IC/89/287

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

LINKING NUMBERS AND VARIATIONAL METHOD

Ichiro Oda

and

Shigeaki Yahikozawa

INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

1989 MIRAMARE - TRIESTE

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

LINKING NUMBERS AND VARIATIONAL METHOD *

Ichiro Oda ** and Shigeaki Yahikozawa
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

The ordinary and generalized linking numbers for two surfaces of dimension p and $n-p-1$ in an n dimensional manifold are derived. We use a variational method based on the properties of topological quantum field theory in order to derive them.

MIRAMARE – TRIESTE
September 1989

* Submitted for publication.
** On leave of absence from: Department of Physics, Chiba University, 1-33 Yayoi-cho, Chiba 260, Japan.

Topological quantum field theories (TQFT's) have recently received much attention. It is now believed that the language of TQFT's provides a beautiful framework for both theoretical physics and mathematics. First, Schwarz has constructed a TQFT and identified the Ray-Singer analytic torsion with a partition function about ten years ago [1]. Then, Witten [2] has constructed a new class of TQFT's in three and four dimensions which are closely related to the Jones polynomials of knot theory [3], Floer cohomology groups of three dimensional manifolds [4] and Donaldson polynomials of four dimensional manifolds [5].

Recently, Horowitz and Srednicki have shown that the two-point function of TQFT of Schwarz produces the linking numbers of two surfaces of dimension p and n-p-1 in an n dimensional manifold [6]. Their derivation method of the linking numbers is heavily based on the Hodge theorem and some results of the eigenform of the Laplacian. Also, other authers deal with the same topic [7].

In this paper, we not only rederive their results [6],[7] in the abelian case but also extend to the nonabelian one and a new class of higher dimensional TQFT's [8] by making use of a variational method based on the properties of TQFT. Actually, the variational method has already proved to be useful for a direct derivation of link invariants in three dimension without relying on the two dimensional conformal field theory [9],[10]. By this direct proof, we could have clear understanding of the equivalence between the three dimensional Chern-Simons action and link polynomials. Therefore we summarize the variational method of Ref.[9] and [10] briefly and make comments on the difference with our variational method. Their method uses the fundamental identity of the Chern-Simons action

$$\frac{\delta S}{\delta A_\mu^a(x)} = \frac{k}{8\pi}\varepsilon^{\mu\nu\rho}F_{\nu\rho}^a(x)$$

and the well-known property of the Wilson line operator under an infinitesimal variation of the path in x produces the $F_{\mu\nu}$ insertion, namely

$$U(x_1,x_2) \longrightarrow U(x_1,x) \cdot \Sigma^{\mu\nu} F^a_{\mu\nu}(x) U(x,x_2)$$

where $U(x_1,x_2) = P\exp\{\int_{x_1}^{x_2} dx^\mu A^a_\mu T^a\}$ is the Wilson line operator associated to the path connecting two arbitrary points x and x . Our variational method shown below is slightly different from one of Ref.[9] and [10], since even if we use a similar identity, we never think of deformation of the Wilson line. Instead, we use the variational method for the direct evaluation of two-point function as we will see later.

We begin by rewritng the mathematical definition of the linking number concisely [11]. Let M be a compact, oriented n dimensional manifold without boundary. Let U and V be disjoint, compact and oriented submanifolds of M with dimension p and n-p-1, respectively. We assume that U and V are boundaries of surfaces of one higher dimension, namely, $U=\partial Y$ and $V=\partial W$. Mathematically, such manifolds are called homologically trivial. We will see later that this condition is essential for the derivation of the linking number. The linking number of U to V, which we write L(U,V), is defined as follows; When U and W intersect in a finite number of points p., the weighted sum over p. with +1 or -1 according to a suitable orientation convention is the linking number L(U,V). Roughly speaking, one can interpret the linking number as the intersection number. One can also define the linking number of V to U, which we write L(V,U). This is the weighed sum over intersection points between V and Y. Then, L(U,V) is bilinear with respect to U and V and has an important non-commutative relation

$$L(U,V) = (-)^{p(n-p-1)+1} L(V,U) \qquad (1)$$

which implies that for instance the linking number $L(S^2,S^2)$ in S^5 is vanishing. Examples

may be useful to help to illustrate the linking number and relation (1) (see Fig.1).

Now we consider the following abelian TQFT [1],[6],[7]

$$S = \int_M B \wedge dC \qquad (2)$$

where B is a p form and C is an n-p-1 form on an n dimensional manifold M. The metric does not appear, thus the action is manifestly invariant under diffeomorphism. In addition, this action has a gauge symmetry $B \to B+dv$ and $C \to C+dw$ where v and w are p-1 and n-p-2 forms,respectively. The equations of motion are dB=0 and dC=0. Let us consider the two-point function

$$\langle \int_U B \int_V C \rangle = \frac{1}{Z} \int \mathcal{D}B \mathcal{D}C \int_U B(x) \int_V C(y)\, e^{iS} \qquad (3)$$

where $Z = \int \mathcal{D}B \mathcal{D}C\, e^{iS}$ is an usual partition function [6]. Since the expression inside the expectation value is clearly gauge invariant and metric independent, the result appears to be a topological invariant of the surfaces U and V. However one might worry the following two things. First, the functional measure needs a metric, so the result may have anomalous metric dependence, which is easily solved in our formalism as we will see later. Second, the two-point function (3) is ill-defined owing to the gauge invariance, thus we have to gauge fix in order to make this expression well-defined. Here we should notice that the action (2) has a reducible gauge symmetry, namely, $v \to v+dv'$, $w \to w+dw'$ and $v' \to v'+dv''$, $w' \to w'+dw''$ etc. Of course, this system has not an infinite but a finite number of reducibility once one fix the dimension n of manifold M. Horowitz and Srednicki have solved these both problems by making use of the Hodge theorem and some results of the eigenform of the Laplacian [6].

We will use a variational method. As a first step, we do the gauge fixing of the action (2) in a covariant gauge. The gauge fixed and BRS invariant action becomes

$$S_{fix} = \int_M [B \wedge dC + N \wedge d*B + L \wedge d*C + d\bar{\alpha} \wedge * d\alpha + d\bar{\beta} \wedge * d\beta + \{ ---- \}] \tag{4}$$

where N and L are Nakanishi-Lautrup fields with form p-1 and n-p-2, and α and β are ghosts with form p-1 and n-p-2, and $\bar{\alpha}$ and $\bar{\beta}$ are the corresponding antighosts with form p-1 and n-p-2, respectively. And * means the Hodge dual operation which contains a metric but is needed to set the covariant gauge. And also, the dotted lines inside the curly bracket denote terms of ghosts, antighosts and Nakanishi-Lautrup fields from the reducibility of gauge symmetry a la Batalin- Vilkovisky [12]. (We omitted to write these terms explicitly, since their concrete forms are not necessary for later discussion.)

As a second step, we derive the variational identities in a component expression,

$$\begin{aligned} \frac{\delta S_{fix}}{\delta B_{\mu_1 \cdots \mu_p}(x)} &= \varepsilon^{\mu_1 \cdots \mu_n} \partial_{\mu_{p+1}} C_{\mu_{p+2} \cdots \mu_n}(x) \\ &\quad - p! \, \partial^{\mu_p} \{ \sqrt{g} \, N^{\mu_1 \cdots \mu_{p-1}}(x) \} \\ &= \varepsilon^{\mu_1 \cdots \mu_n} \partial_{\mu_{p+1}} C_{\mu_{p+2} \cdots \mu_n}(x) \\ &\quad + \{ Q, \partial^{\mu_p} \bar{\alpha}^{\mu_1 \cdots \mu_{p-1}}(x) \} \end{aligned} \tag{5a}$$

$$\begin{aligned} \frac{\delta S_{fix}}{\delta C_{\mu_1 \cdots \mu_{n-p-1}}(x)} &= (-)^{(p+1)(n-p)} \varepsilon^{\mu_1 \cdots \mu_n} \partial_{\mu_{n-p}} B_{\mu_{n-p-1} \cdots \mu_n}(x) \\ &\quad - (n-p-1)! \, \partial^{\mu_{n-p-1}} (\sqrt{g} \, L^{\mu_1 \cdots \mu_{n-p-2}}(x)) \\ &= (-)^{(p+1)(n-p)} \varepsilon^{\mu_1 \cdots \mu_n} \partial_{\mu_{n-p}} B_{\mu_{n-p-1} \cdots \mu_n}(x) \\ &\quad + \{ Q, \partial^{\mu_{n-p-1}} \bar{\beta}^{\mu_1 \cdots \mu_{n-p-2}}(x) \} \end{aligned} \tag{5b}$$

where Q is BRS charge, and we have used the fact that BRS transformation of antighosts becomes Nakanishi-Lautrup fields in the present formalism.

As a third step, we use one of the variational identities (5a), the part integral and the Stokes theorem as follows. Consider the equation

$$\begin{aligned} &\langle B(x) \, dC(y) \rangle \\ &\equiv \frac{1}{Z} \int [\mathcal{D}X] \, B(x) \, dC(y) \, e^{i S_{fix}} \\ &\equiv \frac{1}{Z} \int [\mathcal{D}X] \, B_{\mu_1 \cdots \mu_p}(x) \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \\ &\qquad \otimes \partial_{\mu_{p+1}} C_{\mu_{p+2} \cdots \mu_n}(y) \, dx^{\mu_{p+1}} \wedge \cdots \wedge dy^{\mu_n} \, e^{i S_{fix}} \end{aligned} \tag{6}$$

where $[\mathcal{D}X]$ denotes the functional measure $\mathcal{D}B \mathcal{D}C \mathcal{D}N \mathcal{D}L \mathcal{D}\alpha \mathcal{D}\beta \mathcal{D}\bar{\alpha} \mathcal{D}\bar{\beta} \cdot \mathcal{D}$ (other reducible ghosts, antighosts and NL fields) and $\otimes$ denotes tensor product. Using the identity (5a) except the last term $\{Q, d\bar{\alpha}\}$, which we will specifically take account later

$$\begin{aligned} &\langle B(x) \, dC(y) \rangle \\ &= \frac{1}{Z} \int [\mathcal{D}X] \, \frac{(-)^{p(n-p)}}{p!(n-p)!} B_{\mu_1 \cdots \mu_p}(x) \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \\ &\qquad \otimes \varepsilon_{\mu_{p+1} \cdots \mu_n \nu_1 \cdots \nu_p} \left(\frac{1}{i} \frac{\delta}{\delta B_{\nu_1 \cdots \nu_p}(y)} e^{i S_{fix}} \right) dy^{\mu_{p+1}} \wedge \cdots \wedge dy^{\mu_n} \end{aligned}$$

$$= i \frac{1}{p!(n-p)!} \delta^{(n)}(x-y) \; \epsilon_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$\wedge dy^{\mu_{p+1}} \cdots \wedge dy^{\mu_n} \tag{7}$$

To derive the last equation, we have used the part integral which is the only assumption for the functional measure. It is worth mentioning that the partition function in the denominator is automatically cancelled by the partition function appeared in the numerator, therefore, there is no problem with respect to the metric dependency on the functional measure as mentioned before. After using the Stokes theorem twice and the homological triviality of V, namely, $V = \partial W$, a short calculation shows the following result in a flat metric

$$\langle \int_U B(x) \int_V C(y) \rangle = i \, L(U, V) \tag{8}$$

where

$$L(U,V) = \frac{\Gamma(n/2)}{(2n-4)\pi^{n/2} p!(n-p-1)!} \int_U dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$\int_V dy^{\mu_{p+1}} \wedge \cdots \wedge dy^{\mu_{n-1}} \epsilon_{\mu_1 \cdots \mu_n} \partial^{\mu_n} \frac{1}{|x-y|^{n-2}}$$

which is an integral formula for the linking number of two surfaces of dimension p and n-p-1 in S^n. Incidentally, one can apply another identity (5b), for $\langle dB(x) C(y) \rangle$ and obtain the same result as (7) and (8) in a similar way. And if one define the linking number $L(V,U) = \langle \int_V C \int_U B \rangle$, one can easily prove an important relation (1) in a general curved space without using an explicit expression (8) in a flat metric. It is interesting to realize that the homological triviality of U and V is essentially and naturally needed in our derivation method.

The problem to be remained in order to complete our derivation is that one must show

$$\langle \int_U B(x) \int_W \{ Q, * d\bar{\alpha}(y) \} \rangle = 0 . \tag{9}$$

In fact, this proves to be true because of BRS invariance of the vacuum and $\int_U B$.

Next, we turn our consideration to the nonabelian TQFT [1].

$$S = \int_M \mathrm{Tr}[\, B \wedge DC \,] \tag{10}$$

where D denotes a covariant derivative with flat connection one form for a semi-simple Lie group G, thus $D^2 = 0$. D is defined in a graded mannar, for example, $D\alpha_{2k} = d\alpha_{2k} + [A, \alpha_{2k}]$, but $D\alpha_{2k+1} = d\alpha_{2k+1} + \{A, \alpha_{2k+1}\}$, and B and C are p and n-p-1 forms transforming in a representation under G. The field equations are $DB = 0$ and $DC = 0$, and the gauge transformation is $B \to B + Dv$ and $C \to C + Dw$ owing to $D^2 = 0$.

Let us consider the following gauge invariant two-point function

$$\langle \mathrm{Tr}[\, \int_U B(x) \int_V P e^{\int_x^y A} C(y) P e^{\int_y^x A} \,] \rangle \tag{11}$$ **

where two Wilson line operators connect a point x in U with a point y in V. When proving the gauge invariance of (11), we have to use the fact that the exterior derivative is well-

** The authors of ref.[6] deal with the differnt type of "generalized" linking number from ours.

defined at points x and y. To achieve this fact, we restrict surfaces U,V,Y and W to be holonomically trivial as well as homologically trivial. Although the variational method in this case is also useful and the path of thought proceeds the same as the above abelian case, some complicated features will occur. The gauge fixed and BRS invariant action is equal to the nonabelian version of (4) whose exterior derivative d is replaced by the covariant derivative D. The variational identity for B field becomes

$$\varepsilon^{\mu_1\cdots\mu_n} D_{\mu_{p+1}} C_{\mu_{p+2}\cdots\mu_n}(x) = \frac{\delta S_{fix}}{\delta B_{\mu_1\cdots\mu_p}(x)} - \{ Q , D^{\mu_p}\bar{\alpha}^{\mu_1\cdots\mu_{p-1}}(x) \} \qquad (12)$$

And we use the useful property of two-point function (11)

$$\langle Tr[B(x)\, d^y\{ P e^{\int_x^y A} C(y) P e^{\int_y^x A}\}]\rangle = \langle Tr[B(x)\, P e^{\int_x^y A} DC(y) P e^{\int_y^x A}]\rangle \qquad (13)$$

where d^y denotes the exterior derivative with respect to the variable y. If one substitutes (12) into (13) and use the part integral, and integrate with respect to x over U and y over W, and then use the Stokes theorem, one obtains the following result,

$$\langle Tr[\int_U B(x) \int_V P e^{\int_x^y A} C(y) P e^{\int_y^x A}]\rangle = i L(U,V,A,\gamma_1,\gamma_2) \qquad (14)$$

where

$$L(U,V,A,\gamma_1,\gamma_2) = \int_{U(x)}\int_{W(y)} \frac{1}{p!(n-p)!}\,\delta^{(n)}(x-y) \cdot \epsilon_{\mu_1\cdots\mu_n}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \otimes dy^{\mu_{p+1}}\wedge\cdots\wedge dy^{\mu_n} \cdot Tr[R^a P e^{\int_x^y A} R^a P e^{\int_x^y A}] \qquad (15)$$

where R is a generator matrices of a semi-simple Lie group G.

If one specifically take the fundamental representation of SU(N), one has the famous Fierz identity

$$R^a_{ij} R^a_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl} \qquad (16)$$

Substituting (16) into (15) and taking the flat metric, one obtain the generalized linking number

$$L(U,V,A,\gamma_1,\gamma_2) = \sum_i sign(p_i)\,[\, Tr(P e^{\oint_{\Gamma_i} A})\, Tr(P e^{\oint_{\gamma_i} A}) - \frac{1}{N} Tr(P e^{\oint_{\Gamma_i+\gamma_i} A})\,]\cdot\frac{1}{2} \qquad (17)$$

where $\sum_i sign(p_i)$ denotes ordinary linking number which has been obtained in the abelian theory. Γ_i and γ_i are closed curves having the basepoint x=y.

To visualize the meaning of the generalized linking number (15), let us take a simple case G=U(1). In this case, eq.(15) becomes

$$L(U,V,A,\gamma_1,\gamma_2) = \sum_i sign(p_i)\, e^{\oint_{\Gamma_i+\gamma_i} A} \qquad (18)$$

Some examples are given in Fig.2(a) and Fig.2(b). These figures show two circles U and V in R^3 which have ordinary linking number zero and one, respectively. Let us consider the specific situation where two parallel dotted lines are removed and put on a U(1) connection with phase factor $e^{i\alpha}$ when circulating around two lines according to the orientation shown in the figure. The generalized linking numbers are thus

$$L(U,V,A,\gamma_1,\gamma_2) = 0 \tag{19}$$

for Fig.2(a) and

$$L(U,V,A,\gamma_1,\gamma_2) = e^{i\alpha} \tag{20}$$

for Fig.2(b), where we notice that (19) is equal to ordinary linking number and (20) become equal to ordinary linking number taking the limit $\alpha \to 0$. In these examples, M has a boundary, but we have previously assumed that $\partial M = \emptyset$, so that we should modify these examples to become $\partial M = \emptyset$. First cut these figures by a family of horizontal planes, where every plane has the topology of R^2 except two points. To remove these singular points, we connect the boundaries of the two points by a handle. If we add a point at infinity to this topology, the resulting surface is equal to a torus T^2. And then identify the top surface with the bottom one, we eventually obtain T^3. Therefore, the generalized linking number of these examples corresponds to the linking number in T^3.

Finally, we touch on an application of the variational method for a new class of TQFT [8]. The action is an infinite dimensional generalization of three dimensional Chern-Simons action. In the abelian theory,

$$S = \frac{1}{2}\int_{\mathbb{R}^\infty} \phi\, d\phi \tag{21}$$

where ϕ is the field containing arbitrary forms with commuting coefficients. The gauge invariance of S is $\delta\phi = d\chi$. One can obtain One can obtain the variational identity

$$\frac{\delta S}{\delta\phi(x)} = d\phi(x) \tag{22}$$

If we consider $\langle \phi(x) d\phi(y) \rangle$, use the identity (22) and follow the same path of the above discussion, we can easily have the linking linking numbers which contain the various dimensional surfaces.

In conclusion, we have obtained both ordinary linking number and generalized linking number from the variational method, and applied for more general TQFT. However, we think that the nonabelian theory considered in this paper has an unsatisfactory point, which is to introduce a flat connection field A as a background field by hand in order to have gauge symmetries. We will improve this point by establishing other extended theories in the next work [13]. Anyway, we expect that the variational method provides a nice technique together with the path integral in the direct analysis of higher dimensional TQFT's, where there is in general no counterpart of conformal field theory.

Acknowledgments

The authors would like to thank Professor Abdus Salam, Dr. E. Sezgin, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. (I.O.) would like to thank Professor A. Sugamoto for valuable discussions and encouragements.

REFERENCES

[1]A.S.Schwarz, Lett. Math. Phys.2 (1978) 247; Comm. Math. Phys. 67(1979) 1; A.S.Schwarz and Yu.S.Tyupkin, Nucl. Phys. B242 (1984)436

[2]E.Witten, Comm. Math. Phys.117 (1988) 353; "Quantum Field Theory and the Jones Polynomial", preprint IAS-HEP-88/33, to appear in Comm. Math. Phys.

[3]V.F.R.Jones, Bull. AMS12 (1985) 103; Ann. Math.126 (1987) 335

[4]A.Floer, "An instanton invariant for three manifolds", Courant Institute preprint (1987); Bull AMS16 (1987) 279

[5]S.Donaldson, J. Diff. Geo.18 (1983) 269; ibd 26 (1987) 397

[6]G.T.Horowitz and M.Srednicki, "A Quantum Field Theoretic Description of Linking Numbers and Their Generalization", preprint UCSB-TH-89-14

[7]M.Blau and G.Thompson, "Topological Gauge Theories of Antisymmetric Tensor Fields", preprint SISSA 39 March 89.

[8]R.C.Myers and V.Periwal, Phys. Lett.225B (1989) 352

[9]P.Cotta-Ramusino, E.Guadagnini, M.Martellini and M.Mintchev, "Quantum Field Theory and Link Invariants", preprint CERN-TH 5277/89; E.Guadagnini, M.Martellini and M.Mintchev, "Perturbative Aspects of the Chern-Simons Field Theory' preprint CERN-TH-5234/89

[10]L.Smolin, "Invariants of links and critical points of the Chern-Simons path integrals", Syracuse preprint

[11]D.Rolfsen, "Knots and Links"(Publish or Perish, Berkeley, 1976)

[12]I.A.Batalin and G.A.Vilkovisky, Phys. Rev.D28 (1983) 2567

[13]I.Oda and S.Yahikozawa, "A Systematic Construction of Topological Field Theories", to appear.

FIGURE CAPTIONS

Fig.1.

Examples of ordinary linking number are shown. Both U and V are V are circles in R^3. Both Y and W are disc, therefore the homological triviality, that is, $U = \partial Y$ and $V = \partial W$ hold. Ordinary linking number of each each figure (a),(b),(c) and (d) is zero, one, one and two. One can see that $L(U,V) = L(V,U)$ holds from fig.(b) and (c), for instance.

Fig.2

Examples of generalized linking number are shown. Two points, namely, $u_0 \in U$ and $v_0 \in V$ are selected, and connected by curves γ_1 and γ_2. The generalized linking number of fig.2(a) and 2(b) is zero and $e^{i\alpha}$ respectively. The former corresponds to ordinary linking number but the latter is only equal to ordinary one when $\alpha \rightarrow 0$.

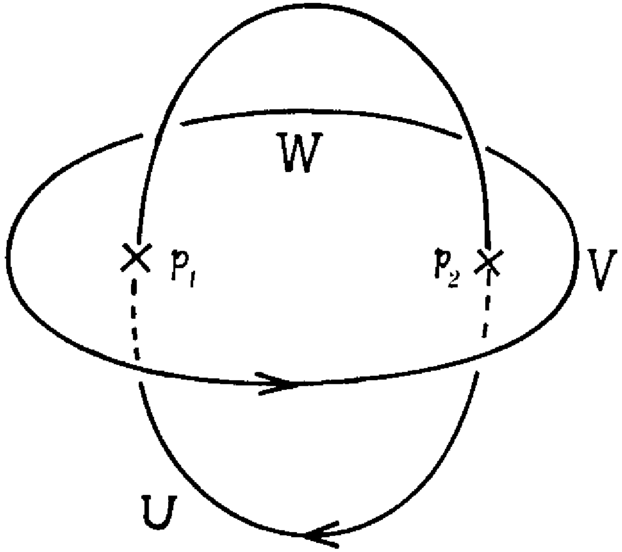

fig.1(a)

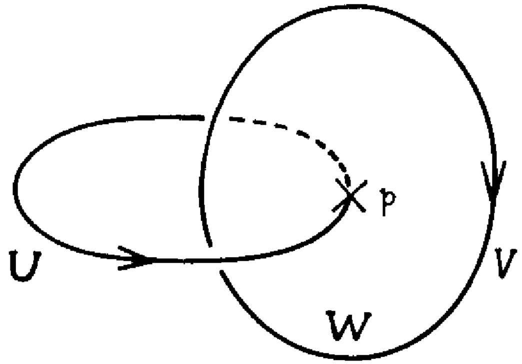

fig.1(b)

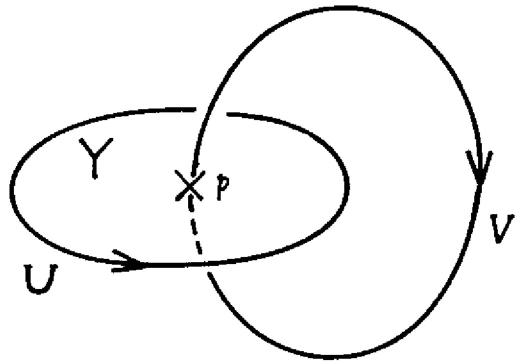

fig.1(c)

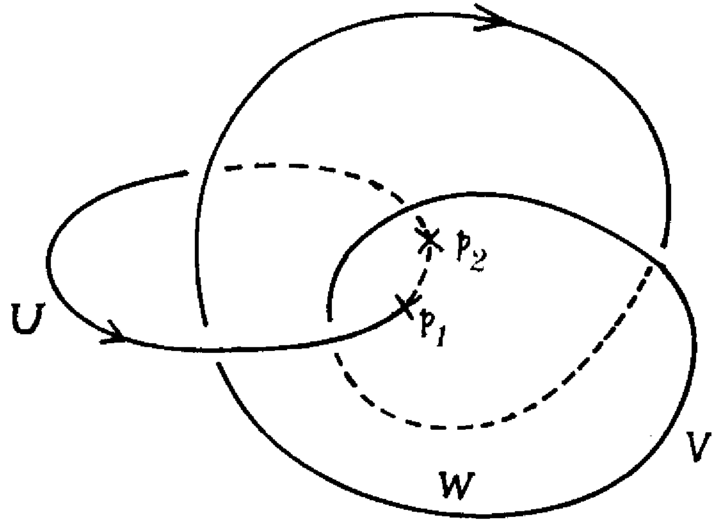

fig.1(d)

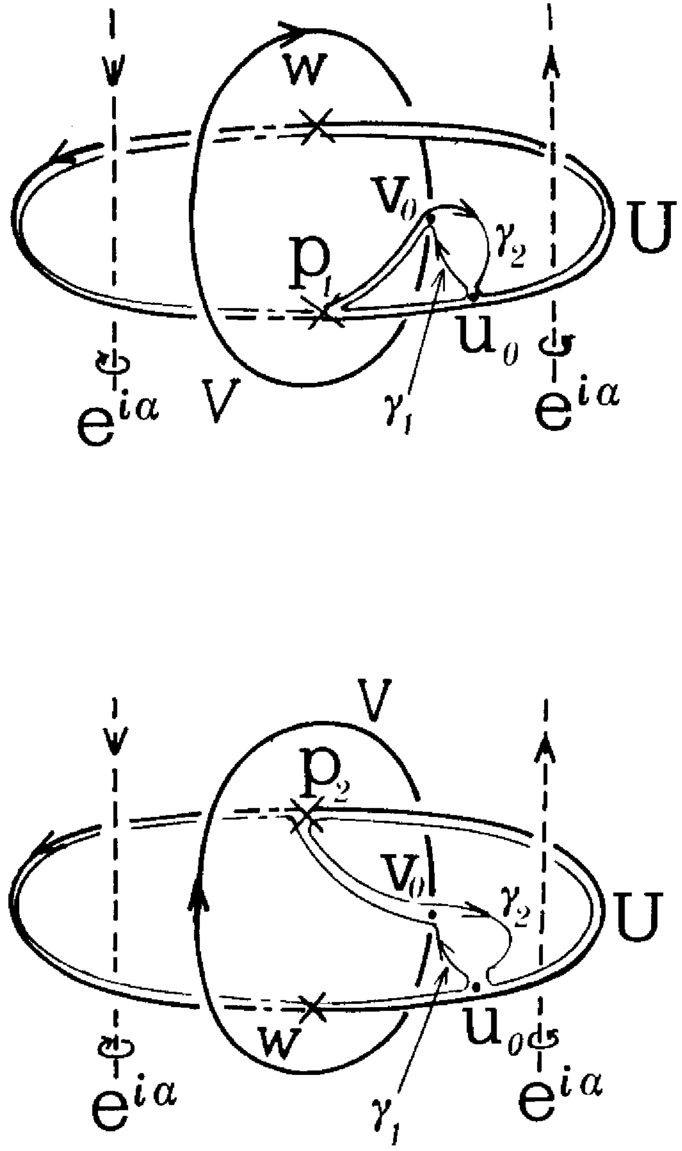

FIG. 2 (a)

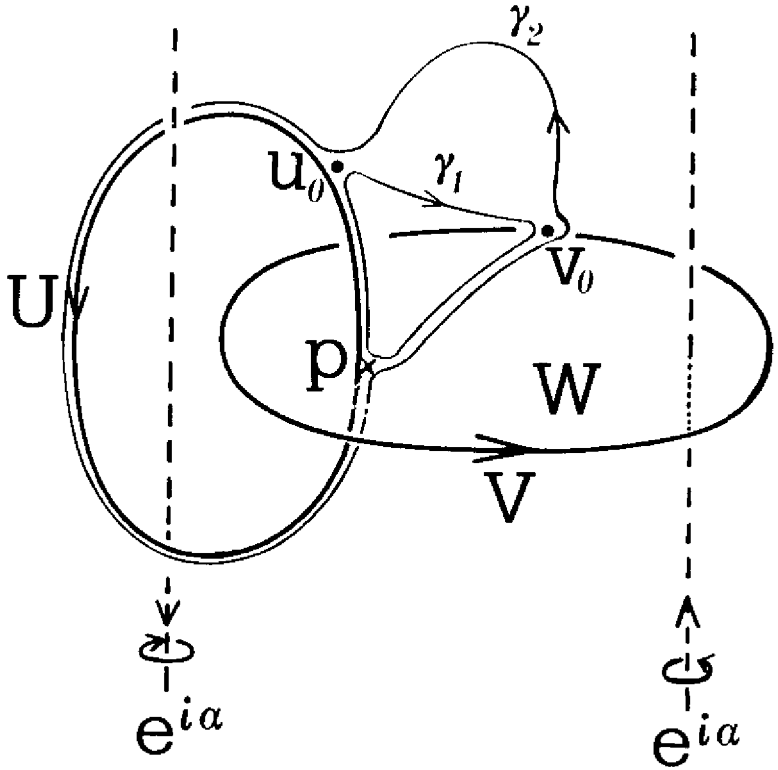

FIG. 2 (b)

Stampato in proprio nella tipografia
del Centro Internazionale di Fisica Teorica