SLAC/AP--77

DE90 003764

CANONICAL FORMALISM FOR COUPLED BEAM OPTICS*

S. A. KHEIFETS

Stanford Linear Accelerator Center (SLAC)
Stanford University, Stanford, CA 94309

ABSTRACT

Beam optics of a lattice with an inter-plane coupling is treated using canonical Hamiltonian formalism. The method developed is equally applicable both to a circular (periodic) machine and to an open transport line. A solution of the equation of a particle motion (and correspondingly transfer matrix between two arbitrary points of the lattice) are described in terms of two amplitude functions (and their derivatives and corresponding phases of oscillations) and four coupling functions, defined by a solution of the system of the first-order nonlinear differential equations derived in the paper. Thus total number of independent parameters is equal to ten.

* Work supported by the Department of Energy, contract DE-AC03-76SF00515)

1. INTRODUCTION

Until recently most of the accelerators and beam lines were built in such a way as to ascertain that the particle motion in the horizontal plane (the plane of the reference trajectory) is decoupled from the motion in the vertical plane at least for the ideal unperturbed machine. In this case the transverse motion of the particle in each plane becomes one-dimensional. The full theory of the betatron oscillations in one dimension for a periodic lattice was developed by Courant and Snyder.[1] Their method with little modification could be extended for any uncoupled transport line.[2]

The misalignments of the machine elements and the errors in the powering and machining of them bring up to a coupling between the planes. Nevertheless this coupling is small and could be treated as a perturbation on the one dimensional particle dynamics.

Nowadays however, there is certain degree of interest for the beam optics which provide strong coupled motion. Firstly, the beam lines of the SLAC Linear Collider (SLC) transporting the beams from the Linac to the Final Focus System lay not in a plane and produce strong inter-plane coupling.[3] Secondly, some existing (for example the SLC Damping Rings[4]) and future (for example the B-factory considered by the Cornell University group[5]) machines are based on a strong coupling to provide round cross section beams.

That makes it desirable to develop a method which describes the coupled transverse motion in a natural and economical way similar to one which exist for the one dimensional case.

There exist several different approaches in description of the motion of a charged particle in a transport line or in a circular accelerator with strong coupling between the horizontal and vertical planes. All these approaches are based on a treatment of the transformation matrix from one point of the lattice to another (so called R-matrix).

The approach used in the TRANSPORT program[6] is based on the multiplication of the transfer matrices for each element all of which are assumed to have the mid plane symmetry. Each such matrix is simplectic in the first order. To accommodate a coupling TRANSPORT allows an instantaneous rotation around the longitudinal axis and a magnet with the solenoidal field. Matrices of these elements are also simplectic. Hence, the transfer matrix defined at the end point of each element of the transport line produces a simplectic transformation from one point of the lattice to another. This method is purely numerical and is more appropriate to a transport line.

Some authors suggest to find the eigenvectors of R-matrix[7] or introduce four beta-functions[8]. In these cases it is difficult to fulfill all six simplectic conditions. A different approach is suggested by E. Forest.[9] He deals with a ma · which is supposed to be found. The method is suitable for numerical calculations in such cases where the Hamiltonian is impossible to construct. All these methods are more appropriate for a circular machine but not to a transport line.

The problem can be solved very naturally by applying a canonical transformation to normal coordinates in which the new Hamiltonian of the motion is block-diagonal. Hence, the simplecticity of the transformation and the corresponding R—matrix is fulfilled automatically. The starting point of the L. Teng's method[10] is writing one particular simplectic transformation to normal coordinates. The matrix of the transformation is parametrized in terms of four functions - a local rotation angle and three other parameters comprising a unimodular (2,2) matrix. Such transformation is called *simplectic rotation.* Since simplectic rotation represents a canonical transformation, its four parameters together with six others which parametrize two normal betatron oscillations satisfy a set of ten first order nonlinear differential equations.

Here I present a similar but more general approach. The starting point is to find a general linear canonical transformation to normal coordinates. Hence, the simplecticity of the transformation and the corresponding R—matrix is fulfilled automatically. The whole class of ten-parametric simplectic matrices is found. An example of such a matrix is given below. Next step is to find a set of four (out of ten) parameters of the transformation which make the new Hamiltonian of the motion block-diagonal.

An example of a solution is given for which the right hand sides of the differential equations defining these four parameters are algebraic expressions with the focusing and coupling functions as their coefficients.

As is shown below, the method developed here is equally applicable to both circular accelerators and transport lines. The difference between these two cases manifests only in the boundary conditions of the differential equations.

Since we are interested here only in the transverse motion, the energy of the particle is assumed to be constant. Furthermore, to simplify our task as much as possible only the linear part of the magnetic force acting on the particle is considered.

As it is practically always a case, the existence of a reference trajectory is assumed. This trajectory in general can deviate from any plane quite arbitrary. That means that the concept of the *horizontal* and the *vertical* directions has no meaning any more. Nevertheless, we will call one deviation from the reference trajectory (measured along one normal direction to it) horizontal and denote it by x. The deviation along the second normal direction to the reference trajectory will be called vertical and will be denoted y. Although these directions are arbitrary, in any given magnetic lattice they are defined quite naturally by the magnetic field configuration. By choosing the coordinates x and y properly, the expressions for focusing and coupling functions (the only functions which depend on the choice of the coordinates) can be simplified greatly.

The distance along the reference trajectory measured from an arbitrary point of it is chosen as independent coordinate s. Hence, all the focusing and coupling strengths are functions of s.

A comment on the definitions used in this paper seems to be in place here. We use everywhere the canonical conjugate coordinates and momenta, the latter being denoted by the letter p with the subscript corresponding to the coordinate, e.g. x and p_x. A particle position in the phase space at each given value of s is described either by a set of its coordinates and momenta x, p_x, y and p_y or by a (4,1) matrix X:

$$X \equiv \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \qquad (1.1)$$

All the matrices are denoted by capital letters.

In Section 2 the Hamiltonian and the corresponding hamiltonian equations of motion are defined. The canonical transformation and its generating function are described. The most general linear transformation is defined in the next Section. Here is also given an example of a general ten-parametric simplectic matrix and corresponding to it generating function A particular four-parametric canonical transformation is given in the next Section. Here can be also found the set of differential equations for the four coupling functions which define transformation to the normal coordinates. In Section 5 each of these normal modes are expressed in terms of betatron function. At last Section 6 contains expressions for R–matrix in terms of two betatron functions for the normal oscillations and four coupling functions.

2. THE HAMILTONIAN OF THE TRANSVERSE MOTION

The hamiltonian of the transverse coupled motion can be written either in a scalar form:[1]

$$h_z = [(p_x - qy)^2 + (p_y + qx)^2 + fx^2 + gy^2 + 2kxy]/2 \tag{2.1}$$

or in an equivalent matrix form:

$$h_z = (X^T H_z X)/2, \tag{2.2}$$

where the superscript T means the transposition of the matrix and H_z is (4,4) matrix:

$$H_z = \begin{pmatrix} f+q^2 & 0 & k & q \\ 0 & 1 & -q & 0 \\ k & -q & g+q^2 & 0 \\ q & 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

The physical meaning of functions $f(s), g(s), k(s)$ and $q(s)$ in these equations depends on the shape of the reference orbit. In the particular case of the flat reference orbit laying in the horizontal plane these functions have meaning of the horizontal focusing strength, the vertical focusing strength, the skew quadrupole strength and the solenoid field, respectively.

The physical inter-plane coupling provided by functions k and q manifest from the formal mathematical point of view in the fact that the Hamiltonian is represented by a non-diagonal matrix (2.3).

The Hamiltonian equations of motion can be written in the matrix form in the following way:

$$X' = SH_z X. \tag{2.4}$$

where the prime means the derivative in respect to s and the (4,4) matrix

$$S = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.5}$$

has the following properties:

$$S^T = -S, \tag{2.6}$$

$$S^2 = -I. \tag{2.7}$$

The (4,4) matrix I is the unity matrix.

To be able to describe the coupled betatron motion as two one-dimensional oscillations one needs to find a representation in which the Hamiltonian is at least block-diagonal, i.e. the elements of the (4,4) matrix in the (2,2) upper right and lower left corners of it are all equal to zero.

Hence, our first goal will be to find such a linear canonical transformation to new variables v, p_v, w and p_w that the new Hamiltonian H_v appears in the block-diagonal form. In the matrix notation the new coordinates are

$$V \equiv \begin{pmatrix} v \\ p_v \\ w \\ p_w \end{pmatrix} \tag{2.8}$$

The old coordinates X are connected to the new ones V by a linear transformation

$$X = TV \tag{2.9}$$

which in general depends on 16 yet unknown functions $t_{ik}(s), i, k = 1, \ldots 4$. However, for the transformation (2.9) to be canonical it must be simplectic:

$$T^T S T = S \tag{2.10}$$

These 6 conditions reduce the number of independent parameters T_{ik} in equation (2.9) from 16 to 10.

The standard way to produce a canonical transformation is to find a *generating function* from which the transformation follows. It appears that a convenient choice of the generating function in our case is the function of the old coordinates and the new momenta:

$$\psi = \psi(x, p_v, y, p_w; s) \tag{2.11}$$

If we introduce two more (4,1) matrices

$$Q \equiv \begin{pmatrix} x \\ p_v \\ y \\ p_w \end{pmatrix} \tag{2.12}$$

and

$$P \equiv \begin{pmatrix} p_x \\ v \\ p_y \\ w \end{pmatrix} \tag{2.13}$$

then the new coordinates and old momenta are obtained from the generating function in the following symbolic way:

$$P = \frac{\partial \psi}{\partial Q}. \tag{2.14}$$

The Hamiltonian in the new coordinates is

$$h_v = h_x + \frac{\partial \psi}{\partial s} \tag{2.15}$$

It is understood certainly, that in equation (2.15) the old coordinates and momenta should be substituted by the new coordinate and momenta using equation (2.9).

3. THE CANONICAL TRANSFORMATION

It follows from equation (2.14) that the generating function ψ is a quadratic form of its variables:

$$\psi = (Q^T U Q)/2, \tag{3.1}$$

where U is a (4,4) symmetric matrix of yet unknown coefficients $u_{ik}(s)$. These coefficients can be expressed in terms of the matrix elements t_{ik} using equation (2.9). Imposing now the symmetry conditions for the matrix U:

$$u_{ik} = u_{ki} \tag{3.2}$$

a set of six elements t_{ik} can be expressed in terms of the other ten. The transformation constructed in such a way is a linear canonical transformation and as such it is guaranteed to be simplectic.

The particular choice of the six matrix elements is a matter of convenience and can be done in a number of different ways. As an example we give here one set of six matrix elements as functions of ten other arbitrary elements:

$$t_{14} = (t_{11}t_{32} - t_{12}t_{31} + t_{13}t_{34})/t_{33} \tag{3.3}$$

$$\begin{aligned} t_{22} = (&t_{11}t_{23}t_{32} + t_{12}t_{21}t_{33} - t_{12}t_{23}t_{31} \\ &- t_{13}t_{21}t_{32} + t_{33})/(t_{11}t_{33} - t_{13}t_{31}) \end{aligned} \tag{3.4}$$

$$\begin{aligned} t_{24} = (&t_{11}t_{21}t_{32}t_{33} - t_{11}t_{23}t_{31}t_{32} + t_{11}t_{23}t_{33}t_{34} \\ &- t_{12}t_{21}t_{31}t_{33} + t_{12}t_{23}t_{31}^2 - t_{13}t_{23}t_{31}t_{34} \\ &- t_{31}t_{13})/[t_{33}(t_{11}t_{33} - t_{13}t_{31})] \end{aligned} \tag{3.5}$$

$$t_{41} = (t_{11}t_{23} - t_{13}t_{21} + t_{31}t_{43})/t_{33} \tag{3.6}$$

$$\begin{aligned} t_{42} = (&t_{11}t_{12}t_{23}t_{33} - t_{11}t_{13}t_{23}t_{32} + t_{11}t_{32}t_{33}t_{43} \\ &- t_{12}t_{13}t_{21}t_{33} + t_{13}^2t_{21}t_{32} - t_{13}t_{31}t_{32}t_{43} \\ &- t_{13}t_{33})/[t_{33}(t_{11}t_{33} - t_{13}t_{31})] \end{aligned} \tag{3.7}$$

$$\begin{aligned} t_{44} = (&t_{11}^2t_{23}t_{32} - t_{11}t_{12}t_{23}t_{31} - t_{11}t_{13}t_{21}t_{32} \\ &+ t_{11}t_{33}t_{34}t_{43} + t_{11}t_{33} + t_{12}t_{13}t_{21}t_{31} \\ &- t_{13}t_{31}t_{34}t_{43})/[t_{33}(t_{11}t_{33} - t_{13}t_{31})] \end{aligned} \tag{3.8}$$

The reader is invited to check that matrix T which is constructed in this way is simplectic with determinant equal to 1. It is interesting to note that the found matrix T is not included into Teng's classification scheme of *simplectic rotations*.[10] Such a transformation which has ten free parameters is more general then a simplectic rotation which has only four of them.

Corresponding to such a choice of matrix T the generating function ψ is defined by the matrix U with the following matrix elements:

$$u_{11} = (t_{21}t_{33} - t_{23}t_{31})/(t_{11}t_{33} - t_{13}t_{31}) \tag{3.9}$$

$$u_{12} = u_{21} = t_{33}/(t_{11}t_{33} - t_{13}t_{31}) \tag{3.10}$$

$$u_{13} = u_{31} = (t_{11}t_{23} - t_{13}t_{21})/(t_{11}t_{33} - t_{13}t_{31}) \tag{3.11}$$

$$u_{14} = u_{41} = -t_{31}/(t_{11}t_{33} - t_{13}t_{31}) \tag{3.12}$$

$$u_{22} = -(t_{12}t_{33} - t_{13}t_{32})/(t_{11}t_{33} - t_{13}t_{31}) \qquad (3.13)$$

$$u_{23} = u_{32} = -t_{13}/(t_{11}t_{33} - t_{13}t_{31}) \qquad (3.14)$$

$$u_{24} = u_{42} = -(t_{11}t_{32} - t_{12}t_{31})/(t_{11}t_{33} - t_{13}t_{31}) \qquad (3.15)$$

$$u_{33} = -(t_{11}t_{13}t_{23} - t_{11}t_{33}t_{13} - t_{13}^2 t_{21} + t_{13}t_{31}t_{13})/(t_{11}t_{33} - t_{13}t_{31}) \qquad (3.16)$$

$$u_{34} = u_{43} = t_{11}/(t_{11}t_{33} - t_{13}t_{31}) \qquad (3.17)$$

$$u_{44} = (t_{11}t_{31}t_{32} - t_{11}t_{33}t_{34} - t_{12}t_{31}^2 + t_{13}t_{31}t_{34})/(t_{11}t_{33} - t_{13}t_{31}) \qquad (3.18)$$

By and large, expressions obtained up to now are sufficient for finding equations for such values of the elements t_{ik} which bring about the matrix corresponding to the new Hamiltonian (2.15) to become block-diagonal. Since this matrix is symmetric one needs to find a set of any four elements t_{ik}. The rest of them then can be chosen quite arbitrary.

However, using this freedom, it is much easier first to choose several matrix elements t_{ik} from consideration of simplicity of the expressions and after that to construct the new Hamiltonian. One possible such a rout is pursued in the next section.

4. A PARTICULAR SOLUTION

From expressions (3.3) through (3.24) follows that a substantial simplification arises by removing their denominators. One way to achieve this is to chose the following values of the two diagonal matrix elements:

$$t_{33} = 1, \quad t_{11} = 1 + t_{13}t_{31}. \tag{4.1}$$

Using the above mentioned freedom of the parameter choice, we also make zero the following non-diagonal matrix elements:

$$t_{12} = 0, \quad t_{21} = 0, \quad t_{34} = 0, \quad t_{43} = 0. \tag{4.2}$$

Eqs. (3.3)-(3.8) and Eqs. (4.1),(4.2) define twelve matrix elements t_{ik} in terms of the four independent matrix elements t_{13}, t_{31}, t_{23} and t_{32}, so that the matrices T and U now look like the following:

$$T = \begin{pmatrix} t_{11} & 0 & t_{13} & t_{32}t_{11} \\ 0 & t_{22} & t_{23} & -t_{31}t_{22} \\ t_{31} & t_{32} & 1 & 0 \\ t_{23}t_{11} & -t_{13}t_{22} & 0 & t_{11}t_{22} \end{pmatrix} \tag{4.3}$$

and

$$U = \begin{pmatrix} -t_{23}t_{31} & 1 & t_{23}t_{11} & -t_{31} \\ 1 & t_{13}t_{32} & -t_{13} & -t_{32}t_{11} \\ t_{23}t_{11} & -t_{13} & -t_{13}t_{23}t_{11} & t_{11} \\ -t_{31} & -t_{32}t_{11} & t_{11} & t_{31}t_{32}t_{11} \end{pmatrix}, \tag{4.4}$$

where the following notation is introduced to make the subsequent formulae more compact:

$$t_{11} = 1 + t_{13} t_{31} \qquad (4.5)$$

$$t_{22} = 1 + t_{23} t_{32} t_{11} \qquad (4.6)$$

The symmetric matrix H_v corresponding to the Hamiltonian h_v for the new coordinates V (*the new Hamiltonian*) can now be found with the help of the matrices T and U:

$$H_v = T^T H_x T + \left(\partial (Q^T U Q) / \partial s \right), \qquad (4.7)$$

where the old coordinates x and y should be substituted by their values from expression (2.9):

$$x = t_{11} v + t_{13} w + t_{32} t_{11} p_w, \qquad (4.8)$$

$$y = w + t_{31} v + t_{32} p_v \qquad (4.9)$$

The matrix elements of H_v are too cumbersome to be presented here. As can be seen from eq. (4.7), each of them is a sum of a linear form of the first derivatives of the independent elements with respect to s and a term which is independent of these derivatives. The coefficients in front of the derivatives and the free term are non-linear algebraic expressions of the same independent elements.

Up to this point the coupling focusing functions $k(s)$ and $q(s)$ were assumed to acquire arbitrary values. In particular, for the uncoupled motion they can be equal to zero identically. Since the Hamiltonian of the uncoupled motion is already diagonal, the matrix T should in the limit $k = 0$ and $q = 0$ go to the unity matrix I.

Now it is straightforward to impose conditions under which matrix H_v becomes block-diagonal:

$$h_{v13} = h_{v14} = h_{v23} = h_{v24} = 0. \qquad (4.10)$$

Solving these equations with respect to derivatives we find the following system of non-linear first-order differential equations for the four independent matrix element (the coupling functions):

$$t'_{13} = -(qt_{11}^2t_{22} - kt_{11}t_{32} + qt_{23}t_{32} - t_{23}t_{22} - gt_{32} - q^2t_{32} + qt_{22})/2t_{22} \qquad (4.11)$$

$$\begin{aligned} t'_{31} = -(&qt_{23}t_{31}^2t_{32} - t_{23}t_{31}^2t_{22} - t_{23}t_{11}^2t_{22} - gt_{31}^2t_{32} - q^2t_{31}^2t_{32} - 2kt_{31}t_{32}t_{11} \\ &+ kt_{31}t_{32} - ft_{32}t_{11}^2 - q^2t_{32}t_{11}^2 - 3qt_{11}t_{22} + qt_{11} + qt_{22})/2t_{22} \end{aligned} \qquad (4.12)$$

$$\begin{aligned} t'_{23} = -(&qt_{13}t_{23}t_{11} + ft_{13}t_{11} + q^2t_{13}t_{11} \\ &- qt_{23}t_{31} + gt_{31} + q^2t_{31} + 2kt_{11} - k)/2t_{11} \end{aligned} \qquad (4.13)$$

$$\begin{aligned} t'_{32} = -(&ft_{13}t_{32}^2t_{11}^2 + q^2t_{13}t_{32}^2t_{11}^2 + 2qt_{13}t_{32}t_{11}t_{22} - qt_{13}t_{32}t_{11} + 2t_{13}t_{11}t_{22}^2 \\ &- t_{13}t_{11}t_{22} + gt_{31}t_{32}^2t_{11} + q^2t_{31}t_{32}^2t_{11} - 2qt_{31}t_{32}t_{22} + qt_{31}t_{32} + 2t_{31}t_{22}^2 \\ &- t_{31}t_{22} + 2kt_{32}^2kt_{11}^2 - 2kt_{32}^2t_{11})/2t_{11}t_{22} \end{aligned} \qquad (4.14)$$

Prime here denotes derivative over s. Note, that for the uncoupled motion $k = 0$ and $q = 0$ the system is consistent with the solution $t_{13} = t_{31} = t_{23} = t_{32} = 0$ which gives the identical transformation $T = I$.

The following remark concerning the denominators of Eqs. (4.11)-(4.14) should be made here. If the coupling functions behave in such a way that at least one of denominators goes through zero the solution will have a singularity. Such cases correspond to the instability of the coupled motion. The investigation of this problem can not be done here and should be a subject of the subsequent study.

To define the solution of this system one still needs to specify initial conditions for the coupling functions t_{13}, t_{31}, t_{23} and t_{32}. We will postpone the discussion of this point and consider these initial conditions together with those for yet another set of differential equations defining β-functions.

After the solution is found, for example numerically, the Hamiltonian h_v is reduced to a sum of two Hamiltonians each describing one-dimensional motion. Representation of each of them in terms of betatron function is derived in the next Section.

5. ONE-DIMENSIONAL CASE

Since we managed to make the Hamiltonian H_0 block-diagonal, the four- dimensional phase space v, p_v, w, p_w is now split into two independent two-dimensional phase spaces. Accordingly, the (4,1) phase space vector V can now be split into two independent (2,1) phase space vectors. To avoid excessive indexing let us denote

$$Z \equiv \begin{pmatrix} z \\ p_z \end{pmatrix} \tag{5.1}$$

where z and p_z stand either for v, p_v or for w, p_w. For each of these sub-spaces the Hamiltonian is represented by a (2,2) matrix H_2:

$$H_2 = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \tag{5.2}$$

The matrix elements of this matrix can be obtained from equation (4.7) by substituting into them expressions (4.11) trough (4.14). They are given by the following expressions. *For the coordinates* v, p_v:

$$a = (qt_{23}t_{11} + kt_{31} + ft_{11} + q^2t_{11})/t_{22} \tag{5.3}$$

$$b = -(qt_{32} - t_{22})/t_{11} \tag{5.4}$$

$$\begin{aligned} c = -(&t_{13}t_{23}t_{11}t_{22} + ft_{13}t_{32}t_{11} + q^2t_{13}t_{32}t_{11} + 2qt_{13}t_{22} - q[illegible] \\ &- qt_{23}t_{31}t_{32} + t_{23}t_{31}t_{22} + gt_{31}t_{32} + q^2t_{31}t_{32} + 2kt_{32}t_{11} - 2kt_{32}k)/t_{22} \end{aligned} \tag{5.5}$$

And *for the coordinates* w, p_w:

$$a = (kt_{13} - qt_{23} + g + q^2)/t_{11}t_{22} \tag{5.6}$$

$$b = t_{11}(qt_{32} + t_{22}) \tag{5.7}$$

$$c = qt_{13} \tag{5.8}$$

The equations of motion corresponding to a Hamiltonian (5.2) $Z' = SH_2Z$ look like:

$$z' = cz + bp_z, \quad p_z' = -az - cp_z \tag{5.9}$$

Although our problem became one-dimensional we still can not apply directly the Courant parametrization in terms of the betatron function. The reason for that is that the Hamiltonian matrix (5.2) differs from the one considered in Ref. 1. Let us derive now the modification of the Courant-Snyder method.[1]

We use slightly modified generating function[11]

$$\psi_2 = -\frac{z^2}{2\beta b}(\tan\phi + \alpha) \tag{5.10}$$

for a canonical transformation from the 'old' coordinates z and momenta p_z to 'new' coordinates ϕ and momenta j:

$$p_z = \frac{\partial\psi_2}{\partial z}, \quad j = -\frac{\partial\psi_2}{\partial\phi} \tag{5.11}$$

Unknown functions $\beta(s)$ and $\alpha(s)$ should be defined in such a way that the 'new' Hamiltonian is represented in the form $H_j = j/\beta(s)$. That is achieved provided that they satisfy the following equations

$$\frac{\beta\beta''}{2} - \frac{(\beta')^2}{4} + (ab - \frac{3(b')^2}{4b^2} + \frac{cb'}{b} - c^2 - c' + \frac{b''}{2b})\beta^2 = 1 \qquad (5.12)$$

and

$$\alpha = -\frac{\beta'}{2} - \frac{\beta b'}{2b} + c\beta \qquad (5.13)$$

Equations of motion in the 'new' coordinates then look like:

$$j' = 0, \quad \phi' = \frac{1}{\beta}. \qquad (5.14)$$

and the solution of the equations of motion in the 'old' coordinates is:

$$z = \sqrt{2j\beta b}\cos(\phi + \theta), \quad p_z = -\sqrt{2j/\beta b}[\sin(\phi + \theta) + \alpha\cos(\phi + \theta)] \qquad (5.15)$$

Two constants of motion j and θ are defined by initial values of z and p_z.

To define the solution of eq. (5.12) one needs yet to add the initial values $\beta(0)$ and $\beta'(0)$. It is reasonable to discuss them together with the initial conditions for Eqs. (4.11) - (4.14).

Up to this point we have not specified the lattice under consideration. Hence all the results obtained up to here are applicable to any lattice. The initial conditions for the solution of the system of differential equations distinguish different cases.

A. *Circular machine.*

Since the lattice of a circular machine is periodic at least with the period of one revolution, all the focusing functions are also periodic and the same should be true for all the β- and coupling functions. Hence, in this case the solution of the system of differential equations is defined by imposing on each unknown function the condition of periodicity.

B. *Transport line.*

For a transport line the initial conditions are defined by the physical conditions of the beam at the entrance of the line. The simplest initial conditions arise in the case when the entrance is chosen at such a point of the lattice where there are no coupling elements and the ellipsoid representing the beam distribution in the phase space is upright, i.e. its main axes are collinear with the directions of the axes of the local coordinate system x, p_x, y, p_y and s.

In this case, the initial values of $t_{ik}, i, k = 1, ..., 4$ are defined from the condition $T(0) = I$, i.e. $t_{ik}(0) = \delta_{ik}$, and the initial values of the beta-functions are defined by the corresponding beam size $\sigma_{x,y}$ at the entrance of the lattice:

$$\beta_v(0) = \sigma_x^2(0)/\epsilon_x \tag{5.16}$$

$$\beta_v'(0) = 0 \tag{5.17}$$

$$\beta_w(0) = \sigma_y^2(0)/\epsilon_y \tag{5.18}$$

$$\beta_w'(0) = 0 \tag{5.19}$$

where $\epsilon_{x,y}$ are the area of the projections of the ellipsoid on the corresponding axis.

Eqs. (5.12) express now the coordinates v, p_v and w, p_w in terms of two betatron functions. The physical coordinates x, p_x, y and p_y can be found using equation (2.9). That is described in the next section.

6. R-MATRIX OF COUPLED MOTION

Let us introduce a (4,1) vector of the initial values:

$$A \equiv \begin{pmatrix} \sqrt{2j_v}\cos\theta_v \\ -\sqrt{2j_v}\sin\theta_v \\ \sqrt{2j_w}\cos\theta_w \\ -\sqrt{2j_w}\sin\theta_w \end{pmatrix} \tag{6.1}$$

In terms of the vector A the solution of the equations of motion for coordinates V can be written in the matrix form $V = MA$, where the block-diagonal matrix M has the following (2,2) matrices on the main diagonal:

$$M_3 = \begin{pmatrix} \sqrt{b\beta}\cos\phi & \sqrt{b\beta}\sin\phi \\ -\frac{\sin\phi+\alpha\cos\phi}{\sqrt{b\beta}} & \frac{\cos\phi-\alpha\sin\phi}{\sqrt{b\beta}} \end{pmatrix} \tag{6.2}$$

The solution of the equations of motion for coordinates X then is

$$X(s) = T(s)M(s)A. \tag{6.3}$$

Since this expression holds for any value of s, also

$$X(0) = T(0)M(0)A. \tag{6.4}$$

Now it is quite easy to find the expression for the transfer matrix (R-matrix) from a point $s = 0$ to any other point s. To do that one only needs to eliminate the initial values A from Eqs. (6.3) and (6.4):

$$X(s) = R(0|s)X(0), \tag{6.5}$$

where

$$R(0|s) = T(s)M(s)M(0)^{-1}T(0)^{-1} \tag{6.6}$$

from eq.(4.3) we get:

$$T^{-1}(0) = \begin{pmatrix} t_{22}(0) & 0 & -t_{13}(0)t_{22}(0) & -t_{32}(0) \\ 0 & t_{11}(0) & -t_{23}(0)t_{11}(0) & t_{31}(0) \\ -t_{31}(0)t_{22}(0) & -t_{32}(0)t_{11}(0) & t_{11}(0)t_{22}(0) & 0 \\ -t_{23}(0) & t_{13}(0) & 0 & 1 \end{pmatrix} \tag{6.7}$$

Furthermore, using eq. (6.2) we find:

$$M_2^{-1}(0) = \begin{pmatrix} \frac{\cos\phi(0)-\alpha(0)\sin\phi(0)}{\sqrt{b(0)\beta(0)}} & -\sqrt{b(0)\beta(0)}\sin\phi(0) \\ \frac{\sin\phi(0)+\alpha(0)\cos\phi(0)}{\sqrt{b(0)\beta(0)}} & \sqrt{b(0)\beta(0)}\cos\phi(0) \end{pmatrix} \tag{6.8}$$

From where

$$M_2(s)M_2^{-1}(0) =$$

$$\begin{pmatrix} \sqrt{\frac{b(s)\beta(s)}{b(0)\beta(0)}}(\cos\Delta\phi + \alpha(0)\sin\Delta\phi) & \sqrt{b(s)b(0)\beta(s)\beta(0)}\sin\Delta\phi \\ -\frac{[1+\alpha(s)\alpha(0)]\sin\Delta\phi+[\alpha(s)-\alpha(0)]\cos\Delta\phi}{\sqrt{b(s)\beta(s)b(0)\beta(0)}} & \sqrt{\frac{b(0)\beta(0)}{b(s)\beta(s)}}(\cos\Delta\phi - \alpha(s)\sin\Delta\phi) \end{pmatrix} \tag{6.9}$$

Here function $b(s)$ is defined by Eqs. (5.4) or (5.7), $\Delta\phi \equiv \phi(s) - \phi(0)$, and functions $\beta(s), \alpha(s)$ and $\phi(s)$ are defined by Eqs. (5.13)-(5.15). Eqs. (6.6) through (6.9) fully define the transfer matrix between two arbitrary points 0 ans s in terms of two β-functions ($\beta_v(s)$ and $\beta_w(s)$) and four coupling functions $t_{13}(s), t_{31}(s), t_{23}(s), t_{32}(s)$.

For the lattice of a circular machine the matrix for the full revolution R_L simplifies considerably. Since $T(L) = T(0)$, where L is the machine circumference we have:

$$R_l = T(0) R_v T(0)^{-1}, \tag{6.10}$$

where R_v is a block-diagonal matrix with the following two (2,2) matrices on its diagonal:

$$R_{2v} = \begin{pmatrix} \cos\mu_v + \alpha_v(0)\sin\mu_v & b_v(0)\beta_v(0)\sin\mu_v \\ -\frac{[1+\alpha_v^2(0)]\sin\mu_v}{b_v(0)\beta_v(0)} & \cos\mu_v - \alpha_v(0)\sin\mu_v \end{pmatrix} \tag{6.11}$$

and

$$R_{2w} = \begin{pmatrix} \cos\mu_w + \alpha_w(0)\sin\mu_w & b_w(0)\beta_w(0)\sin\mu_w \\ -\frac{[1+\alpha_w^2(0)]\sin\mu_w}{b_w(0)\beta_w(0)} & \cos\mu_w - \alpha_w(0)\sin\mu_w \end{pmatrix}. \tag{6.12}$$

Her

$$\mu_v = \int_0^L ds/\beta_v(s), \tag{6.13}$$

$$\mu_w = \int_0^L ds/\beta_w(s), \tag{6.14}$$

and β_v, β_w are defined by eq. (5.13) with the coefficient a, b and c for coordinates v, p_v and w, p_w, respectively.

Matrix R_L has the following property:

$$R_L^m(\mu_v, \mu_w) = R_L(m\mu_v, m\mu_w) \tag{6.15}$$

similar to the property of the R-matrix for the uncoupled motion. Furthermore, since R_L and R_v are connected by a similarity transformation (6.10) their eigenvalues are the same, i.e. they are given by $\exp(\pm\mu_v)$ and $\exp(\pm\mu_w)$.

For a transport line $T^{-1}(0) = I$ so the R-matrix also simplifies somewhat. Nevertheless, expressions for its eigenvalues are very messy and they are not given by μ_v and μ_w.

7. ACKNOWLEDGEMENTS

Discussion with R. Ruth, K. Oide, J. Irwin, S. Heifets, E. Forest and T. Raubenheimer were beneficial in the process of working on the paper. M. Sands and R. Warnock helped to clear two points in the calculations. For that I am grateful to all of them.

REFERENCES

1. E. Courant and H. Snyder, "Theory of the Alternating Gradient Synchrotron", Annals of Physics, 3, pp 1-48,(1958)
2. D. Carey, "The Optics of Charged Particle Beams", Accelerators and Storage Rings, Vol. 6, Harwood Academic Publishers
3. S. Kheifets *et al*, "Beam Optical Design and Studies of the SLC Arcs", Proc. of 13nth Int. Conf. on High Energy Acc., Novosibirsk, USSR, 1986
4. L. Rivkin *et al*, Proc. of the Particle Acc. Conf., Vancouver. IEEE NS-32, No. 5, 1985, p. 2626
5. R. Siemann, "The Accelerator Physics Challenges of B-factories", in the Proc. of 14nth Int. Conf. on High Energy Acc., Tsukuba, Japan, 1989
6. K. Brown et al, "TRANSPORT. A Computer Program for Designing Charged Particle Beam Transport Lines", CERN 80-04, 1980
7. T. O. Raubenheimer and R.D.Ruth., SLAC-PUB 4914, March 1989; T. O. Raubenheimer, SLAC-PUB-4913, April 1989;
8. F. Willeke and G. Ripken, DESY Report DESY 88 114, August 1988

9. E. Forest, SSC Central Design Group, SSC-111, February 1987

10. L.C. Teng, Fermilab Report FN-229, 0100, May 3, 1971; D.A. Edwards and L.C. Teng, IEEE, NS-20, No. 3, p. 885, 1973; L. C. Teng, Fermilab Report FNAL-TM-1556, January 1989

11. R. D. Ruth, "Single Particle Dynamics and Nonlinear Resonances in Circular Accelerators", in *Nonlinear Dynamics Aspects of Particle Accelerators*, ed. J.M.Jowett, M. Month and S. Turner, Springer-Verlag, p. 37