0\theta_+ + m\theta_-) \simeq 0, \tag{32}$$

where the supercharge Q is required to be a function of the variables $\vec{x}, \vec{p}, \vec{\theta}$ only. In the general case we would like to maintain the basic square root relation between S and $\mathcal{H}$, where the constraint

$$\mathcal{H} = p_0 + H(\vec{x}, \vec{p}, \vec{\theta}) \simeq 0 \tag{33}$$

includes interactions now. The already assumed dependence of Q upon the dynamical variables implies that the supercharge Q turns out to be the square root of the physical Hamiltonian H. That is

$$H = \frac{1}{2im}(Q, Q) \,. \tag{34}$$

As a consequence of the Jacobi identity and the equation (34) we obtain

$$(Q, H) = 0 \,. \tag{35}$$

The above equation implies that the constraints S and $\mathcal{H}$ remain first class, exactly as in the non-interacting case.

We recognize the basic structure of Witten supersymmetric quantum mechanics [13] in the equations (34) and (35). It is now possible to promote this supersymmetry to a local one generated by S.

The above construction can also be generalized to include N fermionic constraints S_a, $a = 1, \dots, N$ which are required to satisfy the following algebra

$$(S_a, S_b) = i\delta_{ab}\mathcal{H}, \quad (S_a, \mathcal{H}) = 0 \,. \tag{36}$$

A realization of such algebra is given by extending the definition (32) in an obvious manner,

$$S_a = \frac{1}{\sqrt{2m}}(Q_a + p_0\theta_{+a} + m\theta_{-a}), \tag{37}$$

with Q_a still being function of $\vec{x}, \vec{p}, \vec{\theta}$ only. The N supercharges Q_a are such that

$$\begin{aligned} (Q_a, Q_b) &= 2im\delta_{ab}H \\ (Q_a, H) &= 0 \end{aligned} \tag{38}$$

which again guarantee that S_a and $\mathcal{H}$ are first class constraints.

We will further comment on Eqs. (38), after we discuss the one dimensional locally invariant Witten model as a particular realization of the previous ideas. The Witten model [13] corresponds to $N = 2$, with

$$\begin{aligned} Q_1 &= \theta_1 p + \theta_2 V(x), \\ Q_2 &= \theta_2 p - \theta_1 V(x), \end{aligned} \tag{39}$$

leading to the physical Hamiltonian

$$H = \frac{1}{2}(p^2 + V^2 + 2i\theta_1\theta_2 V'), \tag{40}$$

with $V' = \frac{dV}{dx}$ and $m = 1$.

The action for the system is

$$\begin{aligned} S = \int_0^1 d\tau\, [\, &\dot{q}p + \dot{t}p_0 + \frac{i}{2}\dot{\theta}\theta + \dot{\theta}_{+a}\theta_{-a} \\ &- N\mathcal{H} - M_a S_a] + B.T. \end{aligned} \tag{41}$$

This action possesses the same two local gauge invariances as the free case: i) under the transformations generated by S_a and ii) under time reparametrizations generated by $\mathcal{H}$. This fact, together with the explicit form (40) of the physical Hamiltonian allows us to interpret the action (41) as describing the original Witten model having its global supersymmetry promoted to a gauge supersymmetry.

In order to distinguish the true dynamical degrees of freedom from the gauge variables associated to local supersymmetry, we proceed in complete analogy to the free case and eliminate the variables N, p_0 and M_a from (41). The result is

$$\begin{aligned} S = \int_0^1 d\tau\, [\, &\dot{q}p - H\dot{t} + \frac{i}{2}\dot{\theta}\theta \\ &- i\dot{\theta}_{+a}(Q_a - H\theta_{+a})] + B.T. \end{aligned} \tag{42}$$

Again the variables θ_{-a} automatically drop out from the action and the corresponding boundary conditions α_{-a} are correctly recovered in the same manner as described following Eq.(25).

The action (42) is invariant under the local supersymmetry transformations

$$\begin{aligned} \delta x &= i\varepsilon_a(\theta_a - p\theta_{+a}), \\ \delta p &= i\varepsilon_a(-\bar{\theta}_a V' + \frac{\partial H}{\partial x}\theta_{+a}), \\ \delta t &= 0, \\ \delta\theta_a &= \varepsilon_a p - \bar{\varepsilon}_a V + iV'\bar{\theta}_a\varepsilon_b\theta_{+b}, \end{aligned} \tag{43}$$

generated by the constraints $s_a = \Pi_{+a} + i(Q_a - H\theta_{+a})$ arising from the action (42) in a manner similar to the non-interacting case discussed after Eq.(29). Here $\varepsilon_a = \varepsilon_a(\tau)$ and $\bar{\theta}_a = \varepsilon_{ab}\theta_b$ with $\varepsilon_{12} = -\varepsilon_{21} = 1$. Now we can introduce new variables (denoted by the corresponding capital letter) which are invariant under the transformations (43) and that generalize the analogous expressions of the free case given in (30). They are

$$\begin{aligned} X &= x - i\theta_{+a}\theta_a - iV\theta_{+1}\theta_{+2}, \\ P &= p(1 - iV'\theta_{+1}\theta_{+2}) + iV'(\theta_1\theta_{+2} - \theta_2\theta_{+1}), \\ \Theta_a &= \theta_a - p\theta_{+a} + V\bar{\theta}_{+a} + iV'\theta_a\theta_{+1}\theta_{+2} \end{aligned} \tag{44}$$

and can be shown to satisfy the Poisson brackets corresponding to bosonic and fermionic canonical degrees of freedom. In terms of these variables, the action (42) reduces to

$$S = \int_0^1 d\tau \ \ [\dot{X}P - \tilde{H}\dot{t} + \frac{i}{2}\dot{\Theta}_a\Theta_a] \\ + \frac{i}{2}\Theta_a(1)\Theta_a(0), \qquad (45)$$

where $\tilde{H} \equiv H(x(X,P,\Theta), p((X,P,\Theta), \theta(X,P,\Theta))$. The result is

$$\tilde{H} = \frac{1}{2}(P^2 + V^2(X) + 2i\Theta_1\Theta_2 V'(X)). \qquad (46)$$

Let us observe that the functional form of $\tilde{H}(X,P,\Theta)$ is the same as that of $H(x,p,\theta)$.

If the action (45) is deparametrized, setting $\dot{t} = 1$ in the Lagrangian, the result is the canonical first order action of and unconstrained system with phase space coordinates $X(t), P(t), \Theta_a(t)$ and Hamiltonian given by (46). At this level of the description, all reference to the previous invariance under local supersymmetry is lost and only a global supersymmetry remains. This global invariance cannot be considered as a restriction of the local supersymmetry because, for example, that would be inconsistent with the fixed coordinate conditions at the end points. The algebra of the global symmetry, however, is isomorphic to (36)

$$(\tilde{Q}_a, \tilde{Q}_b) = i\delta_{ab}\tilde{H}, \qquad (47a)$$

$$(\tilde{Q}_a, \tilde{H}) = 0, \qquad (47b)$$

where $\tilde{Q}_a = Q_a - 2iH\theta_{+a}$ is obtained by replacing $x^i \to X^i, p_i \to P_i, \theta^i \to \Theta^i$ in Q_a. It would not surprise that under the change of variables (44) H is form invariant, whereas Q_a is not. The reason is that (44) is a canonical transformation (generated by Q_a) and H commutes with its generator, while Q_a does not. Now it is easy to see that the supersymmetry transformations generated by $\tilde{Q}_a$ on each variable produce a net shift at the end points of the action (45), as appropriate to a global symmetry. These transformations are precisely those of global supersymmetry in the Witten model.

Nevertheless, our calculation has shown that the local supersymmetric Witten model is just a parametrized form of the globally invariant one, and hence they are physically equivalent. In order to understand this point further let us discuss the quantum version of the theory described by the action (41). For simplicity we will assume only one supercharge Q ($a = 1$) so that the classical properties previously discussed will have a more transparent origin. We proceed in the most direct way by changing the Poisson brackets into commutators or anticommutators and by imposing the first class constraints as null conditions upon the wave function which depends on the real coordinates t, x, θ, θ_+. We recall that θ, θ_+ are Grassmann numbers with the derivatives taken from the left and that they anticommute (commute) with every fermionic (bosonic) operator. The wave function then satisfies

$$(\hat{Q} + \hat{p}_0\theta_+ + m\frac{\partial}{\partial\theta_+})\psi = 0, \qquad (48a)$$

$$(\hat{p}_0 + \hat{H})\psi = 0, \qquad (48b)$$

with the Schrödinger equation (48b) being a consequence of equation (48a).

The fact that in the classical case we were able to effectively eliminate the coordinate θ_+ has as counterpart here the fact that equation (48a) has the general solution

$$\psi = e^{\frac{-\theta_+\hat{Q}}{m}}\phi(x,\theta,t). \qquad (49)$$

Substituting (49) in (48a) we obtain

$$\theta_+(\hat{p}_0 + \hat{H})\phi = 0, \qquad (50)$$

which is satisfied in terms of the Schrödinger equation for ϕ obtained from (48b).

The solution (49) suggests that the coordinate θ_+ can be interpreted as a kind of fermionic time whose dynamics is governed by the evolution operator $\hat{Q}$ [14]. In this sense, the description of the system in terms of ψ or ϕ is like going from the Schrödinger to the Heisenberg picture with respect to this fermionic time. To carry the analogy further we can shift the θ_+ independent operators $\hat{a}$ to their Heisenberg picture expression by means of the unitary transformation

$$\hat{A}(\theta_+) = e^{\frac{+\theta_+\hat{Q}}{m}}\hat{a}e^{\frac{-\theta_+\hat{Q}}{m}}. \qquad (51)$$

This expression is the quantum mechanical analogue of the classical variables, invariant under local supersymmetry introduced in (44). It should be stressed that this supersymmetry is the one generated by $\hat{s} = \hat{\Pi}_+ + i(\hat{Q} - i\hat{H}\hat{\theta}_+)$. Using the explicit expression (51) it is possible to verify that $\delta\hat{A} = [\hat{A}, \eta(\tau)\hat{s}]$ is indeed equal to zero for any bosonic or fermionic operator.

The quantum analogue of the description of the system in terms of the classical action given by Eqs.(45) and (46) for the N = 1 case corresponds to the use of a full Heisenberg representation for both the bosonic and fermionic times t and θ_+ respectively. The transition from the Schrödinger picture $(\psi(x,\theta,\theta_+,t), \hat{a})$ to the Heisenberg picture $(\tilde{\phi}(x,\theta), \tilde{A}(t,\theta_+))$ is achieved by means of the following unitary transformation

$$\tilde{\phi}(x,\theta) = V^\dagger\psi(x,\theta,t,\theta_+), \\ \tilde{A}(t,\theta_+) = V\hat{a}V^\dagger \qquad (52)$$

where

$$V = e^{i\hat{H}t}e^{\frac{-\theta_+\hat{Q}}{m}}. \qquad (53)$$

The wave function $\tilde{\phi}(x,\theta)$ describes the initial condition of the system at $t = 0$ and $\theta_+ = 0$ and the arguments x, θ are the eigenvalues of the operators $\mathring{X}, \mathring{\Theta}$ at this particular instant. The dynamical evolution is now fully contained in the operators $\tilde{A}$ which are the quantum analogs of the classical canonical variables $A(t,\theta_+)$ introduced in Eq.(44). In particular, the physical evolution with respect to the real time t is given by the Hamiltonian

$$\mathring{H} = e^{-i\hat{H}t}e^{\frac{\theta_+\hat{Q}}{m}}\hat{H}(x,p,\theta)e^{\frac{-\theta_+\hat{Q}}{m}}e^{i\hat{H}t}. \qquad (54)$$

By explicit application of Eq. (52), we conclude that $\hat{H} = \hat{H}(\hat{X}, \hat{P}, \hat{\Theta})$. On the other hand, using $[\hat{Q}, \hat{H}] = 0$, we obtain $\hat{H} = \hat{H}(x, p, \theta)$. In this way we have recovered in quantum mechanics the form invariance of $\hat{H}$ and $\hat{H}$ which we found previously at the classical level.

The above exercise using a simple model with one supercharge shows that given a globally invariant supersymmetric theory (which means to know the corresponding algebra (38)) it is always possible to construct a locally invariant extension of it by taking the square root of the Schrödinger operator in the form given by Eq.(37). Nevertheless, both theories are in fact the same because the locally invariant extension turns out to be just a Heisenberg picture description of the globally invariant original theory. This result can be easily extended to the general case of a system possessing N–global supersymmetry and any number of physical degrees of freedom. Again, the locally invariant extension is constructed by adding $2N$ extra variables θ_{+a}, θ_{-a} and by defining the generators of local supersymmetry as in Eq.(37). When quantizing the system, there will be one equation of the form (48a) for each of the N constraints and the Schrödinger equation (48b) will be the square of any one them. The general solution of the N fermionic equations will be again of the form (49) with $\theta_+\hat{Q}$ replaced by $\theta_{+a}\hat{Q}_a$ and ϕ depending on the original physical degrees of freedom only. The reason why this method works is that the graded algebra (38) implies $[\theta_{+a}\hat{Q}_a, \theta_{+b}\hat{Q}_b] = 0$ for fixed a and b with $a \neq b$. Now, the local extension of the original theory will correspond to a Heisenberg picture defined by N fermionic times θ_{+a}.

4. CONCLUSIONS

We have presented a systematic way to construct non-relativistic systems having bosonic and fermionic degrees of freedom $(\vec{x}, \vec{p}, \vec{\theta})$ which are invariant under local supersymmetry transformations. This is achieved by reformulating the original bosonic problem in terms of a gauge type action which is invariant under time reparametrizations, in analogy with the relativistic point particle. The associated constraint is the Schrödinger operator. Subsequently, and according to refs. [9, 10], local supersymmetry is introduced by constructing the corresponding generators S_a as square roots of the Schrodinger constraint in the sense of satisfying the graded algebra of first class constraints given in Eq. (36). These generators are obtained as the non-relativistic limit of the Dirac operator and hence, local supersymmetry is understood as the non-internal gauge symmetry associated with these fermionic constraints. Consequently, the fermionic phase space had also to be enlarged by adding the variables θ_{+a} and θ_{-a} for each supersymmetry generator S_a considered, in complete analogy with the bosonic situation.

The basic structure underlying local supersymmetry is understood from the analysis of the free non–relativistic spinning particle. An important point that already arises from the non–interacting situation is that a reparametrization invariant description of non–relativistic systems is correctly achieved only in terms of a velocity $\dot{t}$, according to Eq.(1) for example, instead of a coordinate e as sometimes done in the literature.

The construction is extended to include interactions by redefining the constraints S_a (c.f. Eq.(37)) via the introduction of the supercharges Q_a which are assumed to depend upon the physical degrees of freedom only. The closure of the graded algebra of first class constraints $\mathcal{H}$, S_a then required that the physical Hamiltonian H and the supercharges Q_a satisfy the graded algebra of standard SSQM. This algebra is the usual starting point to discuss global supersymmetry and in our approach is recovered from the locally invariant formulation.

In particular, the local supersymmetric version of the Witten model is analyzed (c.f. Eq.(41)) in order to elucidate better the relation between local and global supersymmetry. The basic idea is to identify the true degrees of freedom of the model by rewriting the action in terms of canonical variables invariant under local supersymmetry. After this is done, a first order action corresponding precisely to the global Witten model described by the physical Hamiltonian $\tilde{H} = \frac{1}{2i}(\tilde{Q}_1, \tilde{Q}_1) = \frac{1}{2i}(\tilde{Q}_2, \tilde{Q}_2)$ is found. Besides, in terms of these new variables, the Hamiltonian $\tilde{H}$ has the same functional form as the one appearing in the original constraint $\mathcal{H}$. All reference to local supersymmetry is then lost and the extra variables introduced previously are completely eliminated from the action.

The above result is an indication that in the non-relativistic case, any system invariant under local supersymmetry is equivalent to the corresponding globally invariant one constructed by using gauge invariant degrees of freedom. In order to test this idea at the quantum level we finally considered the quantization of a simple model with constraints $\mathcal{H}$ and S. In this case, the relation between the locally invariant description and the globally invariant one is achieved by means of a unitary transformation which can be interpreted as describing the passage from a Schrödinger to a Heisenberg picture defined with respect to the real time t together with a ficticious fermionic time θ_+. The new operators in this Heisenberg picture correspond to the canonical variables invariant under local supersymmetry introduced at the classical level. Naturally, the form invariance of the respective Hamiltonians is a direct consequence of the unitary transformation involved. This argument was generalized to arbitrary global systems and no inconsistencies previously reported [6] were found.

The general point of view taken in this article starts with the well–known assertion that any bosonic Lagrangian theory can be written in a parametrized form, for instance introducing an extra time parameter τ as it is done here, or extra space–time coordinates τ, σ^i. The resulting theory naturally possesses a gauge invariance corresponding to the group of reparametrizations and therefore has a set of first class constraints satisfying a closed algebra. The results obtained in the case of SSQM discussed in this article suggest that the square root of such constraints will give rise to a local supersymmetric theory that is merely equivalent to a globally invariant one. The passage from the local to the global theory could then be obtained by deparametrizing the former (e.g. setting $t = \tau, \theta_+ = 0$). This deparametrization cannot be achieved by a gauge transformation (it is not a gauge choice) and corresponds to a canonical transformation to gauge invariant (physical) coordinates in phase space. In quantum mechanics this transformation is seen to correspond to a unitary transformation that can be interpreted as the change from a Schrödinger to a Heisenberg picture in some fermionic time.

Clearly, local and global supersymmetry give rise to different dynamical systems if the former theory could not be deparametrized. This happens, for instance, when the parameters are coordinates on a manifold with dynamics of its own, i.e. strings, membranes, gravity, etc. This explains why in the case of a point particle the two theories are equivalent: the world line of the particle is flat and can sustain no geometrodynamics.

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