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On The Langevin Equation for Stochastic Quantization of Gravity

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Abstract

We study the Langevin equation for stochastic quantization of gravity. By introducing two independent variables with a second-class constraint for the gravitational field, we formulate a pair of the Langevin equations for gravity which couples with white noises. After eliminating the multiplier field for the second-class constraint, we show that the equations leads to stochastic quantization of gravity including an unique superspace metric.

1. Introduction

Stochastic Quantization (in short, SQ) was first introduced by Parisi and Wu[1] as an interesting alternative quantization method.[2] SQ is ordinally formulated by a Langevin equation which is coupled with a white noise. The formulation based on the Langevin equation is appropriate for the numerical calculation, provided that it is coupled with a white noise, while it seems to be not so suitable for the investigation of the symmetry properties of the system (such as renormalizability, unitarity, spontaneous symmetry breaking, etc.).

On the other hand, since a Fokker-Planck equation is an equation for the probability distribution functional of the system, it properly reflects the symmetry properties of the system. In other word, the Fokker-Planck (in short, F-P) equation is constructed from the invariance principle under the symmetries of the system. In this sense, the description in terms of the F-P equation is preferable for the investigation of the symmetry properties in SQ.

This fact becomes more transparent if we consider a stochastic process of a quantum mechanical system on a Riemannian manifold $\{q^A, G_{AB}\}$. The drift force of this system is assumed to be given by the gradient of a scalar function $S[q]$ on the manifold, namely $\partial_A S \equiv \frac{\partial S}{\partial q^A}$. The system is invariant under the general coordinate transformation on the manifold

$$q^A \to q^{A'} \equiv f^A(q). \tag{1.1}$$

From the invariance principle under the transformation (1.1), the F-P equation is given by

$$\dot{P}(q,t) = \frac{1}{\sqrt{G}}\partial_A G^{AB}\sqrt{G}(\partial_B + \partial_B S)P(q,t), \tag{1.2}$$

for the scalar distribution functional which is normalized

$$\int \mathcal{D}q\sqrt{G}P(q,t) = 1. \tag{1.3}$$

In (1.2), G and G^{AB} are $\det G_{AB}$ and the inverse of G_{AB}, respectively. F-P equation (1.2) is manifestly invariant under the general coordinate transformation (1.1). In the sense of Ito's stochastic calculus,[3] the Langevin equation which is in fact equivalent to the F-P equation (1.2) is given by[4]

$$\dot{q}^A = -G^{AB}\partial_B S + \frac{1}{\sqrt{G}}\partial_B(G^{AB}\sqrt{G}) + \xi^A. \tag{1.4}$$

In this equation, the noise variables ξ^A is defined by

$$\begin{aligned} \xi^A &\equiv e^A_I \eta^I, \\ \left\langle \eta^I(t)\eta^J(t')\right\rangle &= 2\delta^{IJ}\delta(t-t'), \end{aligned} \tag{1.5}$$

where the "Veilbeine " e^A_I satisfies $G^{AB} = e^A_I e^B_I$. A peculiar feature of (1.4) is the appearance of a non-covariant term $\frac{1}{\sqrt{G}}\partial_B(G^{AB}\sqrt{G})$. This term is necessary for the covariance of the Langevin equation (1.4) under the transformation (1.1) because $\dot{q}^A$ is not a contravariant vector in the sense of Ito's stochastic calculus.[4]

In Ito's stochastic calculus, we have

$$\begin{aligned} \left\langle \dot{f}^A(q)\right\rangle &= \left\langle \partial_B f^A \dot{q}^B + \frac{1}{2}\partial_B\partial_C f^A \dot{q}^B \dot{q}^C + \ldots\right\rangle, \\ &= \left\langle \partial_B f^A \dot{q}^B\right\rangle + \left\langle \partial_B\partial_C f^A\right\rangle \left\langle G^{BC}\right\rangle. \end{aligned} \tag{1.6}$$

This implies that $\dot{q}^A$ is not a contravariant vector under the general coordinate transformation (1.1). Thus when we study the symmetry properties of the system described by the Langevin equation, we must be careful for the transformation property of $\dot{q}^A$. However note that the Langevin equation (1.4) is totally covariant under the general coordinate transformation (1.1). Although the Langevin equation (1.4) is equivalent to F-P equation (1.2) in the sense of Ito's stochastic calculus, the appearance of the non-covariant term makes the study of symmetry properties of the system somewhat complicated.

In stochastic quantization of gravity, it has been pointed out that the Langevin equation takes the form of (1.4) under the following identification[5]

$$\begin{aligned} q^A &\to g_{\mu\nu}(x), \\ G_{AB} &\to G^{\mu\nu;\rho\sigma}_{(DW)}(\zeta)\delta^4(x;x'), \end{aligned} \tag{1.7}$$

where $G^{\mu\nu;\rho\sigma}_{(DW)}(\zeta)$ is so called DeWitt's superspace metric (a metric tensor in the configuration space) which is defined by[6]

$$G^{\mu\nu;\rho\sigma}_{(DW)}(\zeta) = \frac{1}{2}\sqrt{g}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} + \zeta g^{\mu\nu}g^{\rho\sigma}). \tag{1.8}$$

In general, there remains the non-covariant term in (1.4) except for the special value of the constant $\zeta = -1$ in (1.8).

In this short note, we propose a formulation of stochastic quantization of gravity based on a pair of Langevin equations which couple with white noises. The pair of Langevin equations include two independent variables for the gravitational field for which we require a second-class constraint by introducing a Lagrange multiplier field. A similar system of a pair of Langevin equations has been discussed in Ref.7 where the multiplier field of a constraint remains to be governed by an additional Langevin equation. We show that since the constraint is a second-class one, the multiplier field of the constraint is eliminated by the consistency condition of the constraint. We also show that after eliminating the multiplier field for the second-class constraint, the pair of Langevin equation takes the form (1.4) without the non-covariant term. Our claim is that, in D-dimensional space-time, an appropriate choice of the independent variables in a pair of the Langevin equations gives an unique form of the Langevin equation in which the DeWitt's type superspace metric is slightly different from (1.8) and it satisfies the following conditions

$$\begin{aligned} G \equiv \det G_{AB} &= \text{constant}, \\ \partial_B G^{AB} &= 0. \end{aligned} \tag{1.9}$$

This implies that the non-covariant term in (1.4) vanishes in the Langevin equation for gravity which is equivalent to the pair of the langevin equations coupled with white noises.

2. Invariant Path-Integral Measure and The Langevin Equation for Gravity

We first discuss a metric tensor in the configuration space $\{g_{\mu\nu}\}$ (DeWitt's type superspace metric) in connection with the invariant path-integral measure. A Langevin equation for gravity is formulated in arbitrary dimensions by using an appropriate choice of the field variable for the gravitational field and a metric in the configuration space $\{g_{\mu\nu}\}$ (we call it superspace metric in short).

We start from the DeWitt's superspace metric (1.8) which defines an invariant path-integral

$$\int \Pi_x \sqrt{G_{(DW)}(\zeta)} d[\delta g_{\mu\nu}] \exp(- \int d^4x G^{\mu\nu;\rho\sigma}_{(DW)}(\zeta) \delta g_{\mu\nu} \delta g_{\rho\sigma}) = \text{constant}, \qquad (2.1)$$

where $G_{(DW)}(\zeta)$ denotes the determinant of the DeWitt's metric $G^{\mu\nu;\rho\sigma}_{(DW)}(\zeta)$. Especially in 4-dimensions, the DeWitt's superspace metric satisfies the condition (1.9). More explicitly

$$\det\{G^{\mu\nu;\rho\sigma}_{(DW)}(\zeta)\delta^4(x;x')\} = \text{constant}. \qquad (2.2)$$

Furthermore for a special choice of the constant $\zeta = -1$, it satisfies

$$\frac{\delta}{\delta g_{\rho\sigma}}\{G^{(DW)}_{\mu\nu;\rho\sigma}(\zeta = -1)\delta^4(x;x')\} = 0, \qquad (2.3)$$

where $G^{(DW)}_{\mu\nu;\rho\sigma}(\zeta = -1)$ is the inverse of $G^{\mu\nu;\rho\sigma}_{(DW)}(\zeta = -1)$. These conditions are sufficient to obtain a Langevin equation for gravity which does not include the non-covariant term in (1.4) in the case of 4-dimensional space-time . We note that, in D-dimensions, DeWitt's metric does not satisfy the condition (2.2).

In the following, we want to argue a reason why the superspace metric $G^{\mu\nu;\rho\sigma}_{(DW)}(\zeta)$ satisfies the condition (2.2) only in 4-dimensional space-time. Following the analysis by Fujikawa, the fundamental variables which give a BRS invariant path-integral measure in D-dimensional space-time are defined by[8]

$$\begin{aligned}\tilde{g}_{\mu\nu} &\equiv g^{k} g_{\mu\nu}; \qquad k = \frac{D-4}{4D}, \\ \tilde{g}^{\mu\nu} &\equiv g^{l} g^{\mu\nu}; \qquad l = \frac{D+4}{4D}.\end{aligned} \tag{2.4}$$

In terms of the field variables (2.4), an invariant path-integral in DeWitt's sense may be given by

$$\int \Pi_x \sqrt{\tilde{G}(\zeta_D)}\, d[\delta\tilde{g}_{\mu\nu}] \exp\Big(- \int d^D x\, \tilde{G}^{\mu\nu:\rho\sigma}(\zeta_D)\delta\tilde{g}_{\mu\nu}\delta\tilde{g}_{\rho\sigma}\Big) = \text{constant}. \tag{2.5}$$

Here the most general form of the superspace metric $\tilde{G}^{\mu\nu:\rho\sigma}(\zeta_D)$ is determined from the invariance of (2.5) under the D-dimensional general coordinate transformation as follows

$$\tilde{G}^{\mu\nu:\rho\sigma}(\zeta_D) = \frac{1}{2}\tilde{g}^{-\frac{2}{D}}(\tilde{g}^{\mu\rho}\tilde{g}^{\nu\sigma} + \tilde{g}^{\mu\sigma}\tilde{g}^{\nu\rho} + \zeta_D \tilde{g}^{\mu\nu}\tilde{g}^{\rho\sigma}), \tag{2.6}$$

where $\tilde{g} \equiv \det\tilde{g}_{\mu\nu}$. ζ_D is an arbitrary constant. Since $\Pi_x d[\delta\tilde{g}_{\mu\nu}]$ is just the invariant measure, the superspace metric (2.6) should satisfy the condition corresponding to (2.2) in D-dimensions, namely,

$$\det\{\tilde{G}^{\mu\nu:\rho\sigma}(\zeta_D)\delta^D(x;x')\} = \text{constant}. \tag{2.7}$$

It is now clear why the DeWitt's superspace metric (1.8) satisfies the condition (2.2) only in 4-dimensional space-time: From (2.4), $g_{\mu\nu}$ and $\sqrt{g}g^{\mu\nu}$ are the fundamental variables for the path-integral <u>only</u> in 4-dimensions. In D-dimensional space-time, the variables in (2.4) define a BRS invariant path-integral measure which implies a constant determinant of the superspace metric (2.6).

In D-dimensions, the superspace metric (2.6) also satisfies the condition corresponding to (2.3) for a special value of the constant $\zeta_D = -\frac{4}{D}$. For the convenience in the following discussions we denote $\tilde{G}^{\mu\nu:\rho\sigma}(\zeta_D = -\frac{4}{D}) \equiv \tilde{G}^{\mu\nu:\rho\sigma}$. The inverse of the superspace metric $\tilde{G}^{\mu\nu:\rho\sigma}$ is given by

$$\tilde{G}_{\mu\nu:\rho\sigma} \equiv \frac{1}{2}\tilde{g}^{-\frac{2}{D}}\left(\tilde{g}_{\mu\rho}\tilde{g}_{\nu\sigma} + \tilde{g}_{\mu\sigma}\tilde{g}_{\nu\rho} - \frac{4}{D}\tilde{g}_{\mu\nu}\tilde{g}_{\rho\sigma}\right). \tag{2.8}$$

which satisfies

$$\frac{\delta}{\delta\tilde{g}_{\rho\sigma}}\{\tilde{G}_{\mu\nu:\rho\sigma}\delta^D(x;x')\} = 0. \tag{2.9}$$

In (2.9), the functional derivative is defined by

$$\frac{\delta}{\delta\tilde{g}_{\mu\nu}(x)}\tilde{g}_{\rho\sigma}(x') = \frac{1}{2}(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\mu_\sigma\delta^\nu_\rho)\delta^D(x;x'), \tag{2.10}$$

where δ-function $\delta^D(x;x')$ is a by-scalar density,

$$\int d^Dx\delta^D(x;x') = \int d^Dx'\delta^D(x;x') = 1. \tag{2.11}$$

These two conditions, (2.7) and (2.9), are a sufficient condition to obtain a Langevin equation for gravity without the non-covariant term appearing in (1.4).

A comment on the choice $\zeta_D = -\frac{4}{D}$ is in order. The variations of the two fundamental variables in (2.4) can be connected with the superspace metric if and only if $\zeta_D = -\frac{4}{D}$

$$\delta\tilde{g}^{\mu\nu} = -\tilde{G}^{\mu\nu:\rho\sigma}\delta\tilde{g}_{\rho\sigma}. \tag{2.12}$$

Now we discuss the Langevin equation for gravity. By using the choice $\zeta_D = -\frac{4}{D}$, we obtain a Langevin equation for gravity.

$$\begin{aligned} \dot{\tilde{g}}_{\mu\nu} &= -\alpha\tilde{G}_{\mu\nu:\rho\sigma}\frac{\delta}{\delta\tilde{g}_{\rho\sigma}}S_E + \tilde{\xi}_{\mu\nu}, \\ S_E &\equiv \int d^Dx\,\sqrt{g}R. \end{aligned} \tag{2.13}$$

The functional derivative with respect to $\tilde{g}^{\mu\nu}$ should be taken after rewriting the Einstein action by using the variable $\tilde{g}^{\mu\nu}$. The constant *alpha* should be imaginary if we consider (2.13) in D-dimensional Minkowski space-time, while it is a positive constant in D-dimensional Euclidean space-time. Note that the Langevin equation does not include the non-covariant term because of the appropriate choice of the constant $\zeta_D = -\frac{1}{D}$. The correlation of the noise variable $\tilde{\xi}_{\mu\nu}$ may be given by

$$\left\langle \tilde{\xi}_{\mu\nu}(x,t)\tilde{\xi}_{\rho\sigma}(x',t')\right\rangle = \frac{2}{\beta}\left\langle \tilde{G}_{\mu\nu;\rho\sigma}\right\rangle \delta^D(x;x')\delta(t-t'), \tag{2.14}$$

from the requirement of the invariance under the D-dimensional general coordinate transformation. The expectation value is taken in the sense of the Ito's stochastic calculus. From the relation (2.12), the Langevin equation (2.13) is also rewritten as

$$\dot{\tilde{g}}_{\mu\nu} = \alpha\frac{\delta}{\delta\tilde{g}^{\mu\nu}}S_E + \tilde{\xi}_{\mu\nu}. \tag{2.15}$$

In the following section, we describe how to realize the Langevin equation (2.13) with (2.14) (or equivalently (2.15)) in terms of a pair of Langevin equations which couple with white noises.

3. Stochastic Quantization of Gravity with the Langevin Equation

In order to introduce white noise variables in SQ of the gravitational field, we first consider a system in which the variables $g_{\mu\nu}(x)$ and $\sqrt{g}g^{\mu\nu}(x)$ are independent each other, that is, we introduce two independent field variables $g_{\mu\nu}(x)$ and $h^{\mu\nu}(x)$ which satisfy the constraint

$$h^{\mu\lambda}g_{\lambda\nu} - \sqrt{g}\delta^\mu_\nu = 0, \tag{3.1}$$

where $g \equiv \det g_{\mu\nu}$. By solving the constraint, we have

$$h^{\mu\nu}(x) = \sqrt{g}g^{\mu\nu}(x). \tag{3.2}$$

The system including these two independent variables is invariant under the general coordinate transformation in 4-dimensions provided that the transformation property of these variables $g_{\mu\nu}(x)$ and $h^{\mu\nu}(x)$ are that of a tensor and a tensor density , respectively.

We extend the constraint (3.1) to the case of D-dimensions. For the variables (2.4), the constraint is given by

$$\tilde{h}^{\mu\lambda}\tilde{g}_{\lambda\nu} - \tilde{g}^{\frac{1}{D}}\delta^{\mu}_{\nu} = 0, \tag{3.3}$$

where $\sqrt{g} = \tilde{g}^{\frac{1}{D}}$. This leads

$$\tilde{h}^{\mu\nu}(x) = \tilde{g}^{\mu\nu}(x), \tag{3.4}$$

where $\tilde{g}^{\mu\nu}$ is also defined in (2.4).

For these independent variables, $\tilde{g}_{\mu\nu}(x)$ and $\tilde{h}^{\mu\nu}(x)$, we assume a pair of the Langevin equations with the constraint (3.3)

$$\begin{aligned} \dot{\tilde{g}}_{\mu\nu} &= -\gamma_1 \frac{\Delta S_{cl}}{\Delta \tilde{h}^{\mu\nu}} + \tilde{\eta}_{\mu\nu}, \\ \dot{\tilde{h}}^{\mu\nu} &= -\gamma_2 \frac{\Delta S_{cl}}{\Delta \tilde{g}_{\mu\nu}} + \tilde{\eta}^{\mu\nu}, \end{aligned} \tag{3.5}$$

where γ_1 and γ_2 are some constants. The classical action S_{cl} consists of the Einstein action and the second-class constraint

$$S_{cl} = S_E + \int d^D x \; \phi^{\nu}_{\mu}(\tilde{h}^{\mu\lambda}\tilde{g}_{\lambda\nu} - \tilde{g}^{\frac{1}{D}}\delta^{\mu}_{\nu}). \tag{3.6}$$

Here the auxiliary field ϕ^{μ}_{ν} is introduced as a Lagrange multiplier field of the constraint (3.3). In the derivative (or variation) Δ, the variables $\tilde{g}_{\mu\nu}$ and $\tilde{h}^{\mu\nu}$ are regarded to be independent each other and

$$\frac{\Delta \tilde{g}_{\rho\sigma}(x')}{\Delta \tilde{g}_{\mu\nu}(x)} = \frac{\Delta \tilde{h}^{\mu\nu}(x')}{\Delta \tilde{h}^{\rho\sigma}(x)} = \frac{1}{2}(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} + \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho})\delta^{D}(x;x'). \tag{3.7}$$

The pair of the Langevin equations (3.5) with (3.6) is invariant under the fictitious time independent general coordinate transformation in which the transformation parameter is <u>independent</u> of the fictitious time t, provided that the noise fields $\tilde{\eta}_{\mu\nu}$ and $\tilde{\eta}^{\mu\nu}$ are transformed by the same transformation rules as those of $\tilde{g}_{\mu\nu}$ and $\tilde{g}^{\mu\nu}$, respectively.

The correlations of the white noise variables are given by

$$
\begin{aligned}
\left\langle \tilde{\eta}^{\mu\nu}(x,t)\tilde{\eta}_{\rho\sigma}(x',t')\right\rangle &= \frac{2}{\gamma_3}(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} + \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho})\delta^{D}(x;x')\delta(t-t'), \\
\left\langle \tilde{\eta}^{\mu\nu}(x,t)\tilde{\eta}^{\rho\sigma}(x',t')\right\rangle &= \left\langle \tilde{\eta}_{\mu\nu}(x,t)\tilde{\eta}_{\rho\sigma}(x',t')\right\rangle = 0.
\end{aligned}
\tag{3.8}
$$

γ_1, γ_2 in (3.5) and γ_3 in (3.8) are determined from the requirement that (3.5) should reproduce the Langevin equation of gravity (2.13). Note that the transformation property of the white noises are consistent with the correlation (3.8).

Now we show that the pair of the Langevin equations (3.5) with (3.8) is equivalent to the one (2.13) with (2.14). We first show that the multiplier field ϕ^{μ}_{ν} of the constraint is eliminated by using the consistency condition of the constraint. This implies that the constraint (3.3) is a second-class one.[9] In fact, by the consistency condition

$$\frac{\partial}{\partial t}\left(\tilde{h}^{\mu\lambda}\tilde{g}_{\lambda\nu} - \tilde{g}^{\frac{2}{D}}\delta^{\mu}_{\nu}\right) = 0, \tag{3.9}$$

and the constraint (3.3), we obtain

$$
\begin{aligned}
\dot{\tilde{g}}_{\mu\nu} = &-\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}\left(\frac{\Delta S_E}{\Delta\tilde{h}^{\mu\nu}}\Big|_{\tilde{h}^{\mu\nu}=\tilde{g}^{\mu\nu}} - \tilde{G}_{\mu\nu:\rho\sigma}\frac{\Delta S_E}{\Delta\tilde{g}_{\rho\sigma}}\Big|_{\tilde{h}^{\mu\nu}=\tilde{g}^{\mu\nu}}\right) \\
&+ \frac{1}{\gamma_1+\gamma_2}\left(\gamma_2 \qquad \gamma_1\tilde{G}_{\mu\nu:\rho\sigma}\tilde{\eta}^{\rho\sigma}\right),
\end{aligned}
\tag{3.10}
$$

where $\tilde{G}_{\mu\nu:\rho\sigma}$ is defined in (2.8). The relation (2.12) implies that

$$\frac{\delta}{\delta\tilde{g}^{\mu\nu}}S_E = \frac{\Delta S_E}{\Delta\tilde{h}^{\mu\nu}}\Big|_{\tilde{h}^{\mu\nu}=\tilde{g}^{\mu\nu}} - \tilde{G}_{\mu\nu:\rho\sigma}\frac{\Delta S_E}{\Delta\tilde{g}_{\rho\sigma}}\Big|_{\tilde{h}^{\mu\nu}=\tilde{g}^{\mu\nu}}, \tag{3.11}$$

holds. If we choose the constants $-\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2} = \alpha$, we obtain

$$
\begin{aligned}
\dot{\tilde{g}}_{\mu\nu} &= \alpha\frac{\delta}{\delta\tilde{g}^{\mu\nu}}S_E + \tilde{\xi}_{\mu\nu}, \\
\tilde{\xi}_{\mu\nu} &= \frac{1}{\gamma_1+\gamma_2}\left(\gamma_2\tilde{\eta}_{\mu\nu} - \gamma_1\tilde{G}_{\mu\nu:\rho\sigma}\tilde{\eta}^{\rho\sigma}\right).
\end{aligned}
\tag{3.12}
$$

The correlation of the noise variable $\tilde{\xi}_{\mu\nu}$ is evaluated in the sense of Ito's stochastic calculus as follows

$$\begin{aligned}\left\langle \tilde{\xi}_{\mu\nu}(x,t)\tilde{\xi}_{\rho\sigma}(x',t')\right\rangle &= -\frac{\gamma_1\gamma_2}{(\gamma_1+\gamma_2)^2}\Big\{ \left\langle \tilde{G}_{\mu\nu\ \alpha\beta}\right\rangle \left\langle \tilde{\eta}_{\rho\sigma}\tilde{\eta}^{\alpha\beta}\right\rangle \\ &\qquad + \left\langle \tilde{G}_{\cdot\ \cdot\ \alpha\beta}\right\rangle \left\langle \tilde{\eta}_{\mu\nu}\tilde{\eta}^{\alpha\beta}\right\rangle \Big\} \\ &= \frac{2}{\beta}\left\langle \tilde{G}_{\mu\nu\ \rho\sigma}\right\rangle \delta^D(x;x')\delta(t-t'),\end{aligned} \tag{3.13}$$

where γ_3 in (3.8) is chosen to be $\gamma_3 = \frac{2\alpha\beta}{\gamma_1+\gamma_2}$.

The equations in (3.5) are not independent each other after eliminating the Lagrange multiplier field ϕ^{μ}_{ν}, we also have

$$\begin{aligned}\dot{\tilde{g}}^{\mu\nu} &= \alpha\frac{\delta}{\delta\tilde{g}_{\mu\nu}}S_E + \tilde{\xi}^{\mu\nu}, \\ \left\langle \tilde{\xi}^{\mu\nu}(x,t)\tilde{\xi}^{\rho\sigma}(x',t')\right\rangle &= \frac{2}{\beta}\left\langle \tilde{G}^{\mu\nu\ \rho\sigma}\right\rangle \delta^D(x;x')\delta(t-t').\end{aligned} \tag{3.14}$$

From (3.12) and (3.13), we find that the choice of the superspace metric (2.8) is properly realized in terms of the pair of the Langevin equations coupled with white noises, namely, the metric tensor G^{AB} is given by

$$\{G^{AB}\} = \{\tilde{G}_{\mu\nu\ \rho\sigma}\delta^D(x;x')\}. \tag{3.15}$$

The procedure we have illustrated above is equivalent to that of *non-linear σ-model*, for example $O(N)$ non-linear σ-model. We first consider an enlarged configuration space which is spanned by $\{\tilde{g}_{\mu\nu}\}$ and $\{\tilde{h}^{\mu\nu}\}$. Then we require a second-class constraint. it defines the induced metric $\tilde{G}_{\mu\nu\ \rho\sigma}\delta^D(x;x')$ on the configuration space of the gravitational field which is parametrized by the coordinate $\{\tilde{g}_{\mu\nu}\}$.

4. Conclusions

We studied the Langevin system which describes *stochastic quantization* of gravity. To formulate the Langevin system coupled with white noises, we introduced two *independent* variables, each of them defines a BRS invariant path-integral measure, with a second-class constraint. After eliminating the multiplier field of the second-class constraint, we found the pair of the Langevin equations realizes the Langevin equation of gravity which is manifestly invariant under the general coordinate transformation in *superspace* (i.e., redefinition of the field variables). In constructing the pair of the Langevin equations, we claimed that the choice of the field variables is important. We showed that the appropriate choice of the independent variables *determines uniquely* the superspace metric and gives the Langevin equation which does not include the non-covariant term in arbitrary dimensions.

Our formulation may provide a basic Langevin system for numerical analysis of quantum gravity. One of the main advantages is that it couples with white noises. In order to study the pair of the Langevin equations by solving the second-class constraint on time by time, however, there remains some problems to be solved. First of all, the DeWitt's type superspace metric is not positive definite. This may be improved by a "Wick rotation " of the superspace metric, which is essentially equivalent to the Wick rotation of the conformal mode. Another problem is that the Langevin system is, in general, complex which comes from the complex white noise in (3.8). The situation is slightly different from that of stochastic quantization in Minkowski space-time because the complexity comes from only the complex noise variables. We may also encounter the problem of the renormalizability to obtain some sensible results. Even if we consider quantum gravity by the stochastic quantization approach, the renormalizability may not be improved in a perturbative sense, however, it is still important to clarifies the BRS invariant structure of quantum gravity in the context of stochastic quantization. The analysis of the BRS structure based on our formulation is to appear in elsewhere.[14]

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