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INS-Rep. -765
August 1989

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INS-Rep. -755
August 1989

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Abstract

We present an exact classical saddle point solution in a non-linear sigma model. It has a topological charge 1/2 and mediates the vacuum transition. The quantum fluctuations and the transition rate are also examined.

§1 Introduction

In recent years the so-called sphaleron [1] has attracted an attention of physicists as a possibly better source of baryon number generation. It is a classical finite energy saddle point solution. And it dominates the transition between different vacua at finite temperature. The transition through such a saddle point is an old problem and was first studied by Kramers [2] . Much later Langer [3] found that the negative and zero modes of quantum fluctuation around the saddle point solution are important and they determine the prefactor of the transition rate.

On the other hand we have so far rather few examples of sphaleron solution. They are unfortunately too complicated to perform a rigorous analysis of the transition rate. Situation being as such it is desirable to work on a simple model which possesses an exact classical solution so that a precise calculation of quantum fluctuations and transition rate can be caried out. We have constructed such a model, the analysis of which constitutes the main part of the present paper.

§2 Anisotropic non-linear sigma model

Our model is a non-linear $O(3)$ sigma model with an anisotropy in the third direction, which breaks the symmetry down to $O(2)$. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\{(\partial_t\vec{n})^2 - (\partial_x\vec{n})^2 + C^2(1-n_3^2)\}, \tag{1}$$

where C is a constant. We will adopt in the following a parametrization

$$\vec{n} = (n_1, n_2, n_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{2}$$

The model can be interpreted as a continuum version of the one-dimensional Heisenberg spin system with an on-site anisotropy. A similar model was recently studied by Mottola and Wipf [4] . However their model has a different anisotropy from ours and therefore shows different behaviors. The energy in our model is expressed as

$$E = \frac{1}{2}\int dx\{(\pi_\theta)^2 + (\frac{d\theta}{dx})^2 + \frac{\pi_\phi^2}{\sin^2\theta} + \sin^2\theta(\frac{d\phi}{dx})^2 - C^2\sin^2\theta\}, \qquad (3)$$

where π_θ and π_ϕ are the canonical momenta conjugate to θ and ϕ. In the following we set $C = 1$ without loss of generality.

§3 Saddle point solution

We look for a static finite energy solution. The field equations are, $\pi_\theta = 0$, $\pi_\phi/\sin^2\theta = 0$, and

$$-\frac{d^2\theta}{dx^2} + \frac{1}{2}\sin 2\theta((\frac{d\phi}{dx})^2 - 1) = 0, \qquad (4.a)$$

$$-\frac{d}{dx}(\sin^2\theta\frac{d\phi}{dx}) = 0. \qquad (4.b)$$

From eq.(4.b) one has

$$\frac{d\phi}{dx} = \frac{A}{\sin^2\theta}, \qquad (5)$$

with an integration constant A ($|A| < 1$). Inserting (5) into (4.a) one obtains a saddle point solution

$$\theta(x) = \arccos(\frac{\gamma}{\cosh(\gamma x)}), \qquad (6.a)$$

$$\frac{d\phi}{dx} = A(1 + \frac{\gamma^2}{\cosh^2(\gamma x) - \gamma^2}), \qquad (6.b)$$

where $\gamma = \sqrt{1 - A^2}$. Substituting (6) into (3), we find that the solution has a

finite energy. The result is

$$E_{sph} - E_0 = 2\gamma, \qquad (7)$$

where E_0 is the energy of the vacuum solution

$$\theta(x) = \frac{\pi}{2}, \qquad \phi(x) = Ax. \qquad (8)$$

In the subsequent sections we show that this solution is topologically non- trivial and unstable and thus is the so-called sphaleron.

§4 Topological charge

The topological charge Q is computed in accordance with Klinkhamer and Manton [1] by using [5]

$$Q = \frac{1}{2\pi}\int dx A_1, \qquad (9)$$

where A_1 denotes the space component of the "effective" gauge field $\vec{A}$,

$$\vec{A} = (A_0, A_1) = (\frac{1}{2}\cos\theta\frac{\partial\phi}{\partial t}, \frac{1}{2}\cos\theta\frac{\partial\phi}{\partial x}). \qquad (10)$$

As has been observed by Klinkhamer and Manton, it is necessary to perform a gauge transformation on the classical solution to obtain the correct Q. The suitable gauge transformation in the present context is found to be $\phi \to \phi + x$, which extrapolates our system into fully $O(3)$ symmetric model asymptotically. Then a calculation yields

$$Q = \frac{1}{4\pi}\int dx\cos\theta(\frac{d\phi}{dx} + 1) = \frac{1}{2}, \qquad (11)$$

which implies that our solution is in fact a saddle point connecting the vacua with $Q = 0$ and 1 in the functional phase space.

§5 Negative and zero modes

Next we show that the solution is unstable by solving the eigenvalue equations for the quantum fluctuations around the classical solution. They are derived by a replacement $\theta \to \theta + \delta\theta, \phi \to \phi + \delta\phi$ in (4) and retaining terms linear in $\delta\theta$ and $\delta\phi$. One finds

$$\left\{-\frac{d^2}{dx^2} + \cos 2\theta\left(\frac{A^2}{\sin^4\theta} - 1\right)\right\}\delta\theta + \sin 2\theta \frac{d\phi}{dx}\frac{d}{dx}\delta\phi = \epsilon\delta\theta, \qquad (12.a)$$

$$-\frac{d}{dx}\left(\sin 2\theta \frac{d\phi}{dx}\delta\theta\right) - \frac{d}{dx}\left(\sin^2\theta \frac{d}{dx}\delta\phi\right) = \epsilon\delta\phi, \qquad (12.b)$$

where $\theta(x)$ and $\phi(x)$ are the classical solutions (6) and ϵ denotes the energy eigenvalue.

Unfortunately the eigenfunctions for the negative mode are not found in a closed form. However one can find them by resorting to a series expansion. They are

$$\delta\theta = \sum_{n=1}^{\infty} a_n \mathrm{sech}^{2n}(\gamma x), \qquad (13.a)$$

$$\frac{d}{dx}\delta\phi = \sum_{n=1}^{\infty} b_n \mathrm{sech}^{2n-1}(\gamma x), \qquad (13.b)$$

where the coefficients a_n's and b_n's are determined order by order as $a_1 = \sqrt{1-\gamma^2}, b_1 = \gamma, etc,$ with the exact eigenvalue $\epsilon_- = -\gamma^2$.

There is also a zero mode $\epsilon_0 = 0$ whose eigenfunctions are

$$\delta\theta = \frac{d\theta}{dx}, \qquad \delta\phi = \frac{d\phi}{dx} - A, \qquad (14)$$

where $\theta(x)$ and $\phi(x)$ are the classical solutions (6).

§6 Transition rate

These modes determine the prefactor of the rate of vacuum transition in the Fokker-Planck formalism. The general expression of the rate is given by [3],

$$I = |\kappa|\left(\frac{k_B T}{2\pi|\epsilon_-|}\right)^{\frac{1}{2}} NV \exp[-(F_1 - F_0)/k_B T]. \qquad (15)$$

F_1 and F_0 are the free energy with and without sphaleron. κ is given by

$$\kappa = \begin{cases} -|\epsilon_-|\Gamma/k_B T & \text{overdamping case } (\Gamma/k_B T \gg 1), \\ -|\epsilon_-| & \text{underdamping case } (\Gamma/k_B T \ll 1), \end{cases} \qquad (16)$$

where Γ denotes the friction constant. NV is the normalized volume of the saddle point subspace,

$$NV = \mathrm{L}\left(\int dx(\delta\theta^2 + \delta\phi^2)\right)^{\frac{1}{2}}$$
$$= \mathrm{L}\left(3\sqrt{1-A^2} + \frac{1+4A^2}{A}\left(\frac{\pi}{2} - \arcsin A\right)\right)^{\frac{1}{2}}, \qquad (17)$$

where L denotes the spatial size of the system and $\delta\theta, \delta\phi$ are the eigenfunctions in (14).

§7 Conclusion

We have presented a sphaleron solution in a non-linear sigma model and computed the quantum fluctuations and transition rate. Since the non- linear sigma model is regarded as a low energy effective theory of QCD, our model may hopefully shed some light on QCD, although it is not clear under what conditions the asymmetric term n_3^2 results from QCD.

More on this model and the sphaleron solutions in other models will be reported elsewhere [6] .

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