FINITE SIZE AND DYNAMICAL EFFECTS IN PAIR PRODUCTION
BY AN EXTERNAL FIELD

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Abstract

We evaluate the rate of pair production in a uniform electric field confined into a bounded region in space. Using the Balian-Bloch expansion of Green’s functions we obtain explicit expressions for finite size corrections to Schwinger’s formula. The case of a time-dependent boundary, relevant to describe energy deposition by quark-antiquark pair production in ultrarelativistic collisions, is also investigated. We find that finite size effects are important in nuclear collisions. They decrease when the strength of the chromoelectric field between the nuclei is large. As a result, the rate of energy deposition increases sharply with the mass number \( A \) of the colliding nuclei.

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1 - INTRODUCTION

The production of fermion-antifermion pairs by a classical external field via the so-called Schwinger mechanism [1] is a very general phenomenon. It occurs in a rather wide variety of problems and has been the subject of a large number of studies (c.f. the comprehensive review article by Soffel, Müller and Greiner [2]).

One example, originally discussed by Schwinger, is the problem of electron-positron pair creation out of the Dirac vacuum due to a strong electric field. In this case the pair production rate is expected to be significant whenever the strength eE of the electric field becomes comparable to the square of the electron mass. This condition requires very strong fields. For instance the strength of the electric field on a Bohr orbit around a charge \( Z \) is of order \( eE \sim m^2(Z \alpha)^3 \), i.e. only \( 4 \times 10^{-7} \) for hydrogen [3]. However by performing collisions of heavy nuclei with charges \( Z_1 \) and \( Z_2 \) such that \( Z_1 + Z_2 > 1/\alpha = 137 \) it has been possible to observe electron-positron pair production. This has been carried out for the first time at the Unilac facility at GSI Darmstadt which has been able to reach bombarding energies above the Coulomb barrier for reactions such as \( U + Pb \), \( U + U \) or \( U + Cm \) (about 6 MeV per nucleon for the \( U + Cm \) system) [2,4].

A second example is the instability of the Dirac vacuum in the presence of a strong gravitational field, which is also discussed in ref. [2]. This instability leads to the quantum evaporation of black holes by pair creation, which has been investigated in different physical situations, e.g. a non-rotating uncharged black hole described by a Schwarzschild metric [5] or a rotating electrically charged black hole described by a Kerr-Newman metric [5,6].

A last example, which will be discussed at some length below, is the mechanism of energy deposition by quark-antiquark production in ultrarela-
tivistic collisions. Here the external field is the strong chromoelectric field which develops at the early stage of the collision as a result of gluon exchanges [7-9]. The problem in this case is to perform a reliable calculation of the pair production rate in order to determine the energy density reached during the reaction and the possible formation of a quark-gluon plasma. In the calculations of reference [7] the pair production rate was evaluated from Schwinger's formula, which is valid for an infinite and uniform electric field. However the actual configuration of the field is somewhat different, since the chromoelectric field in a collision is enclosed in the flux tube joining the two receding nuclei. Furthermore the boundary of the field changes with time and one may thus expect corrections to arise for both finite size and dynamical effects. The purpose of this paper is to show that these corrections are conveniently evaluated by means of the Balian-Bloch multiple reflection expansion of the Green's functions [10,11]. A preliminary account of the method was presented in a recent rapid communication [12].

In a series of articles [10] Balian and Bloch have constructed approximation schemes of Green functions in terms of classical paths. They have first investigated the density of modes for the wave equation in a cavity of arbitrary shape. In this case they were able to express the density as a sum over all closed trajectories involving multiple reflections at the boundary. Each closed classical trajectory of length \( L \) provides an oscillating contribution
\[
\sin (kL)
\]

\( k \)

\( L \)

\( \rho(k) \) . When the surface is smooth the successive terms decrease rapidly. Sharp peaks result out of the interference between different paths. An interesting feature of the expansion is that its first term represents a volume contribution while the next two terms correspond respectively to surface and curvature corrections. In a subsequent article Balian and Bloch have generalized their method in order to build an expansion of the Green's function \( G(\vec{r}, \vec{r}') \) for a Schrödinger particle in a smooth potential [11]. In this case dominant contributions to \( G \) arise from
multiple reflections of the wave emitted at $r'$ upon the caustic which is the three-dimensional analog of the turning point. Closed paths of zero length yield the familiar Thomas-Fermi expression for the density of states, together with a smooth correction, while higher order terms provide an oscillating contribution which reflects the shell structure in the spectrum.

In the following we will show how the Balian-Bloch expansion can be combined with Schwinger's proper time method to investigate pair production by an external field. Our article is organized as follows. In section II we derive a general expression for the pair production rate up to one reflection in the Balian-Bloch expansion. In section III we consider the specific case of a boundary parallel to the direction of the electric field while section IV discusses the case of a moving boundary perpendicular to the field. A discussion of these formulae in the context of ultrarelativistic collisions is presented in section V while section VI contains a summary of our main conclusions.

II - THE PRODUCTION RATE UP TO ONE REFLECTION

Let us consider a quantized Dirac field coupled to a classical external abelian potential $A_\mu(x)$ described by the interaction Lagrangian

$$\mathcal{L}_I(x) = -\epsilon \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x).$$

The probability to remain in the ground state, i.e. the probability of emitting no pairs is given by

$$P = |\langle 0|S|0\rangle|^2$$

where $S$ is the $S$-matrix
\[ S = T \exp \left\{ i \int d^4x \mathcal{L}_1(x) \right\} \]  

and \( T \) the time-ordering operator. The probability \( \mathcal{P} \) can also be written as

\[ \mathcal{P} = \exp \left\{ - \int d^4x \, \mathcal{W}(x) \right\}, \]

where

\[ \mathcal{W}(x) = \mathcal{L}_{\text{eff}}(x) \cdot \text{Im}. \]

In equation (5) \( \mathcal{L}_{\text{eff}}(x) \) is the one-loop effective Lagrangian density [1], which includes all orders in the external field (but neglects self-interactions of the matter fields). The quantity \( \mathcal{W}(x) \) can be interpreted as the pair production rate per unit time and unit volume at the space-time point \( \alpha = (x_0, x_1, x_2, x_3) \).

A convenient integral representation of the one loop effective Lagrangian is that of Schwinger [1]

\[ \mathcal{L}_{\text{eff}}(x) = \frac{i}{\epsilon} \int_0^\infty ds \, e^{-im^2s} \text{Trace} \langle \chi \mid e^{iHs} \chi \rangle. \]  

In this equation the mass \( m \) of the fermions is supposed to contain a small negative imaginary part and \( H \) is the \( 4 \times 4 \) matrix

\[ H = (p - eA)^2 + \frac{e}{\epsilon} \sigma_{\mu\nu} F^{\mu\nu}, \]

with the usual notation \( \sigma_{\mu\nu} = i [\gamma_\mu, \gamma_\nu]/2 \). Since the first term in \( H \) is a multiple of the \( 4 \times 4 \) unit matrix the trace in equation (6) is easily carried out. In the case of an external electric field of strength \( E \) along
the third-axis the only non-vanishing components of the field strength tensor $F_{\mu\nu}$ are $F_{30} = F_{03} = E$ and we find that

\[
\text{Trace} \exp \left( i \frac{e}{\hbar} \sigma_{\mu\nu} F^{\mu\nu} \right) = 4 \cos \left( \Delta e E \right). \tag{8}
\]

We are thus left with the evaluation of matrix elements of the form

\[
\mathcal{U}(x, x'; \lambda) = \langle x | \exp \left( i H_0 \lambda \right) | x' \rangle , \tag{9}
\]

where $H_0$ is the operator

\[
H_0 = (p - eA)^2. \tag{10}
\]

The matrix element (9) can be represented in terms of the following Feynman path integral \cite{13,14}

\[
\mathcal{U}(x, x'; \lambda) = \int \mathcal{D}[x(z)] \exp i S[x(z)] \tag{11}
\]

where the action $S$ is defined by

\[
S[x(z)] = \int_{0}^{\lambda} L(x(z)) \, dz \tag{12}
\]

In equation (12) the Lagrangian density $L$ is

\[
L = -\frac{1}{4} \dot{x}^\mu \dot{x}^\mu - e A^\mu \dot{x}_\mu . \tag{13}
\]
A first approximation to the functional integral (11) can be obtained by performing a stationary phase evaluation which gives

\[ U(x, x'; \lambda) = \left( \frac{4}{\sqrt{\pi}} \right)^{\frac{1}{2}} \det \left( D_{\mu\nu} \right) \exp (iS_c). \]  

(14)

In this equation \( S_c \) is the classical action \( S_c (x', x; \lambda) \) corresponding to a trajectory starting at \( x' \) at time zero and ending at \( x \) at time \( T = \lambda \). The quantity \( D_{\mu\nu} \) is the \( 4 \times 4 \) matrix \[ D_{\mu\nu} = \partial^x S_c / \partial x^\mu \partial x'^\nu \]  

(15)

Note that equation (14) does not involve an infinite dimensional matrix but rather a \( 4 \times 4 \) determinant.

Equation (14) actually corresponds to the lowest order term in the Balian-Bloch expansion of \( U(x, x'; \lambda) \) which involves no reflection at the field boundary. In reference [12] we showed that the next term involves a single reflection on the field boundary at a point \( y \) and a time \( \lambda_1 \) such that the total action \( S_1 + S_2 = S_c (x', y; \lambda_1) + S_c (y, x; \lambda - \lambda_1) \) is stationary. The corresponding contribution to the matrix element \( U(x, x'; \lambda) \) is

\[ U_1 (x, x'; \lambda) = (2\pi e^{i\pi/2})^2 \exp i(S_{1c} + S_{2c}) \]

\[ \times \left( \frac{4}{\sqrt{\pi}} \right)^{\frac{1}{2}} \det \left( D_1 D_2 D_3^{-1} \right), \]

(16)

where \( D_1, D_2 \) and \( D_3 \) are the following \( 4 \times 4 \) matrices:
\[ D_{1}^{\mu \nu} = \partial^{\nu} S_{1c} (x, y \neq \lambda) / \partial y \mu \partial x \nu, \]
\[ D_{2}^{\mu \nu} = \partial^{\nu} S_{2c} (y, x \neq \lambda) / \partial x \mu \partial y \nu, \]
\[ D_{3}^{\mu \nu} = \partial^{\nu} (S_{1c} + S_{2c}) / \partial y \mu \partial x \nu. \] (17)

The previous determinants satisfy the relation
\[ \det (D_{1} D_{2} D_{3}^{-1}) = \det (-D_{4}) , \] (18)

where
\[ D_{4}^{\mu \nu} = \partial^{\nu} (S_{1c} + S_{2c}) / \partial x \mu \partial x \nu . \] (19)

Equations (14) and (16) require the knowledge of the classical action defined by eq. 12. For a constant electric field in the \( x_{3} \)-direction of this action is given by
\[ S_{\varepsilon} (x, x' ; \lambda) = \frac{eE}{\xi} \cosh (eE\lambda) \left\{ - (x_{0}-x_{0}')^{t} + (x_{3}-x_{3}')^{t} \right\} \\
+ \frac{eE}{\xi} \left( x_{0} x_{3}' - x_{3} x_{0}' \right) + \frac{1}{4\xi} \left\{ (x_{1}-x_{1}')^{t} + (x_{2}-x_{2}')^{t} \right\}. \] (20)

III - SURFACE EFFECTS FOR A BOUNDARY PARALLEL TO THE FIELD

From now on we shall consider only contributions involving no reflection or terms with one reflection. These contributions are indeed proportional to the volume and to the surface of the system respectively. Two (or more) reflection terms would correspond to curvature (or higher order) corrections. In ref. [12] we considered the case of an electric field parallel to
the third-axis localized in the half-space \( \mathcal{H}_1 \subseteq \mathbb{R}^3 \). In this case the optimization of the total action with respect to the reflection point, leads to the following values

\[
S_{1c} = S_{2c} = \frac{1}{2\Delta} (R - \mathcal{H}_1)^r
\]

(21)

The one-reflection correction to the effective Lagrangian is thus

\[
\mathcal{L}_{\text{eff}}^1 = -\frac{i}{8\pi^2} \int_0^\infty d\sigma \left( e^{E \coth (eE\Delta)} - \frac{1}{\sigma} \right) e^{i\sigma \Delta} e^{i (R - \mathcal{H}_1)^r/\sigma}.
\]

(22)

To calculate the imaginary part of the integral in equation (22), we first perform the change of integration variable \( \tau = i\sigma \). This gives

\[
\mathcal{L}_{\text{eff}}^1 = \frac{1}{8\pi^2} \int_0^{i\infty} \frac{d\tau}{\tau^2} \left( e^{E \coth (eE\tau)} - \frac{1}{\tau} \right) e^{-m^2 \tau - (R - \mathcal{H}_1)^r/\tau},
\]

(23)

where the integration is along the contour \( C_1 \) in Fig. 1. However since the integral along the contour \( C_2 \) goes to zero when its radius becomes large, integrations along the contours \( C_1 \) and \( C_3 \) in this figure give equal contributions. Furthermore, since the integrand along the real axis is real, the only non vanishing contributions to \( \text{Im} \mathcal{L}_{\text{eff}}^1 \) arise from the essential singularity at the origin and the simple poles occurring at \( \tau_n = n\pi/eE \) for \( n = 1, 2, \ldots \). Let us first show that the contribution \( I_0 \) arising from the singularity at the origin vanishes. Indeed by performing the change of variable \( \tau = e^{-i\theta} \) we find

\[
I_0 = \frac{i}{3} e^2 E^r \int_0^{\pi/2} \exp \left\{ -e^{-i\theta} (R - \mathcal{H}_1)^r/e \right\} d\theta.
\]

(24)
The imaginary part of this integral is given by an integral sine function

\[ \text{Im} \ I_0 = -\frac{e^t E^t}{3} \ \text{ai} \left( \frac{(R - x_1)^t}{\varepsilon} \right) \]

which vanishes when \( \varepsilon \) goes to zero. We are thus left with the sum of residues at the poles. This gives the following formula for the pair production rate per unit time and unit volume in the region \( \chi_1 \leq R \).

\[ W(x) = \frac{e^t E^t}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^t} \exp \left( -\frac{n \pi m^t}{e E} \right) \left\{ 1 - \exp \left( -\frac{e E (R - x_1)^t}{n \pi} \right) \right\} \]

The first term in the curly parentheses corresponds to Schwinger's formula [1] which gives a production rate \( W_0 \) proportional to the volume. The second term gives a contribution \( W_1 \) which is important only near the boundary at \( x_1 = R \) where it cancels the first one.

In Fig. 2 and 3 we display the ratio \( (W_1 + W_0)/W_0 \) as a function of the distance \( d = R - x_1 \) from the boundary, for various values of the electric field strength. Figure 2 corresponds to a fermion mass \( m = 10 \text{ MeV} \), while Fig. 3 corresponds to \( m = 200 \text{ MeV} \). From these figures we see that surface effects lead to a significant reduction of the Schwinger pair production rate near the boundary of the field. As an illustrative example (to be discussed later in section V) for a distance \( d = 1 \text{ fm} \) from the surface and for a field strength \( eE = 1 \text{ fm}^{-2} \), the reduction is 80 percent for a mass \( m = 10 \text{ MeV} \) and 70 percent for \( m = 200 \text{ MeV} \). The corresponding figures become 35 percent and 25 percent for a field strength \( eE = 5 \text{ fm}^{-2} \).

Equation (26) can be applied to the case of an infinite flux tube in the \( \chi_3 \)-direction with a radius \( R \) in the transverse direction. By integrating equation (26) over the transverse coordinates \( \chi_1 \) and \( \chi_2 \), we find the following expression for the probability of pair creation per unit time.
and unit length

\[ \int \mathcal{W}(\mathbf{x}) \, d\mathbf{x}, \, d\mathbf{x} = \pi R^2 \, \frac{e^2 E^2}{4 \pi^3} \]

\[ \sum_{n=1}^{\infty} \frac{1}{n!} \, \exp \left( - \frac{n \pi m}{e E} \right) \left\{ 1 - \Phi \left( R \sqrt{\frac{e E}{n \pi}} \right) \right\} \]

(27)

where the function \( \Phi \) is related to the error function

\[ \Phi(u) = - \frac{1}{u} \left[ 1 - \exp(-u^2) \right] + \frac{\sqrt{\pi}}{u} \, \text{erf}(u). \]

(28)

Since near the origin \( \text{erf}(u) \approx 2u / \sqrt{\pi} \) we find \( \Phi(0) = 1 \) and the surface term \( \phi \) thus cancels exactly the volume term in the limit of a small tube. On the contrary, if we have a large tube, then \( \text{erf}(u) \approx 1 \),

\[ \Phi \approx \frac{\sqrt{\pi}}{u} \] and surface effects thus become negligible as soon as

\[ R \gg \sqrt{\pi / e E} \].

The ratio of surface to volume contributions is graphed as a function of the dimensionless variable \( u = R \sqrt{e E / \pi} \) in the case \( m = 0 \) in fig. 4.

IV - SURFACE EFFECTS FOR A BOUNDARY PERPENDICULAR TO THE FIELD

Schwinger's proper time method is explicitly Lorentz invariant as it treats all space-time coordinates \( x_0, x_1, x_t, x_3 \) on the same footing. It is thus equally well adapted to handle boundaries in space, time or in both space and time. In the present section we will apply this method to study pair production in an electric field of the form

\[ E_{\infty} = 0 \quad , \quad E_y = 0 \quad , \quad E_z = E \, \Theta (\nu x_0 - x_3) \]

(29)
Unlike the previous section we now have to consider reflections occurring at the moving boundary \( \chi_3 = \mathcal{U} \chi_0 \). This leads to a modification in the argument of the last exponential in equation (22) arising from the total action \( S_1 + S_2 \). It is straightforward to check that the reflection still occurs at \( \tau = \Lambda/2 \) and that the determinants in eq. 16 remain unchanged. The result for the one-reflection correction to the effective Lagrangian is found to be

\[
\mathcal{L}^{\text{eff}} (\chi) = -\frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^2} \left( eE \cot \left( \frac{eE \Lambda}{2} \right) - \frac{1}{\Lambda} \right) \exp \left\{ -im_s \Lambda + i \frac{eE}{\Lambda} d^\epsilon \cot \left( \frac{eE \Lambda}{2} \right) \right\}
\]

(30)

where

\[
d^\epsilon = \gamma^\epsilon (\mathcal{U} \chi_0 - \chi_3)^\epsilon
\]

(31)

with the usual notation \( \gamma = \sqrt{1 - \mathcal{U}^2/c^2} \). In the static case \( \mathcal{U} = 0 \) equation (30) still holds and in this case \( d^\epsilon \) is merely the distance to the boundary. The evaluation of the imaginary part of \( \mathcal{L}^{\text{eff}} \) given by equation (30) is more difficult than in the case of a boundary parallel to the field. Indeed an important difference is that the singularities occurring at \( \tau = \ln \frac{i\pi}{eE} \) are now essential singularities so that the method used in the previous section is no longer applicable.

To simplify let us consider the case \( m = 0 \). In order to evaluate the integral (30) numerically we first perform the change of integration variable \( t = \coth \left( \frac{eE \Lambda}{2} \right) \) which yields
\[ \text{Im } \mathcal{L}^\prime_{\text{eff}} = -\frac{1}{32\pi^2} e^t e^{\frac{t^2}{2}} \int_0^{\infty} \phi(t) \sin\left(\frac{e E}{2} d^2 t\right) dt, \]\hspace{1cm} (32)

where the function \( \phi(t) \) is

\[ \phi(t) = \frac{4}{(t^2 - 1) \left(\log \frac{t + 1}{t - 1}\right)^t \left(t + \frac{1}{t} - \frac{e}{\log \frac{t + 1}{t - 1}}\right)} \]\hspace{1cm} (33)

The function \( \text{Im}(\mathcal{L}^\prime + \mathcal{L}^0)/\text{Im } \mathcal{L}^0 \) is plotted in Fig. 5 as a function of the dimensionless variable \( d \sqrt{e E} \). For large values of the dimensionless parameter \( d \sqrt{e E} \), the asymptotic behaviour of \( \text{Im } \mathcal{L}^\prime_{\text{eff}} \) is found to be

\[ \text{Im } \mathcal{L}^\prime_{\text{eff}} \approx -\frac{1}{8\pi^2} e^t e^{\frac{t^2}{2}} \sin\left(e E d^2 t\right) + o\left(\frac{4}{\log \left(e E d^2 t\right)}\right) \]

while at short distances it is given by

\[ \text{Im } \mathcal{L}^\prime_{\text{eff}} \approx -\frac{e^t E^t}{48\pi^2} + \frac{e^t E^t}{8\pi^2} \left(\frac{e E d^2 t}{t}\right) C \]

with \( C = \frac{1}{3} - \int_1^{\infty} \left( t \phi(t) - \frac{1}{3} \right) dt \approx -0.10 \)

(34)

Note that the first term in eq. 35 is simply \( -\text{Im } \mathcal{L}^0 \) so that the surface term cancels the volume term on the boundary exactly.

From equation (34) surface corrections away from the boundary of the field vanish only logarithmically. Although the corresponding decrease is weaker than it was in eq. (26) one should remember that in the present

...
case the relevant variable $d_L$ is the distance to the boundary dilated by a Lorentz factor $\gamma$. As a consequence surface corrections vanish for a boundary moving at the speed of light. This result can easily be understood as a consequence of causality. Indeed, classical trajectories starting from an arbitrary point $\mathbf{x}$ in space-time will never be able to reach such a boundary. As a result the contribution of the term with one reflection vanishes in this case.

By comparing Figs 2, 3 and 5 it may be noted that the pair production rate exhibits oscillations as a function of the distance to the boundary when this boundary is perpendicular to the field. No such oscillations occur for a boundary parallel to the field. This difference can be understood by considering the value of the optimal action $S_{1c} + S_{2c}$ occurring in equation (16). Indeed one finds for the first case

$$S_{1c} + S_{2c} = \frac{eE}{\ell} \coth\left(\frac{eE\ell}{\ell}\right) d^\ell$$

while in the second case equation (21) gives

$$S_{1c} + S_{2c} = \frac{(R - \alpha_1)^2}{\Delta}$$

Large values of $\Delta$, associated with the long trajectories which produce sharp oscillations in the Balian-Bloch method, are thus suppressed in the second case and not in the first. Indeed, these large values contribute a factor $\exp(iS_{1c} + iS_{2c}) \approx \exp(i e E d^\ell) / \ell$ in the first case and 1 in the second case.

In the special case $\mathcal{U} = 0$ finite size effects in the Schwinger pair production mechanism have been calculated exactly by Wang and Wong [16] who solved the Dirac equation in a linear potential in terms of hypergeometric functions. This calculation provides only the production rate per
unit time, transverse area, transverse momentum and energy interval, which has no exact relation to the rate \( W(\pi) \) per unit time and unit volume. In ref. [16] however, approximate relations were used to extract a value of \( W(\pi) \). Keeping in mind the fact that some of these approximations may be questionable, and the fact that a finite value of the mass is used in reference [16], we can only conclude that there is a qualitative agreement between our results and those of Wang and Wong. In particular we also find large deviations from Schwinger's result for finite systems especially at the field boundary where surface contributions cancel the volume term. Away from the boundary we obtain sharper oscillations than in reference [16]. This difference is however not significant since the approximations made in [16] to derive \( W(\pi) \) imply that only integrated rates can be meaningfully compared.

V - ENERGY DEPOSITION IN ULTRARELATIVISTIC COLLISIONS

Ultrarelativistic heavy-ion collisions have been successfully described [7,17] by means of the flux tube model introduced by Low [8] and Nussinov [9] for hadron-hadron collisions. In this model the two Lorentz contracted nuclei are described as two color charged capacitor plates and the strong color field between them polarizes the vacuum by producing quark-antiquark pairs. The energy deposition in the associated flux tube is subsequently determined by calculating the pair production rate using Schwinger's formula, which is valid for a uniform infinite electric field. With the results of the previous sections it is interesting to study how the boundaries in the longitudinal and transverse directions of the tube modify the energy deposition. The corrections to Schwinger's formula depend on the radius \( R \) of the tube, on the strength \( eE \) of the chromoelectric field and on the speed \( \sigma \) of the nuclei in their center-of-mass frame (i.e. on the incident energy).
The NA38 experiment at CERN used 200 GeV per nucleon oxygen-16 and sulfur-32 beams. In these collisions the Lorentz factor $\gamma$ to be used in equation (31) is about 6. For lead-lead collisions at the same energy $\gamma$ is about 10. According to the results of the previous section, surface effects due to the presence of the boundary in the longitudinal direction become small beyond 0.2 fm from the nuclei for $\gamma = 10$ and a field strength $eE = 1 \text{ fm}^{-2}$. Longitudinal surface effects would be even smaller for the value $eE = 5 \text{ fm}^{-2}$ adopted in Ref. [16].

Finite size effects in the transverse direction lead to more important corrections. Indeed, in the flux tube model of heavy-ion collisions the radius of the tube is typically of the size of 1 fm rather than the size of the nucleus. This is because the spatial variations in color orientation which occur during the color charging process have a coherence length in the radial direction which is approximately the size of a nucleon [18]. The collision thus leads to the formation of a configuration of neighbouring tubes with multiple colors whose transverse size is about 1 fm (which has been called a color rope by Biro, Nielsen and Knoll [18]). Evidence for a transverse size of about 1 fm is substantiated by the values of the average transverse momenta observed in the Helios and NA35 experiments [19,20].

From the results in Figure 4 surface effects in the case of a flux tube with a radius $R = 1$ fm reduce significantly the pair production rate per unit time and unit length. Indeed the corresponding reduction is 90 percent for a field strength $eE = 1 \text{ fm}^{-2}$ and is still 25 percent for a field strength $eE = 15 \text{ fm}^{-2}$. Such corrections may explain why observed multiplicities in the central rapidity region vary like $A_T^{1/3} A_P^{2/3}$ (where $A_P$ and $A_T$ denote the projectile and target mass respectively) [21] while the flux tube model with Schwinger's formula predicts an $A_T^{1/6} A_P^{5/6}$ dependence (see eq. (17) of ref. [17]). If this is the case it would be important to use heavy nuclei in order
to reach large energy densities in ultrarelativistic collisions. Indeed for
\(A_p = A_T = A\) the field \(eE\) scales like \(A^{1/3}\) \([7,17,18]\). Therefore if we assume
that a typical field strength \(eE = 5 \text{ fm}^{-2}\) is reached in sulfur-32 + sulfur-32 colli-
sions, we find from Fig. 4 that the rate of pair production (which is related
to the rate of energy deposition) will be multiplied by a factor of about five
for lead-208 + lead-208 collisions. This makes the perspective of developing
200 GeV per nucleon lead beams at CERN especially attractive.

VI - CONCLUSION

The Schwinger proper time method combined with the Balian-
Bloch multiple reflection expansion of Green's functions has been demonstrated
to be a powerful and elegant tool to investigate corrections to the Schwinger
pair production formula. Indeed it allows one to work out analytically the
case of static boundaries parallel to the field or perpendicular to the field,
as well as boundaries which evolve with time. We have shown that finite size
corrections are large and almost cancel the volume term within a distance
of order \(1/(eE)^{1/2}\) from the boundary, where \(E\) is the field strength. Finite
size corrections are reduced when the field boundary moves with a velocity
\(\sigma\) and vanishes as a consequence of causality when \(\sigma = c\). In the case of
ultrarelativistic nuclear, collisions we argued that energy deposition by pair
production was significantly reduced as compared to Schwinger's formula because
of the transverse dimension of the chromoelectric flux tube. We have pointed
out that in order to reach high energy densities in ultrarelativistic collisions
it is of great interest to develop beams of heavy nuclei since the large field
involved imply a reduction of surface corrections too.
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References


FIGURE CAPTIONS

Figure 1: Contours of integration for evaluating the surface term in the effective Lagrangian $\mathcal{L}_{\text{eff}}^{\gamma}$ given by equation 22.

Figure 2: Ratio $r$ of the total production rate (volume plus surface contributions) to the volume contribution, for a boundary parallel to the field as a function of the distance $d$ to the boundary. The fermion mass is $m = 10$ MeV and the various curves correspond to different strengths of the electric field $\mathcal{E}$.

Figure 3: Same as figure 2 for a fermion mass $m = 200$ MeV.

Figure 4: Ratio $r$ of surface to volume contributions for pair production in a cylinder of radius $R$ as a function of the dimensionless variable $u = R \sqrt{\mathcal{E} / \eta}$.

Figure 5: Ratio $r$ of the total production rate (volume plus surface contributions) to the volume contribution for a moving boundary ($\mathbf{x}_0 = \mathbf{x}_0$) perpendicular to the field, as a function of the dimensionless variable $u = (e \mathcal{E})^{1/2} (\nu \mathbf{x}_0 - \mathbf{x}_3)$. The fermion mass is $m = 0$. The ratio $r$ is small and negative near $u = 0$ (cf. eq. 36). This may result from our semi-classical approximation.
Fig. 1
Fig. 3
Fig. 5