FR9011189

LAPP - TH - 214/88
March 1988

Fermionic Construction of Vertex Operators for Twisted Affine Algebras

L. Frappat [†‡], A. Sciarrino [§], P. Sorba [†]

Abstract

We construct vertex operator representations of the twisted affine algebras in terms of fermionic (or parafermionic in some cases) elementary fields. The folding method applied to the extended Dynkin diagrams of the affine algebras allows us to determine explicitly these fermionic fields as vertex operators.

† L.A.P.P., B.P. 909, 74019 Annecy-le-Vieux, France

§ Dipartimento di Scienze Fisiche, 80125 Napoli, Italy
and I.N.F.N., Sezione di Napoli, Italy

‡ also at Université de Savoie, 74000 Annecy, France

1- INTRODUCTION

The Frenkel-Kac-Segal construction [1], via vertex operators, of level one representations for untwisted affine algebras $\mathcal{G}^{(1)}$ relative to simple simply laced Lie algebras $\mathcal{G}$ constitute today an unavoidable section in any review as well as on string theory as on Kac-Moody algebras. Since then, the case of vertex operators for non simply laced algebras has been considered [2,3] and also the case of twisted vertex operators [4,5]. This last study [5,6] is of first interest in the context of orbifold compactification and explicit breaking in superstring theory. Moreover when the twist is associated to an outer automorphism of the root lattice Λ of the simply laced algebra $\mathcal{G}$, the developed method [5,6] gives rise to a direct construction of the vertex representation for the twisted Kac-Moody algebras (see also ref. [3]).

Hereafter we would like to present a quite different construction of level one representations of twisted affine algebras. We will call our construction fermionic since fermionic (or parafermionic in some cases) auxiliary fields will be explicitly introduced to compensate the fermionic nature of the "bare" vertex operators associated to the (affine) short roots. Such an approach has already been used in ref. [2] in order to build vertex operators relative to non simply laced algebras. Taking as an example the case of B_l or $\mathcal{SO}(2l+1)$ algebras with the set of roots $\{\pm e_i \pm e_j, \pm e_i\}$ with $1 \leq i \neq j \leq l$, one knows that the vertex representation of $\mathcal{SO}(2l)^{(1)}$ can be written in terms of bilinear in fermions :

$$E^{e_i+e_j} = \Psi^{e_i}(z).\Psi^{e_j}(z) \qquad i \neq j$$

while for the short roots an auxiliary field $\Gamma(z)$ allows to recover for the short roots the bosons :

$$E^{e_i} = \Psi^{e_i}(z).\Gamma(z)$$

with a conformal weight one and the moments of which satisfying the expected commutation relations of the $\mathcal{SO}(2l+1)^{(1)}$ algebra. Actually our construction is a sort of generalization to the case of twisted affine algebras of the method of ref. [2] for non simply laced algebras. Indeed, denoting α_L, α_S and $\alpha_S + \delta/2$ the long, short and affine short root respectively of a twisted algebra $\mathcal{G}^{(2)}$, one will have to introduce now two types of auxiliary fields $\Gamma(z)$ and $\Gamma'(z)$ such that :

$$E^{\alpha_S} = \Psi^{\alpha_S}(z).\Gamma(z)$$

$$E^{\alpha_S+\delta/2} = \Psi^{\alpha_S}(z).\Gamma'(z)$$

the number of different $\Gamma(z)$ and $\Gamma'(z)$ being equal to the number of orbits of the short roots under the Weyl group generated by the long ones.

In our approach, symmetries of the Dynkin diagrams (or DD) relative to untwisted affine simply laced algebras $\mathcal{G}^{(1)}$ will be considered. Such symmetries are associated to outer automorphisms of $\mathcal{G}^{(1)}$ and the set of twisted Kac-Moody algebras $A_l^{(2)}$, $D_l^{(2)}$, $E_6^{(2)}$ and $D_4^{(3)}$ can be obtained as superalgebras of $\mathcal{G}^{(1)}$ invariant under such automorphisms.

The aim of this paper is two fold. First, due to the fundamental role of fermions in the up to now proposed unified string theories [7,8], one may think such a fermionic construction to be well adapted for a practical use in string physics. Moreover, we hope that the simple properties on twisted affine algebras developed hereafter, especially with the help of the symmetries of the extended Dynkin diagrams (or EDD), will lead to a better understanding on their structure and their possible relevance.

The paper is organized as follows. After recalling some definitions and basic properties on the Kac-Moody algebras in section 2, we show in section 3 how special symmetries of the EDD of an affine algebra $\mathcal{G}^{(1)}$ easily allow us to obtain the twisted affine algebras as $\mathcal{G}^{(1)}$ subalgebras left invariant under such discrete symmetries. This folding techniques is then used in section 4 to build vertex operators for $\tilde{\mathcal{G}}^{(m)}$ $(m \neq 1)$ as bilinear in fermionic fields. Finally, for each type of twisted algebra, an explicit construction is performed in section 5 where in particular the cocycle problem is worked out in detail.

2- A REMINDER ABOUT KAC-MOODY ALGEBRAS

Let us briefly recall the definition of untwisted and twisted affine algebras [9]. A Kac-Moody algebra $\mathcal{G}^{(1)}$ constructed from a simple Lie algebra $\mathcal{G}$ is the loop algebra

$$\mathcal{G}^{(1)} = C(t,t^{-1}) \otimes \mathcal{G} \quad \oplus \quad C\, c \qquad [2.1]$$

in which we denote by $C(t,t^{-1})$ the algebra of Laurent polynomials in the complex variable t and by c the central extension term. Commutation relations among generators of $\mathcal{G}^{(1)}$ are :

$$[t^m \otimes a, t^n \otimes b] = t^{m+n} \otimes [a,b] + m\ (a,b)\ \delta_{m+n,0}\ c \qquad [2.2]$$

where a, b are $\mathcal{G}$ generators and (a,b) denotes the usual Killing form on $\mathcal{G}$.

A twisted affine algebra $\mathcal{G}^{(m)}$ $(m \neq 1)$ is defined with the help of an outer automorphism τ of $\mathcal{G}$ of order m (i.e. m is the smallest positive integer such that $\tau^m = 1$) in such a way that its elements

$$t^n \otimes a \ \equiv\ |t|^n e^{in\theta} \otimes a \ \equiv\ f(\theta) \otimes a \qquad [2.3]$$

satisfy

$$f(\theta + 2\pi) \otimes a = f(\theta) \otimes \tau(a) \qquad [2.4]$$

Setting

$$\mathcal{G}_k = \{a \in \mathcal{G} \mid \tau(a) = e^{2i\pi k/m} a\} \qquad [2.5]$$

where the quantities $e^{2i\pi k/m}$ $(k = 0, 1, ..., m-1)$ are the m eigenvalues of τ , one has for $\mathcal{G}^{(m)}$ the decomposition

$$\mathcal{G}^{(m)} = \bigoplus_{k=0\ ,\ n\in Z}^{m-1} t^{n+k/m} \otimes \mathcal{G}_k \qquad [2.6]$$

and for

$$\mathcal{G} = \mathcal{G}_o \oplus \mathcal{G}_1 \oplus ... \oplus \mathcal{G}_{m-1} \qquad [2.7]$$

the Z/mZ gradation :

$$[\mathcal{G}_k, \mathcal{G}_l] \subset \mathcal{G}_{k+l} \qquad [\text{mod } m] \qquad [2.8]$$

Two remarks are in order. First $\mathcal{G}^{(m)}$ is actually a subalgebra of $\mathcal{G}^{(1)}$ (this is obvious when replacing in eq. [2.2] the t variable by z^m in order for the $\mathcal{G}^{(m)}$-generators to appear with integer powers in the complex variable). Secondly, if τ were not an outer but an inner automorphism of $\mathcal{G}$, then $\mathcal{G}^{(m)}$ would be isomorphic to $\mathcal{G}^{(1)}$ itself. By the same kind of arguments, affine algebras associated with a simple Lie algebra $\mathcal{G}$ will be in one to one correspondance with the quotient $\frac{Aut(\mathcal{G})}{Int(\mathcal{G})}$ where $Aut(\mathcal{G})$ (resp. $Int(\mathcal{G})$) denotes the group of automorphisms (resp. inner automorphisms) of $\mathcal{G}$. If Δ_R denotes the root lattice of $\mathcal{G}$ and W its corresponding Weyl group, one has the well-known isomorphisms :

$$\frac{Aut(\mathcal{G})}{Int(\mathcal{G})} \simeq \frac{Aut(\Delta_R(\mathcal{G}))}{Int(\Delta_R(\mathcal{G}))} = \mathcal{F}(\mathcal{G}) \qquad [2.9]$$

where $\mathcal{F}(\mathcal{G})$ is the symmetry group of the Dynkin diagram of $\mathcal{G}$. In particular, since $\mathcal{F}(\mathcal{G})$ is the identity when $\mathcal{G}$ is non simply laced or equal to E_7 or E_8, only untwisted affine algebras $\mathcal{G}^{(1)}$ will be associated to such a $\mathcal{G}$, while the $Z_2 = \{\pm 1\}$ symmetry of the A_l, D_l $(l \neq 4)$ or E_6 Dynkin diagrams leads to the existence of affine algebras $A_l^{(m)}$, $D_l^{(m)}$, $E_6^{(m)}$ with $m = 1, 2$, and finally the S_3 symmetry group of D_4 allows the construction of $D_4^{(m)}$ with $m = 1, 2, 3$.

3- FOLDING AND SYMMETRIES OF EXTENDED DYNKIN DIAGRAMS

3-1- Folding of extended Dynkin diagrams

Actually the Dynkin diagrams relative to twisted affine algebras can be obtained by folding of the EDD of an untwisted affine one. Let us precise how it works in the general case.

We consider an affine Kac-Moody algebra $\mathcal{G}^{(1)}$ with its EDD associated to a simple root system R :

$$R = \{\alpha_o = \delta - \alpha_h,\ \alpha_i,\ 1 \leq i \leq l = \text{rank } \mathcal{G}\} \qquad [3.1]$$

α_o is the affine root, δ is the isotropic direction and α_h is the highest root of $\mathcal{G}$. If Δ_o is the root system for the horizontal algebra $\mathcal{G}$ and Δ the root system for the affine algebra $\mathcal{G}^{(1)}$, one has

$$\Delta = \{\Delta_o + m\delta,\ n\delta,\ m \in \mathbf{Z},\ n \in \mathbf{Z}^*\} \qquad [3.2]$$

We assume that the EDD of $\mathcal{G}^{(1)}$ has a symmetry, which can be related to an outer automorphism τ of $\mathcal{G}^{(1)}$. If p is the order of the automorphism τ (i.e. the smallest integer such that $\tau^p = 1$), the set of roots of the affine algebra invariant under τ is given by :

$$\tilde{\Delta} = \left\{ \frac{1}{p}(\overline{\beta} + \tau(\overline{\beta}) + ... + \tau^{p-1}(\overline{\beta}))\ ,\ \overline{\beta} \in \Delta \right\} \qquad [3.3]$$

Let us note $\tilde{\mathcal{G}}^{(m)}$ this invariant algebra. The simple root system of $\tilde{\mathcal{G}}^{(m)}$ is given by the invariant combinations of simple roots of $\mathcal{G}^{(1)}$ under τ (m is an integer equal to 1 or p , see further) :

$$\tilde{R} = \left\{ \frac{1}{p}(\overline{\beta} + \tau(\overline{\beta}) + ... + \tau^{p-1}(\overline{\beta}))\ ,\ \overline{\beta} \in R \right\} \qquad [3.4]$$

and the associated DD is obtained by folding of the DD of $\mathcal{G}^{(1)}$ according to the symmetry corresponding to τ.

At this point, one has to consider two different cases :

1) The automorphism τ has a trivial action on the affine root α_o (i.e. α_o is invariant under τ). In this case, the automorphism τ is actually an outer automorphism of the horizontal algebra $\mathcal{G}$. One obtains an affine invariant algebra $\tilde{\mathcal{G}}^{(1)}$, the horizontal part being a non simply laced algebra.

2) The automorphism τ has a non trivial action on the affine root (i.e. it mixes the affine direction and the horizontal roots of $\mathcal{G}$). In this case, τ is no more an

outer automorphism of $\mathcal{G}$. One obtains as invariant algebra under τ, a twisted affine algebra $\tilde{\mathcal{G}}^{(p)}$, where p is the order of τ. Notice however, that if there is no outer automorphism of the corresponding Lie algebra $\tilde{\mathcal{G}}$, the twisted algebra $\tilde{\mathcal{G}}^{(p)}$ is isomorphic to the affine algebra $\tilde{\mathcal{G}}^{(1)}$.

In the following, we will write the commutation relations of the affine algebra in the Cartan-Weyl basis, i.e. :

$$[H^i_m, H^j_n] = m\ \delta_{m+n}\ \delta_{ij} \tag{3.5}$$

$$[H^i_m, E^\alpha_n] = \alpha^i E^\alpha_{m+n} \tag{3.6}$$

$$[E^\alpha_m, E^\beta_n] = \begin{cases} \epsilon(\alpha,\beta)E^{\alpha+\beta}_{m+n} & \text{if } \alpha+\beta \text{ is a root} \\ \alpha.H_{m+n} + m\ \delta_{m+n} & \text{if } \beta = -\alpha \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

• Case 1 :

Take for $\mathcal{G}^{(1)}$ a simply laced Kac-Moody algebra. As mentioned above, if α_o is invariant under τ, the problem is equivalent as the folding of a simply laced Lie algebra. The folding leads in this case to a non simply laced algebra. Although this has been already done [2], let us remind in few lines the philosophy of the method (for simplicity, we treat only the case of an automorphism of order 2).

One can always divide the root system Δ_o of $\mathcal{G}$ into two separate subsets Δ^L_o and Δ^S_o such that :

$$\Delta^L_o = \{\beta \in \Delta_o \mid \beta = \tau(\beta)\} \tag{3.8}$$

$$\Delta^S_o = \{\beta \in \Delta_o \mid \beta.\tau(\beta) = 0\} \tag{3.9}$$

(i.e. we assume that the Dynkin diagrams considered in this construction are such that two connected roots cannot be related to each other by the automorphism τ).

If β is a root of $\mathcal{G}$, we set $\overline{\beta} = \beta + m\delta$ ($m \in \mathbf{Z}$) and $\overline{\beta}$ is a root of $\mathcal{G}^{(1)}$. One can consider that the step operator E^β_m corresponds to this root $\overline{\beta}$.

Now the automorphism τ is extended from the root system Δ of $\mathcal{G}^{(1)}$ to the algebra $\mathcal{G}^{(1)}$ by

$$\tau(E^\beta_m) = E^{\tau(\beta)}_m \tag{3.10}$$

$$x.\tau(H_m) = \tau(x).H_m \tag{3.11}$$

the 2-cocycle being chosen invariant under τ

$$\epsilon(\beta_1,\beta_2) = \epsilon(\tau(\beta_1),\tau(\beta_2)) \tag{3.12}$$

which is still possible, since $\beta = \tau(\beta)$ or $\beta.\tau(\beta) = 0$.
Then taking y invariant under τ, one has

$$[y.H_n, E_m^\beta + E_m^{\tau(\beta)}] = \frac{1}{2} y.(\beta + \tau(\beta))(E_m^\beta + E_m^{\tau(\beta)}) \qquad [3.13]$$

$$[d, E_m^\beta + E_m^{\tau(\beta)}] = m(E_m^\beta + E_m^{\tau(\beta)}) \qquad [3.14]$$

(d is the derivation operator)
which means that the generator attached to the root $\frac{1}{2}(\beta + \tau(\beta))$ of $\mathcal{G}^{(1)}$ at level m is

$$E_m^\beta \quad \text{if} \quad \beta = \tau(\beta) \qquad [3.15]$$

$$\frac{1}{\sqrt{2}}(E_m^\beta + E_m^{\tau(\beta)}) \quad \text{if} \quad \beta.\tau(\beta) = 0 \qquad [3.16]$$

• Case 2 :
We suppose for simplicity that τ is of order 2. As before, one starts from an affine algebra $\mathcal{G}^{(1)}$. Now the automorphism τ does not leave the affine root invariant : $\tau(\alpha_o) = \gamma$ where γ is a simple root of $\mathcal{G}$. Notice that $\mathcal{G}$ is not necessarily a simply laced algebra.
If Δ is the root system for the affine algebra $\mathcal{G}^{(1)}$, the root system of the affine algebra $\tilde{\mathcal{G}}^{(m)}$ invariant under τ is

$$\tilde{\Delta} = \left\{ \frac{1}{2}(\overline{\beta} + \tau(\overline{\beta}) \mid \beta \in \Delta \right\} \qquad [3.17]$$

In particular, if $R = \{\alpha_o, \alpha_1, ..., \alpha_l\}$ is the simple root system of $\mathcal{G}^{(1)}$, the simple root system of $\tilde{\mathcal{G}}^{(m)}$ is

$$\tilde{R} = \left\{ \alpha_i' = \frac{1}{2}(\alpha_i + \tau(\alpha_i)) \ , \ 0 \leq i \leq l = \text{rank } \mathcal{G} \right\} \qquad [3.18]$$

and the affine root is $\alpha_o' = \frac{1}{2}(\alpha_o + \alpha_1)$.
The roots of $\mathcal{G}^{(1)}$ can be decomposed in the R-basis :

$$\overline{\beta} = \sum_{i=0}^{l} m_i \alpha_i \qquad [3.19]$$

whereas the roots of $\tilde{\mathcal{G}}^{(m)}$ are decomposed in the $\tilde{R}$-basis :

$$\overline{\beta}' = \frac{1}{2}(\overline{\beta} + \tau(\overline{\beta})) = \sum_{i=0}^{l'} m_i' \alpha_i' \qquad l' = \text{rank } \tilde{\mathcal{G}}^I \qquad [3.20]$$

($\tilde{\mathcal{G}}^I$ is the invariant integral subalgebra of $\tilde{\mathcal{G}}^{(m)}$).
Using these decompositions, one can divide the root system $\tilde{\Delta}$ of the affine algebra $\tilde{\mathcal{G}}^{(m)}$ invariant under τ in three subsets :

$$\tilde{\Delta}^I_L = \left\{ \frac{1}{2}(\overline{\beta} + \tau(\overline{\beta})) \mid \overline{\beta} = \tau(\overline{\beta}) \right\} \qquad [3.21]$$

which corresponds to the long roots of $\tilde{\mathcal{G}}^I$, appearing at each integral level.

$$\tilde{\Delta}^I_S = \left\{ \frac{1}{2}(\overline{\beta} + \tau(\overline{\beta})) \mid \overline{\beta}.\tau(\overline{\beta}) = 0 \ , \ m'_o = 0 \ \ [\text{mod } 2] \right\} \qquad [3.22]$$

which corresponds to the short roots of $\tilde{\mathcal{G}}^I$, appearing at each integral level.

$$\tilde{\Delta}^T = \left\{ \frac{1}{2}(\overline{\beta} + \tau(\overline{\beta})) \mid \overline{\beta}.\tau(\overline{\beta}) = 0 \ , \ m'_o = 1 \ \ [\text{mod } 2] \right\} \qquad [3.23]$$

which corresponds to the roots of the twisted part ($\tilde{\mathcal{G}}^I$-modules), appearing at each half-integral level.
One obtains therefore in this way a twisted affine algebra $\tilde{\mathcal{G}}^{(2)}$ (i.e. $m = 2$) (†) .
As before the automorphism τ is extended from the root system to the algebra $\mathcal{G}^{(1)}$ by

$$\tau(E^{\beta}_m) = E^{\tau(\beta)}_m \qquad [3.24]$$

$$x.\tau(H_m) = \tau(x).H_m \qquad [3.25]$$

the 2-cocycle being chosen invariant under τ : $\epsilon(\beta_1, \beta_2) = \epsilon(\tau(\beta_1), \tau(\beta_2))$.
Then taking y invariant under τ, one has

$$[y.H_n, E^{\beta}_m + E^{\tau(\beta)}_m] = \frac{1}{2} y.(\beta + \tau(\beta))(E^{\beta}_m + E^{\tau(\beta)}_m) \qquad [3.26]$$

$$[d, E^{\beta}_m + E^{\tau(\beta)}_m] = m(E^{\beta}_m + E^{\tau(\beta)}_m) \qquad [3.27]$$

For the consistency of these relations, one has to take $m \in \mathbf{Z}$ for $\frac{1}{2}(\beta + \tau(\beta)) \in \tilde{\Delta}^I$ but $m \in \mathbf{Z} + 1/2$ for $\frac{1}{2}(\beta + \tau(\beta)) \in \tilde{\Delta}^T$.
The step operators associated to the root $\frac{1}{2}(\beta + \tau(\beta))$ of $\tilde{\mathcal{G}}^{(2)}$ are therefore

$$E^{\beta}_m \quad \text{if} \quad \beta = \tau(\beta) \qquad (m \in \mathbf{Z}) \qquad [3.28]$$

(†) This discussion can be done actually for an automorphism τ of order p in the same way. One finds then that the twisted affine algebra obtained is $\tilde{\mathcal{G}}^{(p)}$. The case $p = 3$ will be treated as an example in the paragraph 5.4. where we discuss the folding $E^{(1)}_6 \rightarrow D^{(3)}_4$.

$$\frac{1}{\sqrt{2}}(E_m^{\beta} + E_m^{\tau(\beta)}) \quad \text{if} \quad \beta.\tau(\beta) = 0 \qquad (m \in \mathbf{Z} + 1/2) \qquad [3.29]$$

3-2- Symmetry group of extended Dynkin diagrams

It might be worthwhile to emphasize on the role of the symmetry group

$$\mathcal{F}(\mathcal{G}^{(1)}) \simeq \frac{Aut\ \Delta_R(\mathcal{G}^{(1)})}{W(\mathcal{G}^{(1)})}$$

of the Dynkin diagram of $\mathcal{G}^{(1)}$, where $\Delta_R(\mathcal{G}^{(1)})$ is the root lattice of $\mathcal{G}^{(1)}$ and $W(\mathcal{G}^{(1)})$ its Weyl group - also called affine Weyl group. Denoting $Z(\overline{G})$ the center of the universal covering group $\overline{G}$ of G [10], the Lie algebra of which is $\mathcal{G}$, itself isomorphic to the quotient $\Delta_W(\mathcal{G})/\Delta_R(\mathcal{G})$ of the weight lattice by the root lattice of $\mathcal{G}$ [11], one has the isomorphism :

$$\frac{\mathcal{F}(\mathcal{G}^{(1)})}{\mathcal{F}(\mathcal{G})} \simeq Z(\overline{G}) \qquad [3.30]$$

(see table 1 which contains the list of these finite groups for each simple Lie algebra).

Note that the symmetries of the $\mathcal{G}^{(1)}$ Dynkin diagrams have already been used, for example in the reduction of Toda field equations [10] and more recently in the construction of modular invariant partition functions for strings [12,13]. Let us add that elements of $\mathcal{F}(\mathcal{G}^{(1)})$ acting non trivially on the affine root α_o (i.e. $\tau(\alpha_o) \neq \alpha_o$) induce a changing from a homogeneous gradation into another one [3,12]. A homogeneous gradation corresponds to define an euclidean hyperplane orthogonal to the isotropic root δ in which lies the root sustem of $\mathcal{G}$. The number of homogeneous gradations of $\mathcal{G}^{(1)}$ is therefore equal to the order of $Z(\overline{G})$.

Table 1 : Symmetries of Extended Dynkin Diagrams

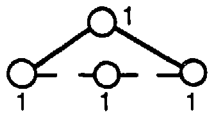
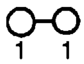
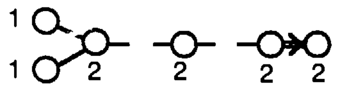
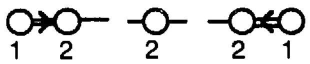
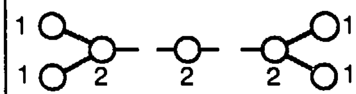
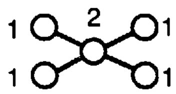
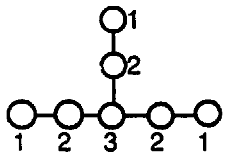
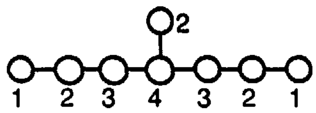
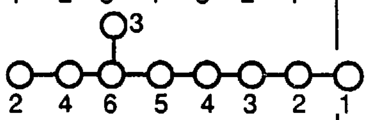
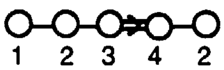
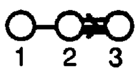
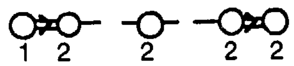
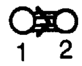
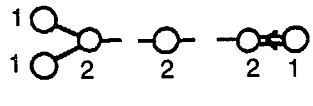
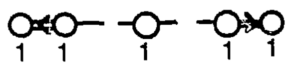
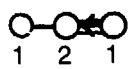
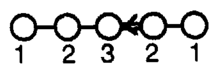
Affine algebra $\mathcal{G}^{(1)}$	*Dynkin diagram*	*Auto-morphism group* $\mathcal{F}(\mathcal{G}^{(1)})$	*Center* $Z(\overline{G})$	*Auto-morphism group* $\mathcal{F}(\mathcal{G})$
$A_l^{(1)}$ $l \geq 2$		D_{l+1}	Z_{l+1}	Z_2
$A_2^{(1)}$		Z_2	Z_2	1
$B_l^{(1)}$ $l \geq 2$		Z_2	Z_2	1
$C_l^{(1)}$ $l \geq 3$		Z_2	Z_2	1
$D_l^{(1)}$ $l > 4$		D_4	$Z_2 \times Z_2$ $(l$ *even*$)$	Z_2
		D_4	Z_4 $(l$ *odd*$)$	Z_2
$D_4^{(1)}$		S_4	$Z_2 \times Z_2$	S_3
$E_6^{(1)}$		S_3	Z_3	Z_2
$E_7^{(1)}$		Z_2	Z_2	1
$E_8^{(1)}$		1	1	1
$F_4^{(1)}$		1	1	1
$G_2^{(1)}$		1	1	1

Table 1 (cont'd)

Twisted algebra $\mathcal{G}^{(m)}$	*Dynkin diagram*	*Automorphism group* $\mathcal{F}(\mathcal{G}^{(m)})$
$A_{2l}^{(2)}$ $l \geq 2$		1
$A_2^{(2)}$		1
$A_{2l-1}^{(2)}$ $l \geq 3$		Z_2
$D_{l+1}^{(2)}$ $l \geq 2$		Z_2
$D_4^{(3)}$		1
$E_6^{(2)}$		1

We note that Z_n is the cyclic group of order n, S_n the permutation group of n objects and D_n the dihedral group with $2n$ elements [10].
Algebras labelled by the index l have DD with $l+1$ vertices.

4- VERTEX OPERATORS FOR TWISTED AFFINE ALGEBRAS

Now we will consider the construction of vertex operators for twisted affine algebras, using the folding method of the previous section. The possibility of writing the generators of a folded algebra as linear combinations of the generators of the non folded algebra allows us to construct the corresponding vertex operators for the twisted affine algebra, once the vertex operators for untwisted affine algebras are known.

As in the previous section, we will consider separately the cases where the automorphism τ acts trivially or not on the affine root.

• Case 1 :

The outer automorphism τ acts trivially on the affine root. The folding of the affine algebra $\mathcal{G}^{(1)}$ leads to another affine algebra $\tilde{\mathcal{G}}^{(1)}$, such that $\tilde{\mathcal{G}}$ is a non simply laced algebra.

The vertex operators are constructed in the following way :

If $Q^i(z)$ are Fubini-Veneziano fields in number rank $\mathcal{G}$

$$Q^i(z) = q^i - ip^i \ln z + i \sum_{m\neq 0} \frac{\alpha^i_m}{m} z^{-m} \qquad [4.1]$$

we define the vertex operator $U(\beta, z)$ where β is a root of $\mathcal{G}$ by

$$U(\beta, z) = z^{\beta^2/2} : \exp i\beta.Q(z) : \qquad [4.2]$$

The momenta p are belonging to the weight lattice of the Lie algebra $\mathcal{G}$. Then the step operators associated to the folded roots can be written as follows :

- The algebra $\tilde{\mathcal{G}}^{(1)}_L$ generated by the long roots of $\tilde{\mathcal{G}}^{(1)}$ is a subalgebra of $\mathcal{G}^{(1)}$. The vertex operators corresponding to the long roots are thus

$$E(\beta, z) = U(\beta, z)c_\beta \qquad [4.3]$$

- The step operator at level m for the short roots being $1/\sqrt{2}(E^\beta_m + E^{\tau(\beta)}_m)$, the corresponding vertex operator is

$$E(\gamma_+, z) = \frac{1}{\sqrt{2}}\Big(U(\beta, z)c_\beta + U(\tau(\beta), z)c_{\tau(\beta)}\Big) \qquad [4.4]$$

where $\gamma_\pm = \frac{1}{2}(\beta \pm \tau(\beta))$.

Since $\beta.\tau(\beta) = 0$ in this case, one has $\gamma_+^2 = \gamma_-^2 = 1$ and $\gamma_+.\gamma_- = 0$, and the vertex operator part factorizes as :

$$U(\beta, z) = U(\gamma_+, z)U(\gamma_-, z) \qquad [4.5]$$

$$U(\tau(\beta), z) = U(\gamma_+, z)U(-\gamma_-, z)$$

We have also to deal with the factorization properties of the cocycle. More precisely, we would like to write the same factorization relation than above, i.e.

$$c_\beta = c_{\gamma_+}.c_{\gamma_-} \qquad c_{\tau(\beta)} = c_{\gamma_+}.c_{-\gamma_-} \qquad [4.6]$$

To do that, one has to extend c_γ with $\gamma \in \Lambda_R(\mathcal{G})$ to $c_{\gamma_\pm}$ with $\gamma_\pm = \frac{1}{2}(\gamma \pm \tau(\gamma)) \in \Lambda_\pm$. The construction of the cocycle operator on the lattice $\Lambda_\pm$ will be worked out in each case. Notice however that the root lattice $\Lambda_R(\tilde{\mathcal{G}}_L)$ of the algebra generated by

the long roots of $\tilde{\mathcal{G}}$ is a sublattice of Λ_+. It follows that the cocycle operator in eq. [4.3] can also be constructed as a cocycle operator on the lattice Λ_+.
Assuming for the moment that the cocycle can be factorized as eq. [4.6], one obtains for the step operator

$$E(\gamma_+, z) = U(\gamma_+, z) c_{\gamma_+} \Gamma_{\gamma_-}(z) \qquad [4.7]$$

with

$$\Gamma_{\gamma_-}(z) = \frac{1}{\sqrt{2}}\Big(U(\gamma_-, z) c_{\gamma_-} + U(-\gamma_-, z) c_{-\gamma_-} \Big) \qquad [4.8]$$

$\Gamma_{\gamma_-}(z)$ is an auxiliary field associated to the short root γ_+. One can verify that $U(\gamma_+, z)$ and $\Gamma_{\gamma_-}(z)$ have the conformal weight 1/2, which gives the right conformal weight 1 for the generator $E(\gamma_+, z)$.
Notice that in general the same auxiliary field can be related to different short roots. Actually all the short roots related to each other by a Weyl reflection with respect to a long root have the same auxiliary field $\Gamma(z)$. In other words, if the set of short roots of $\tilde{\mathcal{G}}$ is divided into orbits Ω under the Weyl group generated by the long roots, one associates one auxiliary field $\Gamma_\Omega(z)$ for each orbit Ω. These auxiliary fields are not necessarily independent of each other because if $\gamma \in \Omega$, $\gamma' \in \Omega'$ and $\gamma + \gamma' \in \Omega''$, one must have

$$\Gamma_\Omega(z)\Gamma_{\Omega'}(w) = z^{1/2}(z-w)^{\Omega.\Omega'}\Gamma_{\Omega''}(z) + \text{regular terms in } z-w \qquad [4.9]$$

• Case 2 :

The outer automorphism τ does not act trivially on the affine root. We assume that τ is of order 2. The folding of the affine algebra $\mathcal{G}^{(1)}$ leads to the twisted affine algebra $\tilde{\mathcal{G}}^{(2)}$. The vertex operators corresponding to the generators with integral moments (invariant integral subalgebra $\tilde{\mathcal{G}}_I^{(1)}$ of $\tilde{\mathcal{G}}^{(2)}$) are constructed as for the case 1. The main problem is to construct vertex operators associated to the roots of $\tilde{\Delta}^T$ at half-integral levels, i.e. vertex operators with half-integral moments, instead of integral moments as for the untwisted Kac-Moody algebras.
To do this, consider the invariant integral subalgebra of $\tilde{\mathcal{G}}^{(2)}$, we have denoted by $\tilde{\mathcal{G}}_I^{(1)}$ above. Let Λ be the weight lattice of $\tilde{\mathcal{G}}^I$, the horizontal algebra of $\tilde{\mathcal{G}}_I^{(1)}$. One extends this lattice to the lorentzian lattice $\overline{\Lambda}$, by adding to the lattice Λ the isotropic direction δ : if $(\epsilon_i)_{1\leq i\leq \dim\,\Lambda}$ is a basis of Λ, one has

$$\delta^2 = 0 \qquad \text{and} \qquad \delta.\epsilon_i = 0 \qquad [4.10]$$

If the $Q^i(z)$'s are the Fubini-Veneziano fields introduced in eq. [4.1.], we extend the number of components of the oscillators α^i_m and p^i from dim Λ to dim $\overline{\Lambda}$ with the following conditions :

$$\delta.\alpha_m = 0 \quad (m \neq 0) \qquad \text{and} \qquad \delta.\alpha_o = \delta.p = 1 \qquad [4.11]$$

We denote by $\tilde{Q}^i(z)$ the obtained Fubini-Veneziano fields.
Then if β belongs to Λ, the extended vertex operators

$$U(\beta, z) = z^{\beta^2/2} : \exp i\beta.\tilde{Q}(z) : \qquad [4.12]$$

and

$$U(\beta + \delta/2, z) = z^{\beta^2/2} : \exp i(\beta + \delta/2).\tilde{Q}(z) : \qquad [4.13]$$

have series expansion either in integral or in half-integral powers of z, but are always of opposite Ramond or Neveu-Schwarz character. More precisely, if $\beta^2 = 1$, $U(\beta, z)$ is of NS character and $U(\beta + \delta/2, z)$ of R character if the momentum p has only integral components, and $U(\beta, z)$ is of R character and $U(\beta + \delta/2, z)$ of NS character if the momentum p has only half-integral components.
Now we are in position to write a vertex operator for the roots of $\tilde{\Delta}_T$:

$$E'(\gamma_+, z) = \frac{1}{\sqrt{2}}\Big(U(\beta + \delta/2, z)c_\beta + U(\tau(\beta) + \delta/2, z)c_{\tau(\beta)}\Big) \qquad [4.14]$$

where $\gamma_\pm = \frac{1}{2}(\beta \pm \tau(\beta))$ as before.
Since $\overline{\beta}.\tau(\overline{\beta}) = 0$, one has $\beta.\tau(\beta) = 0$ and thus $\gamma_+.\gamma_- = 0$. The vertex operator part factorizes therefore as in the untwisted case :

$$U(\beta + \delta/2, z) = U(\gamma_+, z)U(\gamma_- + \delta/2, z) \qquad [4.15]$$

$$U(\tau(\beta) + \delta/2, z) = U(\gamma_+, z)U(-\gamma_- + \delta/2, z)$$

Moreover, the factorization properties of the cocycle are unchanged in comparison to the untwisted case, since it depends only on the root lattice of the horizontal algebra. One has consequently :

$$c_\beta = c_{\gamma_+}.c_{\gamma_-} \qquad c_{\tau(\beta)} = c_{\gamma_+}.c_{-\gamma_-} \qquad [4.16]$$

The vertex operator can therefore be written as a product of two vertex operators of fermionic character :

$$E(\gamma_+, z) = U(\gamma_+, z)c_{\gamma_+}\Gamma'_{\gamma_-}(z) \qquad [4.17]$$

with

$$\Gamma'_{\gamma_-}(z) = \frac{1}{\sqrt{2}}\Big(U(\gamma_- + \delta/2, z)c_{\gamma_-} + U(-\gamma_- + \delta/2, z)c_{-\gamma_-}\Big) \qquad [4.18]$$

$\Gamma'_{\gamma_-}(z)$ is an auxiliary field associated to the short root γ_+. One can show, as in the untwisted case, that all the short roots related to γ_+ by a Weyl transformation with respect to the long roots of $\tilde{\mathcal{G}}_I^{(1)}$ have the same auxiliary fermionic field $\Gamma_{\gamma_-}(z)$.

Therefore $\Gamma_{\gamma_-}(z)$ is associated to the orbit Ω of the short roots under the Weyl group generated by the long roots of $\tilde{\mathcal{G}}_I$, which γ_+ belongs to : $\Gamma_{\gamma_-}(z) = \Gamma_{\Omega(\gamma_+)}(z)$.

Remark : the short roots α_S of $\tilde{\mathcal{G}}^{(2)}$ appear both with integral and half-integral levels, associated with the generators

$$E(\alpha_S, z) = U(\alpha_S, z) c_{\alpha_S} \Gamma_{\Omega(\alpha_S)}(z) \qquad [4.19]$$

$$E'(\alpha_S, z) = U(\alpha_S, z) c_{\alpha_S} \Gamma'_{\Omega(\alpha_S)}(z) \qquad [4.20]$$

Because of the structure of the twisted algebra $\tilde{\mathcal{G}}^{(2)}$, the auxiliary fields $\Gamma_{\Omega(\alpha_S)}(z)$ and $\Gamma'_{\Omega(\alpha_S)}(z)$ are in general not independent of each other but satisfy the following O.P.E.'s :

$$\Gamma_{\Omega}(z)\Gamma_{\Omega'}(w) = z^{1/2}(z-w)^{\Omega.\Omega'}\Gamma_{\Omega''}(z) + \text{regular terms in } z-w \qquad [4.21]$$

$$\Gamma_{\Omega}(z)\Gamma'_{\Omega'}(w) = z^{1/2}(z-w)^{\Omega.\Omega'}\Gamma'_{\Omega''}(z) + \text{regular terms in } z-w \qquad [4.22]$$

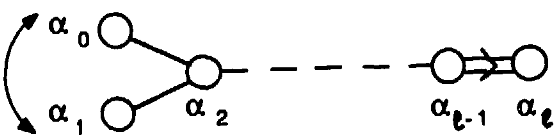
$$\Gamma'_{\Omega}(z)\Gamma'_{\Omega'}(w) = z^{1/2}(z-w)^{\Omega.\Omega'}\Gamma_{\Omega''}(z) + \text{regular terms in } z-w \qquad [4.23]$$

if $\gamma \in \Omega$, $\gamma' \in \Omega'$ and $\gamma + \gamma' \in \Omega''$.

5- EXPLICIT CONSTRUCTIONS OF THE VERTEX OPERATORS

5-1- Folding $B_l^{(1)} \to D_l^{(2)}$

Consider the non simply laced affine algebra $B_l^{(1)}$, whose extended DD has a Z_2-symmetry :



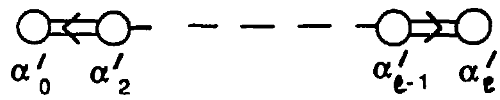
The outer automorphism of order 2 related to this symmetry is defined by

$$\tau(\alpha_o) = \alpha_1 \qquad \tau(\alpha_i) = \alpha_i \qquad (2 \le i \le l) \qquad [5.1]$$

The root system of the folded algebra is

$$\tilde{\Delta} = \{\pm e_i \pm e_j + m\delta,\ \pm e_i + m\delta/2,\ n\delta/2,\ 2 \le i \ne j \le l,\ m \in \mathbf{Z},\ n \in \mathbf{Z}^*\} \qquad [5.2]$$

which corresponds to the folded DD



with $\alpha'_o = \frac{1}{2}(\alpha_o + \alpha_1) = \delta/2 - e_2$.
One obtains therefore the twisted Kac-Moody algebra $D_l^{(2)}$, whose invariant integral subalgebra is $B_{l-1}^{(1)}$.
The vertex operators are consequently :
• for the invariant part (corresponding to long and short roots at integral level)

$$E(\pm e_i \pm e_j, z) = U(\pm e_i \pm e_j, z) c_{\pm e_i, \pm e_j} \qquad [5.3]$$

c being the cocycle operator defined on the $\mathbf{Z}^{l-1}$ sublattice of the $\mathbf{Z}^l$ root lattice of B_l, and

$$E(\pm e_i, z) = U(\pm e_i, z) c_{\pm e_i} \Gamma(z) \qquad [5.4]$$

$\Gamma(z)$ being an auxiliary fermionic field (which can be constructed for instance by folding of the affine algebra $D_{l+1}^{(1)}$).
• for the twisted part (corresponding to short roots at half-integral level) one has following the previous general discussion

$$E'(\pm e_i, z) = \frac{1}{\sqrt{2}} \Big(U(\pm e_i + e_1 + \delta/2, z) c_{\pm e_i, +e_1} + U(\pm e_i - e_1 + \delta/2, z) c_{\pm e_i, -e_1} \Big) \qquad [5.5]$$

the cocycle being defined on the $\mathbf{Z}^l$ lattice of B_l, one can factorize it immediately :

$$c_{\pm e_i, \pm e_1} = c_{\pm e_i} . c_{\pm e_1} \qquad [5.6]$$

and one obtains

$$E'(\pm e_i, z) = U(\pm e_i, z) c_{\pm e_i} \Gamma'(z) \qquad [5.7]$$

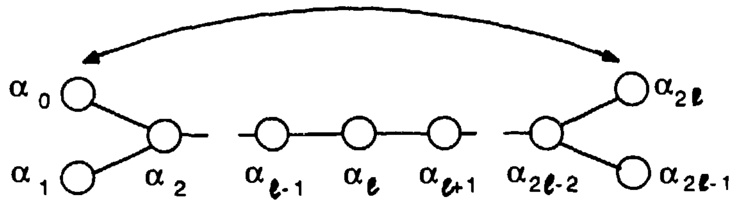
where

$$\Gamma'(z) = \frac{1}{\sqrt{2}} \Big(U(e_1 + \delta/2, z) c_{e_1} + U(-e_1 + \delta/2, z) c_{-e_1} \Big) \qquad [5.8]$$

is another auxiliary fermionic field, independent of $\Gamma(z)$ and of opposite Neveu-Schwarz or Ramond character.
$c_{\pm e_i}$ is a cocycle operator defined on the $\mathbf{Z}^{l-1}$ root lattice of B_{l-1} and $c_{\pm e_1}$ is a cocycle operator defined on the $\mathbf{Z}$ lattice orthogonal to the previous one.

5-2- Folding $D_{2l}^{(1)} \rightarrow A_{2l-1}^{(2)}$

Consider the affine algebra $D_{2l}^{(1)}$ with the EDD



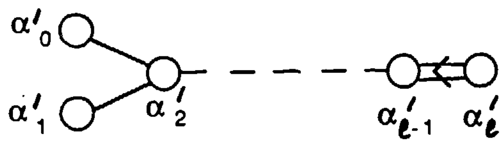
The outer automorphism of order 2 which defines the folding is

$$\tau(\alpha_l) = \alpha_l \qquad \tau(\alpha_i) = \alpha_{2l-i} \qquad (0 \le i \le l-1) \tag{5.9}$$

The simple root system of the folded algebra is given in terms of the rescaled folded roots $\eta_i = \frac{1}{\sqrt{2}}(e_i - e_{2l+1-i}) \quad (1 \le i \le l)$ by

$$\tilde{R} = \left\{ \alpha'_o = \delta/2 - \frac{1}{\sqrt{2}}(\eta_1 + \eta_2)\ ,\ \alpha'_i = \frac{1}{\sqrt{2}}(\eta_i - \eta_{i+1}) \quad (1 \le i \le l-1)\ ,\ \alpha'_l = \frac{2}{\sqrt{2}}\eta_l \right\} \tag{5.10}$$

which corresponds to the folded DD



One obtains therefore the twisted affine algebra $A_{2l-1}^{(2)}$ with invariant integral subalgebra C_l.

The root system of $A_{2l-1}^{(2)}$ is

$$\tilde{\Delta} = \left\{ \frac{\pm 2}{\sqrt{2}}\eta_i + m\delta,\ \frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j) + m\delta/2,\ n\delta/2,\ 1 \le i \ne j \le l,\ m \in \mathbf{Z},\ n \in \mathbf{Z}^* \right\} \tag{5.11}$$

Construction of the vertex operators :

• for the long roots of the invariant part, one has

$$\pm\frac{2}{\sqrt{2}}\eta_i = \pm(e_i - e_{2l+1-i}) = \pm(\alpha_i + \ldots + \alpha_{2l-i}) \tag{5.12}$$

and the long roots at integral level correspond to roots of $D_{2l}^{(1)}$ invariant under τ. Therefore the vertex operator for the long roots is given by

$$E(\pm\frac{2}{\sqrt{2}}\eta_i, z) = U(\pm\frac{2}{\sqrt{2}}\eta_i, z)c_{\pm\frac{2}{\sqrt{2}}\eta_i} = U(\pm(e_i - e_{2l+1-i}), z)c_{\pm(e_i - e_{2l+1-i})} \tag{5.13}$$

• for the short roots of the invariant part, one has

$$\frac{1}{\sqrt{2}}(\eta_i - \eta_j) = \frac{1}{2}(e_i - e_{2l+1-i} - e_j + e_{2l+1-j}) = \frac{1}{2}(\beta + \tau(\beta)) \tag{5.14}$$

with $\beta = e_i - e_j$ and $\tau(\beta) = e_{2l+1-j} - e_{2l+1-i}$.
Therefore

$$E(\frac{1}{\sqrt{2}}(\eta_i - \eta_j), z) = \frac{1}{\sqrt{2}}\Big(U(e_i - e_j, z)c_{e_i - e_j} + U(e_{2l+1-j} - e_{2l+1-i}, z)c_{e_{2l+1-j} - e_{2l+1-i}}\Big) \qquad [5.15]$$

Similarly, one has

$$\frac{1}{\sqrt{2}}(\eta_i + \eta_j) = \frac{1}{2}(e_i - e_{2l+1-i} + e_j - e_{2l+1-j}) = \frac{1}{2}(\beta + \tau(\beta)) \qquad [5.16]$$

with $\beta = e_i - e_{2l+1-j}$ and $\tau(\beta) = e_j - e_{2l+1-i}$.
Therefore

$$E(\frac{1}{\sqrt{2}}(\eta_i + \eta_j), z) = \frac{1}{\sqrt{2}}\Big(U(e_i - e_{2l+1-j}, z)c_{e_i - e_{2l+1-j}} + U(e_j - e_{2l+1-i}, z)c_{e_j - e_{2l+1-i}}\Big) \qquad [5.17]$$

If we define the vectors $\xi_i = \frac{1}{\sqrt{2}}(e_i + e_{2l+1-i})$, orthogonal to the η_i's, one can factorize easily the vertex operator part :

$$\begin{aligned}
U(e_i - e_j, z) &= U(\frac{1}{\sqrt{2}}(\eta_i - \eta_j), z).U(\frac{1}{\sqrt{2}}(\xi_i - \xi_j), z) \\
U(e_{2l+1-j} - e_{2l+1-i}, z) &= U(\frac{1}{\sqrt{2}}(\eta_i - \eta_j), z).U(\frac{-1}{\sqrt{2}}(\xi_i - \xi_j), z) \\
U(e_i - e_{2l+1-j}, z) &= U(\frac{1}{\sqrt{2}}(\eta_i + \eta_j), z).U(\frac{1}{\sqrt{2}}(\xi_i - \xi_j), z) \\
U(e_j - e_{2l+1-i}, z) &= U(\frac{1}{\sqrt{2}}(\eta_i + \eta_j), z).U(\frac{-1}{\sqrt{2}}(\xi_i - \xi_j), z)
\end{aligned} \qquad [5.18]$$

Now we have to deal with the factorization of the cocycle operator. To perform this factorization, one must extend c_γ with $\gamma \in \Lambda_R(D_{2l})$ to $c_{\gamma_\pm}$ where $\gamma_\pm = \frac{1}{2}(\gamma \pm \tau(\gamma))$. Let $(e_i)_{1 \le i \le 2l}$ be a basis of $\gamma \in \Lambda_R(D_{2l})$. A lattice vector is defined by

$$\gamma = \sum_{i=1}^{2l} \gamma_i e_i \quad \text{with} \quad n_i \in \mathbf{Z} \quad \text{and} \quad \sum n_i \in 2\mathbf{Z} \qquad [5.19]$$

A direct calculation shows that

$$\gamma_+ = \frac{1}{2}\sum_{i=1}^{2l} n_i(e_i - e_{2l+1-i}) = \frac{1}{\sqrt{2}}\sum_{i=1}^{l} p_i \eta_i \quad \text{with} \quad p_i \in \mathbf{Z} \quad \sum p_i \in 2\mathbf{Z} \qquad [5.20]$$

$$\gamma_- = \frac{1}{2}\sum_{i=1}^{2l} n_i(e_i + e_{2l+1-i}) = \frac{1}{\sqrt{2}}\sum_{i=1}^{l} q_i \xi_i \quad \text{with} \quad q_i \in \mathbf{Z} \quad \sum q_i \in 2\mathbf{Z} \qquad [5.21]$$

This implies that γ_+ and γ_- belong to two orthogonal lattices $\Lambda_\pm$, each of them being isomorphic to the root lattice of D_l rescaled by $1/\sqrt{2}$.
The cocycle operator factorizes then as

$$c_\gamma = c_{\gamma_+}.c_{\gamma_-} \qquad c_{\tau(\gamma)} = c_{\gamma_+}.c_{-\gamma_-} \qquad [5.22]$$

once one is able to define the cocyle operator on $\Lambda_{\pm}$.
To do that, it is necessary and sufficient to define a symmetry factor $S(x,y)$ on the lattice $\frac{1}{\sqrt{2}}\Lambda_R(D_l)$. Let $(\gamma_1,...,\gamma_l)$ be a basis of simple roots of D_l, rescaled by the factor $1/\sqrt{2}$, such that $\gamma_i^2 = 1$. Then, on this basis we define the symmetry factor S by :

$$S_{ij} = \begin{cases} S(\gamma_i,\gamma_i) = 1 & \\ S(\gamma_i,\gamma_j) = (-1)^{\gamma_i\gamma_j} & \gamma_i\gamma_j \in \mathbf{Z}\ ,\ i \neq j \\ S(\gamma_i,\gamma_j) = -S(\gamma_j,\gamma_i) = e^{i\pi\gamma_i\gamma_j} & \gamma_i\gamma_j \in \mathbf{Z}+1/2\ ,\ i<j \end{cases} \qquad [5.23]$$

Then, if $x = \sum x_i\gamma_i$ and $y = \sum y_i\gamma_i$ are some points of $\frac{1}{\sqrt{2}}\Lambda_R(D_l)$, one sets

$$S(x,y) = \prod_{i,j} S_{ij}^{x_iy_j} \qquad [5.24]$$

It is obvious to verify that S possesses the properties of a symmetry factor :

$$\begin{aligned} &S(x,x) = 1 \\ &S(x,y)S(y,x) = 1 \\ &S(x,y+z) = S(x,y)S(x,z) \end{aligned} \qquad [5.25]$$

This allows us to construct the corresponding cocycle $\epsilon(x,y)$ and the corresponding cocycle operator on the lattice $\frac{1}{\sqrt{2}}\Lambda_R(D_l)$ (see ref. [2] for e.g.).
Now the factorization of the vertex operator for the short roots is complete and one can write finally

$$E(\frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j), z) = U(\frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j), z)c_{\frac{1}{\sqrt{2}}(\pm\eta_i,\pm\eta_j)}\Gamma_{ij}(z) \qquad [5.26]$$

with

$$\Gamma_{ij}(z) = \frac{1}{\sqrt{2}}\left(U(\frac{1}{\sqrt{2}}(\xi_i - \xi_j), z)c_{\frac{1}{\sqrt{2}}(\xi_i,-\xi_j)} + U(\frac{-1}{\sqrt{2}}(\xi_i - \xi_j), z)c_{\frac{-1}{\sqrt{2}}(\xi_i,-\xi_j)}\right) \qquad [5.27]$$

$\Gamma_{ij}(z)$ is an auxiliary fermionic field, which depends on the orbit Ω_{ij} to which the short root belongs to :

$$\Omega_{ij} = \left\{\frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j)\right\} \qquad [5.28]$$

There is $l(l-1)/2$ such orbits for $A_{2l-1}^{(2)}$ and therefore one needs $l(l-1)/2$ auxiliary fields $\Gamma_{ij}(z)$ to construct the vertex operators associated to the short roots at integral level.

- for the twisted part

The short roots $\frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j)$ appear also at half-integral levels (recall that the twisted part forms a representation of the invariant integral subalgebra).
One has

$$\frac{\delta}{2} - \frac{1}{\sqrt{2}}(\eta_i + \eta_j) = \frac{\delta}{2} + \frac{1}{2}(-e_i + e_{2l+1-i} - e_j + e_{2l+1-j}) = \frac{1}{2}(\beta + \tau(\beta)) \quad [5.29]$$

with $\beta = \delta - e_i - e_j$ and $\tau(\beta) = e_{2l+1-j} + e_{2l+1-i}$.
Therefore

$$E'(\frac{1}{\sqrt{2}}(-\eta_i - \eta_j), z) = \frac{1}{\sqrt{2}}\Big(U(-e_i - e_j + \delta/2, z)c_{-e_i,-e_j} + U(e_{2l+1-j} + e_{2l+1-i} + \delta/2, z)c_{e_{2l+1-j}+e_{2l+1-i}}\Big) \quad [5.30]$$

according to then general form of the twisted vertex operator.
Similarly, one has

$$\frac{\delta}{2} - \frac{1}{\sqrt{2}}(\eta_i - \eta_j) = \frac{\delta}{2} + \frac{1}{2}(-e_i + e_{2l+1-i} + e_j - e_{2l+1-j}) = \frac{1}{2}(\beta + \tau(\beta)) \quad [5.31]$$

with $\beta = \delta - e_i - e_{2l+1-j}$ and $\tau(\beta) = e_j + e_{2l+1-i}$.
Therefore

$$E'(\frac{1}{\sqrt{2}}(-\eta_i + \eta_j), z) = \frac{1}{\sqrt{2}}\Big(U(\delta/2 - e_i - e_{2l+1-j}, z)c_{-e_i,-e_{2l+1-j}} + U(\delta/2 + e_j + e_{2l+1-i}, z)c_{e_j+e_{2l+1-i}}\Big) \quad [5.32]$$

The vertex operator part factorizes as follows :

$$\begin{aligned}
U(e_{2l+1-j} + e_{2l+1-i} + \delta/2, z) &= U(\frac{1}{\sqrt{2}}(-\eta_i - \eta_j), z).U(\frac{1}{\sqrt{2}}(\xi_i + \xi_j) + \delta/2, z)\\
U(-e_i - e_j + \delta/2, z) &= U(\frac{1}{\sqrt{2}}(-\eta_i - \eta_j), z).U(\frac{-1}{\sqrt{2}}(\xi_i + \xi_j) + \delta/2, z)\\
U(-e_j + e_{2l+1-i} + \delta/2, z) &= U(\frac{1}{\sqrt{2}}(\eta_i + \eta_j), z).U(\frac{1}{\sqrt{2}}(\xi_i + \xi_j) + \delta/2, z)\\
U(-e_i - e_{2l+1-j}, z) &= U(\frac{1}{\sqrt{2}}(-\eta_i + \eta_j), z).U(\frac{-1}{\sqrt{2}}(\xi_i + \xi_j) + \delta/2, z)
\end{aligned} \quad [5.33]$$

The previous discussion on the cocycle shows that it factorizes as in the untwisted sector. It follows that the vertex operator for the twisted part can be written as :

$$E'(\frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j), z) = U(\frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j), z)c_{\frac{1}{\sqrt{2}}(\pm\eta_i,\pm\eta_j)}\Gamma'_{ij}(z) \quad [5.34]$$

with

$$\Gamma'_{ij}(z) = \frac{1}{\sqrt{2}}\left(U(\frac{1}{\sqrt{2}}(\xi_i+\xi_j)+\delta/2,z)c_{\frac{1}{\sqrt{2}}(\xi_i+\xi_j)}+U(\frac{-1}{\sqrt{2}}(\xi_i+\xi_j)+\delta/2,z)c_{\frac{-1}{\sqrt{2}}(\xi_i+\xi_j)}\right) \quad [5.35]$$

The number of auxiliary fermionic "twisted" fields $\Gamma'_{ij}(z)$ is still given by the number of orbits Ω_{ij} of the short roots under the Weyl group generated by the long ones. One needs also $l(l-1)/2$ auxiliary fermionic twisted fields $\Gamma'_{ij}(z)$ for the twisted part of $A^{(2)}_{2l-1}$.

In summary, one has for $A^{(2)}_{2l-1}$:

• invariant part :

generators associated to the long roots at integral level :

$$E(\pm\frac{2}{\sqrt{2}}\eta_i,z) = U(\pm\frac{2}{\sqrt{2}}\eta_i,z)c_{\pm\frac{2}{\sqrt{2}}\eta_i} = U(\pm(e_i-e_{2l+1-i}),z)c_{\pm(e_i-e_{2l+1-i})} \quad [5.36]$$

generators associated to the short roots at integral level :

$$E(\frac{1}{\sqrt{2}}(\pm\eta_i\pm\eta_j),z) = U(\frac{1}{\sqrt{2}}(\pm\eta_i\pm\eta_j),z)c_{\frac{1}{\sqrt{2}}(\pm\eta_i,\pm\eta_j)}\Gamma_{ij}(z) \quad [5.37]$$

with

$$\Gamma_{ij}(z) = \frac{1}{\sqrt{2}}\left(U(\frac{1}{\sqrt{2}}(\xi_i-\xi_j),z)c_{\frac{1}{\sqrt{2}}(\xi_i-\xi_j)} + U(\frac{-1}{\sqrt{2}}(\xi_i-\xi_j),z)c_{\frac{-1}{\sqrt{2}}(\xi_i-\xi_j)}\right) \quad [5.38]$$

• twisted part :

generators associated to the short roots at half-integral level :

$$E'(\frac{1}{\sqrt{2}}(\pm\eta_i\pm\eta_j),z) = U(\frac{1}{\sqrt{2}}(\pm\eta_i\pm\eta_j),z)c_{\frac{1}{\sqrt{2}}(\pm\eta_i,\pm\eta_j)}\Gamma'_{ij}(z) \quad [5.39]$$

with

$$\Gamma'_{ij}(z) = \frac{1}{\sqrt{2}}\left(U(\frac{1}{\sqrt{2}}(\xi_i+\xi_j)+\delta/2,z)c_{\frac{1}{\sqrt{2}}(\xi_i+\xi_j)}+U(\frac{-1}{\sqrt{2}}(\xi_i+\xi_j)+\delta/2,z)c_{\frac{-1}{\sqrt{2}}(\xi_i+\xi_j)}\right) \quad [5.40]$$

The auxiliary fields $\Gamma_{ij}(z)$ and $\Gamma'_{ij}(z)$ are not independent in the sense they must respect the Z_2-grading of the twisted algebra, i.e.

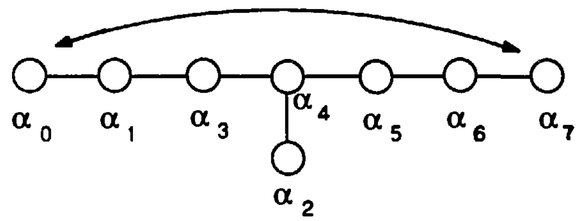
$$\Gamma_{ij}(z)\Gamma_{jk}(w) = (z-w)^{-1/2}(\Gamma_{ik}(w) + \text{regular terms in} z-w) \quad [5.41]$$

$$\Gamma_{ij}(z)\Gamma'_{jk}(w) = (z-w)^{-1/2}(\Gamma'_{ik}(w) + \text{regular terms in} z-w) \quad [5.42]$$

$$\Gamma'_{ij}(z)\Gamma'_{jk}(w) = (z-w)^{-1/2}(\Gamma_{ik}(w) + \text{regular terms in} z-w) \quad [5.43]$$

5-3- Folding $E_7^{(1)} \to E_6^{(2)}$

Consider the case of the exceptionnal Lie algebra E_7 whose EDD has a Z_2 symmetry :



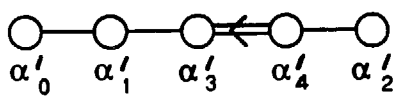
with the simple root system

$$R = \begin{cases} \alpha_o = \delta + e_7 - e_8 \\ \alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8) \\ \alpha_2 = e_1 + e_2 \\ \alpha_3 = -e_1 + e_2 \\ \alpha_4 = -e_2 + e_3 \\ \alpha_5 = -e_3 + e_4 \\ \alpha_6 = -e_4 + e_5 \\ \alpha_7 = -e_5 + e_6 \end{cases} \qquad [5.44]$$

The outer automorphism τ of order 2 which defines the folding is

$$\tau(\alpha_o) = \alpha_7 \qquad \tau(\alpha_1) = \alpha_6 \qquad \tau(\alpha_3) = \alpha_5 \qquad \tau(\alpha_2) = \alpha_2 \qquad \tau(\alpha_4) = \alpha_4 \quad [5.45]$$

One obtains the folded EDD



with the corresponding simple root system

$$\tilde{R} = \begin{cases} \alpha'_o = \delta/2 - \eta_1 \\ \alpha'_1 = \frac{1}{2}(\eta_1 - \eta_2 - \eta_3 - \eta_4) \\ \alpha'_2 = \eta_2 - \eta_3 \\ \alpha'_3 = \eta_4 \\ \alpha'_4 = \eta_3 - \eta_4 \end{cases} \qquad [5.46]$$

where the η_i's are the rescaled roots

$$\begin{aligned}
\eta_1 &= \frac{1}{2}(e_5 - e_6 - e_7 + e_8) \\
\eta_2 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \\
\eta_3 &= \frac{1}{2}(-e_1 - e_2 + e_3 + e_4) \\
\eta_4 &= \frac{1}{2}(-e_1 + e_2 - e_3 + e_4)
\end{aligned} \qquad [5.47]$$

The folded root system, expressed in terms of the η_i's is

$$\tilde{\Delta} = \left\{ \pm\eta_i \pm \eta_j + m\delta,\ \pm\eta_i + m\delta/2,\ \frac{1}{2}(\pm\eta_1 \pm \eta_2 \pm \eta_3 \pm \eta_4) + m\delta/2,\ n\delta/2, \right.$$

$$\left. 1 \leq i \neq j \leq 4,\ m \in \mathbf{Z},\ n \in \mathbf{Z}^* \right\} \qquad [5.48]$$

It appears that the invariant integral subalgebra of $E_6^{(6)}$ is the Lie algebra F_4, and that the twisted part is generated by the short roots of F_4.

Construction of the vertex operator :

• invariant integral algebra

Since the invariant integral subalgebra of $E_6^{(2)}$ is F_4, we will find the generators associated to the long and short roots of F_4 with integral moments, the short roots splitting into three different orbits under the Weyl group generated by the long roots :

$$\Omega_1 = \{\pm\eta_i \quad , \quad 1 \leq i \leq 4\}$$

$$\Omega_2 = \{\frac{1}{2}(\pm\eta_1 \pm \eta_2 \pm \eta_3 \pm \eta_4) \quad , \quad \text{even number of + signs}\}$$

$$\Omega_3 = \{\frac{1}{2}(\pm\eta_1 \pm \eta_2 \pm \eta_3 \pm \eta_4) \quad , \quad \text{odd number of + signs}\} \qquad [5.49]$$

Vertex operators associated to the long roots :

The long roots of F_4 are roots of E_7 invariant under the automorphism τ. One has simply :

$$E(\pm\eta_i \pm \eta_j, z) = U(\pm\eta_i \pm \eta_j, z) c_{\pm\eta_i \pm\eta_j} \qquad [5.50]$$

with $\pm\eta_i \pm \eta_j$ expressed in terms of invariant roots of E_7.

Since the long roots $\pm\eta_i \pm \eta_j$ constitute the root system of D_4, it is obvious to construct a cocycle operator $c_{\pm\eta_i \pm\eta_j}$ on the root lattice of D_4.

Vertex operators associated to the short roots :

The main problem is to construct a cocycle operator on the lattices $\Lambda_\pm = \{\frac{1}{2}(\gamma \pm \tau(\gamma)) \mid \gamma \in \Lambda_R(E_7)\}$, such that the factorization of the cocycle be possible

$$c_\gamma = c_{\gamma_+}.c_{\gamma_-} \qquad c_{\tau(\gamma)} = c_{\gamma_+}.c_{-\gamma_-} \tag{5.51}$$

Let γ be a lattice vector of $\Lambda_R(E_7)$. A direct calculation shows that

$$\gamma_+ = \frac{1}{2}\sum_{i=1}^{4} p_i\eta_i \qquad \text{with} \quad p_i \in \mathbf{Z} \quad \sum p_i \in 2\mathbf{Z} \tag{5.52}$$

$$\gamma_- = \frac{1}{\sqrt{2}}\sum_{i=1}^{4} q_i\xi_i \qquad \text{with} \quad q_i \in \mathbf{Z} \quad \sum q_i = 0 \tag{5.53}$$

where the ξ_i's are linear combinations of the e_i's, orthogonal to the η_i's.
This implies that γ_+ belongs to the lattice $\Lambda_+ = \frac{1}{2}\Lambda_R(D_4)$ and γ_- belongs to the lattice $\Lambda_- = \frac{1}{\sqrt{2}}\Lambda_R(A_3)$, the two lattices $\Lambda_\pm$ being orthogonal.
To construct the cocycle c_{γ_+} on the lattice Λ_+, it is necessary and sufficient to define the corresponding symmetry factor $S(x,y)$. Let $(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ be a basis of simple roots of D_4, rescaled by the factor $1/2$, such that $\gamma_i^2 = 1/2$. Since a point $x \in \Lambda_+$ is defined by $x = \sum x_i\gamma_i$ with $x_i \in \mathbf{Z}$ and $\sum x_i \in 2\mathbf{Z}$, the basis vectors γ_i do not belong to the lattice Λ_+, but the vectors $2\gamma_i$ do. It follows that the symmetry factor $S(x,y)$ should satisfy on the basis (γ_i)

$$S(2\gamma_i, 2\gamma_i) = 1 \Longrightarrow S(\gamma_i,\gamma_i) \in \{\pm 1, \pm i\} \tag{5.54}$$

One is led to define the symmetry factor S on the basis (γ_i) by :

$$S_{ij} = \begin{cases} S(\gamma_i,\gamma_i) = -i & \\ S(\gamma_i,\gamma_j) = (-1)^{\gamma_i\gamma_j} & \gamma_i\gamma_j \in \mathbf{Z}\ ,\ i \neq j \\ S(\gamma_i,\gamma_j) = S(\gamma_j,\gamma_i)^{-1} = e^{i\pi\gamma_i\gamma_j} & \gamma_i\gamma_j \in \mathbf{Z}+1/2 \text{ or} \mathbf{Z}+1/4\ ,\ i<j \end{cases} \tag{5.55}$$

Therefore, if $x = \sum x_i\gamma_i$ and $y = \sum y_i\gamma_i$ are two points of $\frac{1}{2}\Lambda_R(D_4)$, one has

$$S(x,y) = \prod_{i,j} S_{ij}^{x_i y_j} \tag{5.56}$$

This symmetry factor satisfies the usual properties

$$\begin{aligned} &S(x,x) = 1 \\ &S(x,y)S(y,x) = 1 \\ &S(x,y+z) = S(x,y)S(x,z) \end{aligned} \tag{5.57}$$

For the construction of the cocycle operator on the lattice $\Lambda_- = \frac{1}{\sqrt{2}}\Lambda_R(A_3)$, see ref. [2].
One finds finally the expression of the vertex operators for the short roots associated to the three differents orbits :

$$E(\pm\eta_i, z) = U(\pm\eta_i, z)c_{\pm\eta_i}\Gamma_1(z) \qquad [5.58]$$

with

$$\Gamma_1(z) = \frac{1}{\sqrt{2}}\Big(U(\frac{1}{\sqrt{2}}(\xi_3-\xi_2), z)c_{\frac{1}{\sqrt{2}}(\xi_3-\xi_2)} + U(\frac{1}{\sqrt{2}}(\xi_2-\xi_3), z)c_{\frac{1}{\sqrt{2}}(\xi_2-\xi_3)}\Big) \qquad [5.59]$$

$$E(\frac{1}{2}(\pm\eta_1\pm\eta_2\pm\eta_3\pm\eta_4), z) = U(\frac{1}{2}(\pm\eta_1\pm\eta_2\pm\eta_3\pm\eta_4), z)c_{\frac{1}{2}(\pm\eta_1\pm\eta_2\pm\eta_3\pm\eta_4)}\Gamma_2(z) \qquad [5.60]$$

(even number of + signs)
with

$$\Gamma_2(z) = \frac{1}{\sqrt{2}}\Big(U(\frac{1}{\sqrt{2}}(\xi_1-\xi_3), z)c_{\frac{1}{\sqrt{2}}(\xi_1-\xi_3)} + U(\frac{1}{\sqrt{2}}(\xi_3-\xi_1), z)c_{\frac{1}{\sqrt{2}}(\xi_3-\xi_1)}\Big) \qquad [5.61]$$

$$E(\frac{1}{2}(\pm\eta_1\pm\eta_2\pm\eta_3\pm\eta_4), z) = U(\frac{1}{2}(\pm\eta_1\pm\eta_2\pm\eta_3\pm\eta_4), z)c_{\frac{1}{2}(\pm\eta_1\pm\eta_2\pm\eta_3\pm\eta_4)}\Gamma_3(z) \qquad [5.62]$$

(odd number of + signs)
with

$$\Gamma_3(z) = \frac{1}{\sqrt{2}}\Big(U(\frac{1}{\sqrt{2}}(\xi_2-\xi_1), z)c_{\frac{1}{\sqrt{2}}(\xi_2-\xi_1)} + U(\frac{1}{\sqrt{2}}(\xi_1-\xi_2), z)c_{\frac{1}{\sqrt{2}}(\xi_1-\xi_2)}\Big) \qquad [5.63]$$

Actually this can be easily understood when invoking the "triality". In fact, the subalgebra corresponding to the long roots of F_4 is D_4 which possess the property of triality. The three different orbits Ω_i of the short roots under the Weyl group generated by the long ones are related to the three representations 8_V (orbit Ω_1), 8_S (orbit Ω_2) and $8'_S$ (orbit Ω_3) of D_4. The triality implies therefore some relations when computing the O.P.E.'s between the auxiliary fields $\Gamma_i(z)$.

- twisted part

The twisted part forms a representation of the invariant subalgebra F_4. Thus one finds at each half-integral level the short roots of F_4, associated to the twisted generators, which split again into the thre different orbit Ω_i $(i = 1, 2, 3)$. One has

$$E'(\pm\eta_i, z) = U(\pm\eta_i, z)c_{\pm\eta_i}\Gamma'_1(z) \qquad [5.64]$$

with

$$\Gamma'_1(z) = \frac{1}{\sqrt{2}}\Big(U(\frac{1}{\sqrt{2}}(\xi_1-\xi_4)+\delta/2, z)c_{\frac{1}{\sqrt{2}}(\xi_1-\xi_4)} + U(\frac{1}{\sqrt{2}}(\xi_4-\xi_1)+\delta/2, z)c_{\frac{1}{\sqrt{2}}(\xi_4-\xi_1)}\Big) \qquad [5.65]$$

$$E'(\frac{1}{2}(\pm\eta_1 \pm \eta_2 \pm \eta_3 \pm \eta_4), z) = U(\frac{1}{2}(\pm\eta_1 \pm \eta_2 \pm \eta_3 \pm \eta_4), z)c_{\frac{1}{2}(\pm\eta_1 \pm\eta_2\pm\eta_3\pm\eta_4)}\Gamma'_2(z) \quad [5.66]$$

(even number of + signs)
with

$$\Gamma'_2(z) = \frac{1}{\sqrt{2}}\left(U(\frac{1}{\sqrt{2}}(\xi_2-\xi_4)+\delta/2, z)c_{\frac{1}{\sqrt{2}}(\xi_2-\xi_4)}+U(\frac{1}{\sqrt{2}}(\xi_4-\xi_2)+\delta/2, z)c_{\frac{1}{\sqrt{2}}(\xi_4-\xi_2)}\right) \quad [5.67]$$

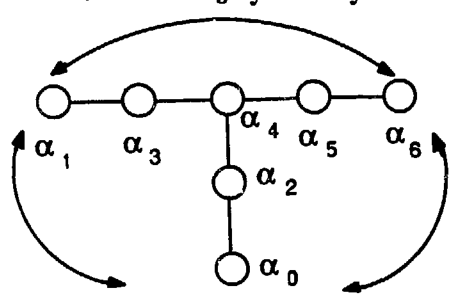
$$E'(\frac{1}{2}(\pm\eta_1 \pm \eta_2 \pm \eta_3 \pm \eta_4), z) = U(\frac{1}{2}(\pm\eta_1 \pm \eta_2 \pm \eta_3 \pm \eta_4), z)c_{\frac{1}{2}(\pm\eta_1\pm\eta_2\pm\eta_3\pm\eta_4)}\Gamma'_3(z) \quad [5.68]$$

(odd number of + signs)
with

$$\Gamma'_3(z) = \frac{1}{\sqrt{2}}\left(U(\frac{1}{\sqrt{2}}(\xi_3-\xi_4)+\delta/2, z)c_{\frac{1}{\sqrt{2}}(\xi_3-\xi_4)}+U(\frac{1}{\sqrt{2}}(\xi_4-\xi_3)+\delta/2, z)c_{\frac{1}{\sqrt{2}}(\xi_4-\xi_3)}\right) \quad [5.69]$$

5-4- Folding $E_6^{(1)} \to D_4^{(3)}$

Let us now examine the case of an automorphism τ of order 3. This case arises for the affine algebra $E_6^{(1)}$ whose EDD exhibits a Z_3-symmetry :



with the simple root system

$$R = \begin{cases} \alpha_o = \delta - \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8) \\ \alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8) \\ \alpha_2 = e_1 + e_2 \\ \alpha_3 = -e_1 + e_2 \\ \alpha_4 = -e_2 + e_3 \\ \alpha_5 = -e_3 + e_4 \\ \alpha_6 = -e_4 + e_5 \end{cases} \quad [5.70]$$

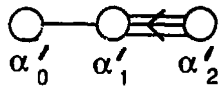
The outer automorphism τ of order 3 associated to this symmetry is defined by

$$\tau^2(\alpha_o) = \tau(\alpha_1) = \alpha_6 \qquad \tau^2(\alpha_2) = \tau(\alpha_3) = \alpha_5 \qquad \tau(\alpha_4) = \alpha_4 \quad [5.71]$$

If β is a root of the horizontal algebra E_6, we set $\overline{\beta} = \beta + m\delta$, which is a root of the affine algebra $E_6^{(1)}$. The root system of the affine algebra invariant under τ is

$$\tilde{\Delta} = \left\{ \frac{1}{3}(\overline{\beta} + \tau(\overline{\beta}) + +\tau^2(\overline{\beta})) \ , \ \overline{\beta} \in \Delta \right\} \qquad [5.72]$$

where Δ is the root system of $E_6^{(1)}$.
The folded EDD is



with the corresponding simple root system

$$\tilde{R} = \begin{cases} \alpha'_o = \delta/3 - \frac{1}{3}(\eta_2 + \eta_3 - 2\eta_1) \\ \alpha'_1 = \frac{1}{3}(2\eta_2 - \eta_1 - \eta_3) \\ \alpha'_2 = \eta_3 - \eta_2 \end{cases} \qquad [5.73]$$

with $\eta_1 = -e_4 \ , \ \eta_4 = e_1 \ , \ \eta_i = e_i \ (i \neq 1, 4)$.
One obtains the twisted affine algebra $D_4^{(3)}$.
If we write an affine root $\overline{\beta}$ of $E_6^{(1)}$ as

$$\overline{\beta} = \sum_{i=0}^{6} m_i \alpha_i \qquad [5.74]$$

and the folded root $\overline{\beta}'$ of $D_4^{(3)}$ as

$$\overline{\beta}' = \frac{1}{3}(\overline{\beta} + \tau(\overline{\beta}) + \tau^2(\overline{\beta})) = \sum_{i=0}^{2} m'_i \alpha'_i \qquad [5.75]$$

one can divide $\tilde{\Delta}$ into three different subsets :

$$\tilde{\Delta}_L^I = \left\{ \frac{1}{3}(\overline{\beta} + \tau(\overline{\beta}) + \tau^2(\overline{\beta})) \mid \overline{\beta} = \tau(\overline{\beta}) \right\} \qquad [5.76]$$

which corresponds to the long roots of the invariant integral subalgebra G_2, appearing at each integral level.

$$\tilde{\Delta}_S^I = \left\{ \frac{1}{3}(\overline{\beta} + \tau(\overline{\beta}) + \tau^2(\overline{\beta})) \mid \overline{\beta}.\tau(\overline{\beta}) = 0 \ , \ m'_o = 0 \ \ [\text{mod } 3] \right\} \qquad [5.77]$$

which corresponds to the short roots of the invariant integral subalgebra G_2, appearing at each integral level.

$$\tilde{\Delta}^T = \left\{ \frac{1}{3}(\overline{\beta} + \tau(\overline{\beta}) + \tau^2(\overline{\beta})) \mid \overline{\beta}.\tau(\overline{\beta}) = 0 \ , \ m'_o \neq 0 \ \ [\text{mod } 3] \right\} \qquad [5.78]$$

which corresponds to the roots of the twisted part, which appear at level $\mathbf{Z}+1/3$ and $\mathbf{Z}+2/3$.
The root system of $D_4^{(3)}$ is therefore

$$\tilde{\Delta} = \Big\{ \pm(\eta_i - \eta_j) + m\delta,\ \pm\frac{1}{3}(2\eta_i - \eta_j - \eta_k) + m\delta/3,\ n\delta/3,$$

$$1 \leq i \neq j \neq k \leq 3,\ m \in \mathbf{Z},\ n \in \mathbf{Z}^* \Big\} \qquad [5.79]$$

The automorphism τ is now extended from the root sustem $\tilde{\Delta}$ to the affine algebra by the relations

$$\tau(E_m^\beta) = E_m^{\tau(\beta)} \qquad [5.80]$$

$$x.\tau(H_m) = \tau^{-1}(x).H_m \qquad [5.81]$$

the 2-cocycle being chosen invariant under τ.
Now taking y invariant under τ, one has

$$[y.H_n, E_m^\beta + E_m^{\tau(\beta)} + E_m^{\tau^2(\beta)}] = \frac{1}{3}y.(\beta + \tau(\beta) + \tau^2(\beta))(E_m^\beta + E_m^{\tau(\beta)} + E_m^{\tau^2(\beta)}) \quad [5.82]$$

$$[d, E_m^\beta + E_m^{\tau(\beta)} + E_m^{\tau^2(\beta)}] = m(E_m^\beta + E_m^{\tau(\beta)} + E_m^{\tau^2(\beta)}) \qquad [5.83]$$

The step operator attached to the root $\frac{1}{3}(\overline{\beta} + \tau(\overline{\beta}) + \tau^2(\overline{\beta}))$ of the folded affine algebra at level m is therefore

$$E_m^\beta \qquad \text{if} \qquad \overline{\beta} = \tau(\overline{\beta}) \qquad (m \in \mathbf{Z}) \qquad [5.84]$$

$$\frac{1}{\sqrt{3}}(E_m^\beta + E_m^{\tau(\beta)} + E_m^{\tau^2(\beta)}) \qquad \text{if} \qquad \overline{\beta}.\tau(\overline{\beta}) = 0 \qquad [5.85]$$

In this last case, $m \in \mathbf{Z} + p/3$ where $m'_o = p \pmod 3$.

To construct the corresponding vertex operators, we use the same trick as in the case of twisted algebras $\tilde{\mathcal{G}}^{(2)}$. One extends the weight lattice Λ of the invariant integral subalgebra G_2 of the folded algebra to a lorentzian lattice $\overline{\Lambda}$.
The vertex operators for the roots of $\tilde{\Delta}_S^I$ are

$$E(\beta, z) = U(\beta, z)c_\beta \qquad [5.86]$$

the vertex operators for the roots of $\tilde{\Delta}_S^I$ are

$$E(\frac{1}{3}(\overline{\beta} + \tau(\overline{\beta}) + \tau^2(\overline{\beta})), z) = \frac{1}{\sqrt{3}}\Big(U(\beta, z)c_\beta + U(\tau(\beta), z)c_{\tau(\beta)} + U(\tau^2(\beta), z)c_{\tau^2(\beta)}\Big)$$
[5.87]

and finally the vertex operators for the roots of $\tilde{\Delta}^T$ are

$$E'(\frac{1}{3}(\overline{\beta}+\tau(\overline{\beta})+\tau^2(\overline{\beta})),z)=\frac{1}{\sqrt{3}}\Big(U(\beta+\delta/3,z)c_\beta+U(\tau(\beta)+\delta/3,z)c_{\tau(\beta)}+$$

$$+U(\tau^2(\beta)+\delta/3,z)c_{\tau^2(\beta)}\Big) \qquad \text{if } m'_o=1 \quad [\text{mod } 3] \tag{5.88}$$

$$E''(\frac{1}{3}(\overline{\beta}+\tau(\overline{\beta})+\tau^2(\overline{\beta})),z)=\frac{1}{\sqrt{3}}\Big(U(\beta+2\delta/3,z)c_\beta+U(\tau(\beta)+2\delta/3,z)c_{\tau(\beta)}+$$

$$+U(\tau^2(\beta)+2\delta/3,z)c_{\tau^2(\beta)}\Big) \qquad \text{if } m'_o=2 \quad [\text{mod } 3] \tag{5.89}$$

c_β is here the cocycle operator constructed on the root lattice $\Lambda_R(E_6)$ of the non folded Lie algebra.

Construction of the vertex operators :

• for the invariant part

The long roots of G_2 at integral level are invariant roots of $E_6^{(1)}$ under the automorphism τ. Therefore

$$E(\pm(\eta_i-\eta_j),z)=U(\pm(\eta_i-\eta_j),z)c_{\pm(\eta_i-\eta_j)} \tag{5.90}$$

The short roots of G_2 at integral level divide into two distincts orbits Ω_1 and Ω_2 under the Weyl group generated by the long roots :

$$\Omega_1=\Big\{\frac{1}{3}(2\eta_i-\eta_j-\eta_k) \quad , \quad 1\le i\ne j\ne k\le 3\Big\}$$

$$\Omega_2=\Big\{-\frac{1}{3}(2\eta_i-\eta_j-\eta_k) \quad , \quad 1\le i\ne j\ne k\le 3\Big\} \tag{5.91}$$

The vertex operator part factorizes easily as :

$$E(\alpha_S,z)=U(\alpha_S,z)\Gamma_i(z) \qquad \text{if} \quad \alpha_s\in\Omega_i \tag{5.92}$$

with

$$\Gamma_1(z)=\frac{1}{\sqrt{3}}\Big(U(\xi_1,z)+U(\xi_2,z)+U(\xi_3,z)\Big) \tag{5.93}$$

$$\Gamma_2(z)=\frac{1}{\sqrt{3}}\Big(U(-\xi_1,z)+U(-\xi_2,z)+U(-\xi_3,z)\Big) \tag{5.94}$$

where

$$\begin{aligned}\xi_1&=\frac{1}{3}(\eta_1+\eta_2+\eta_3+3\eta_4)\\ \xi_2&=\frac{1}{3}(\eta_1+\eta_2+\eta_3-3\eta_4)\\ \xi_3&=-\frac{2}{3}(\eta_1+\eta_2+\eta_3)\end{aligned} \tag{5.95}$$

However, one must also examine the factorization properties of the cocycle ! Actually, the cocycle problem in this case has been studied in ref. [2,3].
Let β be a long root of the underlying D_4 in E_6. Then, the short roots β_S at integral level can be written as

$$\beta_S = \frac{1}{3}(\beta + \tau(\beta) + \tau^2(\beta)) \qquad \beta \text{ root of } D_4 \subset E_6, \text{ with } \beta.\tau(\beta) = 0 \qquad [5.96]$$

.

For $\alpha, \beta \in \Lambda_R(D_4)$, one has

$$(-1)^{\alpha.\beta} = (-1)^{3\alpha_S\beta_S} \qquad [5.97]$$

It follows from this that one can construct a suitable cocycle operator c_{β_S} for the short roots by using a cocycle operator of D_4 associated to the corresponding root β of D_4, related by [5.96]. In other words, the auxiliary fields $\Gamma_1(z)$ and $\Gamma_2(z)$ don't contain cocycle operator in their expression (or it reduces to the trivial unity operator).
Therefore the vertex operator associated to the short roots can be written as

$$E(\alpha_S, z) = U(\alpha_S, z)c_\alpha\Gamma_i(z) \qquad [5.98]$$

if $\alpha_S \in \Omega_i$ and $\alpha_S = \frac{1}{3}(\alpha + \tau(\alpha) + \tau^2(\alpha))$.
• for the twisted part
The twisted part is constituted by the generators associated to the short roots of G_2, appearing at level **Z**+1/3 and **Z**+2/3. As above, the short roots decouple in the two orbits Ω_1 and Ω_2.
The vertex operators are determined by the usual method. One has to introduce vertex operators with moments at level **Z**+1/3 and **Z**+2/3. This is achieved by considering the vertex operators of the form $U(\alpha_S + \delta/3, z)$ and $U(\alpha_S + 2\delta/3, z)$.
Therefore the vertex operators write :
- for the short roots of Ω_1 at level **Z**+1/3

$$E'(\alpha_S, z) = U(\alpha_S, z)c_\alpha\Gamma'_1(z) \qquad [5.99]$$

with

$$\Gamma'_1(z) = \frac{1}{\sqrt{3}}\Big(U(\xi'_1 + \delta/3, z) + U(\xi'_2 + \delta/3, z) + U(\xi'_3 + \delta/3, z)\Big) \qquad [5.100]$$

- for the short roots of Ω_1 at level **Z**+2/3

$$E''(\alpha_S, z) = U(\alpha_S, z)c_\alpha\Gamma''_1(z) \qquad [5.101]$$

with

$$\Gamma_1''(z) = \frac{1}{\sqrt{3}}\Big(U(\xi_1''+2\delta/3,z)+U(\xi_2''+2\delta/3,z)+U(\xi_3''+2\delta/3,z)\Big) \qquad [5.102]$$

- for the short roots of Ω_2 at level $\mathbf{Z}$+1/3

$$E'(\alpha_S,z) = U(\alpha_S,z)c_\alpha\Gamma_2''(z) \qquad [5.103]$$

with

$$\Gamma_2''(z) = \frac{1}{\sqrt{3}}\Big(U(-\xi_1''+\delta/3,z)+U(-\xi_2''+\delta/3,z)+U(-\xi_3''+\delta/3,z)\Big) \qquad [5.104]$$

- for the short roots of Ω_2 at level $\mathbf{Z}$+2/3

$$E''(\alpha_S,z) = U(\alpha_S,z)c_\alpha\Gamma_2'(z) \qquad [5.105]$$

$$\Gamma_2'(z) = \frac{1}{\sqrt{3}}\Big(U(-\xi_1'+2\delta/3,z)+U(-\xi_2'+2\delta/3,z)+U(-\xi_3'+2\delta/3,z)\Big) \qquad [5.106]$$

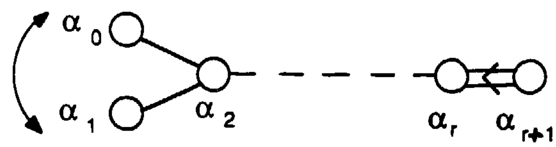
where

$$\begin{aligned}
\xi_1' &= \frac{1}{6}(-\eta_1-\eta_2-\eta_3-3\eta_4-3\eta_5+3\eta_6+3\eta_7-3\eta_8)\\
\xi_2' &= \frac{1}{6}(-\eta_1-\eta_2-\eta_3+3\eta_4-3\eta_5-3\eta_6-3\eta_7+3\eta_8)\\
\xi_3' &= \frac{1}{3}(\eta_1+\eta_2+\eta_3+3\eta_5)\\
\xi_1'' &= \frac{1}{6}(-\eta_1-\eta_2-\eta_3-3\eta_4+3\eta_5-3\eta_6-3\eta_7+3\eta_8)\\
\xi_2'' &= \frac{1}{6}(-\eta_1-\eta_2-\eta_3+3\eta_4+3\eta_5+3\eta_6+3\eta_7-3\eta_8)\\
\xi_3'' &= \frac{1}{3}(\eta_1+\eta_2+\eta_3-3\eta_5)
\end{aligned} \qquad [5.107]$$

Notice that the different auxiliary fields $\Gamma_i(z)$, $\Gamma_i'(z)$ and $\Gamma_i''(z)$ are not independent since $\xi_i+\xi_i'+\xi_i''=0$ for $i=1,2,3$.
These auxiliary fields, together with the elementary vertex operators associated to the short roots of $D_4^{(3)}$, are no longer fermionic fields since the short roots α_S have squared length 2/3 and the vectors ξ_i, ξ_i', ξ_i'' have squared length 4/3. Therefore the corresponding vertex operators $U(\alpha_S,z)$ have conformal weight 1/3 and $U(\xi_i,z)$, $U(\xi_i',z)$, $U(\xi_i'',z)$ have conformal weight 2/3, the O.P.E.'s between these fields having branching points rather than poles. One can say that one obtains in this case a parafermionic construction of the vertex operators of the twisted algebra $D_4^{(3)}$.

5-5- Folding $A^{(2)}_{2l+1} \to A^{(4)}_{2l}$

Finally we will study the case where one can exploit the symmetry of the EDD of a twisted affine algebra. One starts from the EDD of $A^{(2)}_{2l+1}$:



associated to the simple root system

$$R = \left\{ \alpha_o = \delta/2 - \frac{1}{\sqrt{2}}(\eta_1+\eta_2)\,,\ \alpha_i = \frac{1}{\sqrt{2}}(\eta_i - \eta_{i+1}) \quad (1 \le i \le l)\,,\ \alpha_{l+1} = \frac{2}{\sqrt{2}}\eta_{l+1} \right\} \qquad [5.108]$$

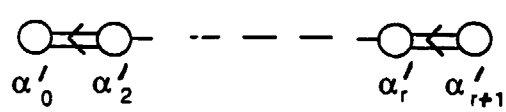
This diagram has a Z_2 symmetry defined by the automorphism τ :

$$\tau(\alpha_o) = \alpha_1 \qquad \tau(\alpha_i) = \alpha_i \quad (2 \le i \le l+1) \qquad [5.109]$$

The simple root system of the folded algebra is

$$\tilde{R} = \left\{ \alpha'_o = \delta/4 - \frac{1}{\sqrt{2}}\eta_2\,,\ \alpha_i = \frac{1}{\sqrt{2}}(\eta_i - \eta_{i+1}) \quad (2 \le i \le l)\,,\ \alpha_{l+1} = \frac{2}{\sqrt{2}}\eta_{l+1} \right\} \qquad [5.110]$$

which corresponds to the folded DD



One obtains therefore the twisted affine algebra $A^{(4)}_{2l}$ with invariant integral subalgebra C_l.
The root system of $A^{(4)}_{2l}$ is

$$\tilde{\Delta} = \left\{ \frac{\pm 2}{\sqrt{2}}\eta_i + m\delta,\ \frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j) + m\delta/2, \frac{\pm 1}{\sqrt{2}}\eta_i + (m+1/2)\delta/2, \right.$$

$$\left. n\delta/4,\ 2 \le i \ne j \le l+1,\ m \in \mathbf{Z},\ n \in \mathbf{Z}^* \right\} \qquad [5.111]$$

Construction of the vertex operators :
Notice first that $A^{(2)}_{2l-1}$ is a regular subalgebra of $A^{(4)}_{2l}$. Therefore, the corresponding vertex operators are those which were constructed in paragraph 5.2. in the case of

$A^{(2)}_{2l-1}$.

• for the invariant part

The invariant part is given by the roots β such that $\beta = \tau(\beta)$. It corresponds to the roots

$$\frac{\pm 2}{\sqrt{2}}\eta_i, \ \frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j), \ 2 \leq i \neq j \leq l+1 \qquad [5.112]$$

which appear at each integral level.

The vertex operators are

$$E(\frac{\pm 2}{\sqrt{2}}\eta_i, z) = U(\frac{\pm 2}{\sqrt{2}}\eta_i, z)c_{\frac{\pm 2}{\sqrt{2}}\eta_i} \qquad [5.113]$$

$$E(\frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j), z) = U(\frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j), z)c_{\frac{1}{\sqrt{2}}(\pm\eta_i, \pm\eta_j)}\Gamma_{ij}(z) \qquad [5.114]$$

where $\Gamma_{ij}(z)$ is a fermionic auxiliary field depending on the orbit Ω_{ij} (see paragraph 5.2.)

$$\Omega_{ij} = \left\{\frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j)\right\} \qquad [5.115]$$

• for the twisted part at level **Z**+1/2

The twisted part at level **Z**+1/2 is constituted by the generators associated with short roots of A_{2l-1}. The vertex operators are

$$E'(\frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j), z) = U(\frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j), z)c_{\frac{1}{\sqrt{2}}(\pm\eta_i, \pm\eta_j)}\Gamma'_{ij}(z) \qquad [5.116]$$

where $\Gamma'_{ij}(z)$ is a fermionic auxiliary field depending on the orbit Ω_{ij}, on opposite Ramond or Neveu-Schwarz character than $\Gamma_{ij}(z)$ (see paragraph 5.2.).

• for the "very short" roots at level **Z**+1/4 and **Z**+3/4

The "very short" roots are the roots $\frac{\pm 1}{\sqrt{2}}\eta_i$ which appear at level **Z**+1/4 and **Z**+3/4. The vertex operators for these roots at level **Z**+1/4 are

$$E'(\frac{\pm 1}{\sqrt{2}}\eta_i, z) = \frac{1}{2}U(\frac{\pm 1}{\sqrt{2}}\eta_i, z)\Big(U(\frac{1}{\sqrt{2}}\eta_1 + \delta/4, z)\Gamma_{1i}(z) + U(-\frac{1}{\sqrt{2}}\eta_1 - \delta/4, z)\Gamma'_{1i}(z)\Big) \qquad [5.117]$$

The generators associated to the bosonic roots $\frac{\pm 1}{\sqrt{2}}\eta_i$ at level **Z**+3/4 are obtained by action of the generators associated to the short roots $\frac{1}{\sqrt{2}}(\pm\eta_i \pm \eta_j)$ at level **Z**+1/2 on the generators associated to the bosonic roots $\frac{\pm 1}{\sqrt{2}}\eta_i$ at level **Z**+1/4. One finds

$$E''(\frac{\pm 1}{\sqrt{2}}\eta_i, z) = \frac{1}{2}U(\frac{\pm 1}{\sqrt{2}}\eta_i, z)\Big(U(\frac{1}{\sqrt{2}}\eta_1 + \delta/4, z)\Gamma'_{1i}(z) + U(-\frac{1}{\sqrt{2}}\eta_1 - \delta/4, z)\Gamma''_{1i}(z)\Big) \qquad [5.118]$$

The auxiliary fields are defined by (see paragraph 5.2)

$$\Gamma_{ij}(z) = \frac{1}{\sqrt{2}}\Big(U(\frac{1}{\sqrt{2}}(\xi_i - \xi_j), z)c_{\frac{1}{\sqrt{2}}(\xi_i, -\xi_j)} + U(\frac{-1}{\sqrt{2}}(\xi_i - \xi_j), z)c_{\frac{-1}{\sqrt{2}}(\xi_i, -\xi_j)}\Big) \qquad [5.119]$$

$$\Gamma'_{ij}(z) = \frac{1}{\sqrt{2}} \quad U(\frac{1}{\sqrt{2}}(\xi_i+\xi_j)+\delta/2, z)c_{\frac{1}{\sqrt{2}}(\xi_i+\xi_j)}+U(\frac{-1}{\sqrt{2}}(\xi_i+\xi_j)+\delta/2, z)c_{\frac{-1}{\sqrt{2}}(\xi_i+\xi_j)} \qquad [5.120]$$

$$\Gamma''_{ij}(z) = \frac{1}{\sqrt{2}}\left(U(\frac{1}{\sqrt{2}}(\xi_i - \xi_j) + \delta, z)c_{\frac{1}{\sqrt{2}}(\xi_i-\xi_j)} + U(\frac{-1}{\sqrt{2}}(\xi_i - \xi_j) + \delta, z)c_{\frac{-1}{\sqrt{2}}(\xi_i-\xi_j)}\right) \qquad [5.121]$$

The properties of $\Gamma_{1i}(z)$, $\Gamma'_{1i}(z)$ and $\Gamma''_{1i}(z)$ insures that one has

$$[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j} \qquad [\text{mod } 4] \qquad [5.122]$$

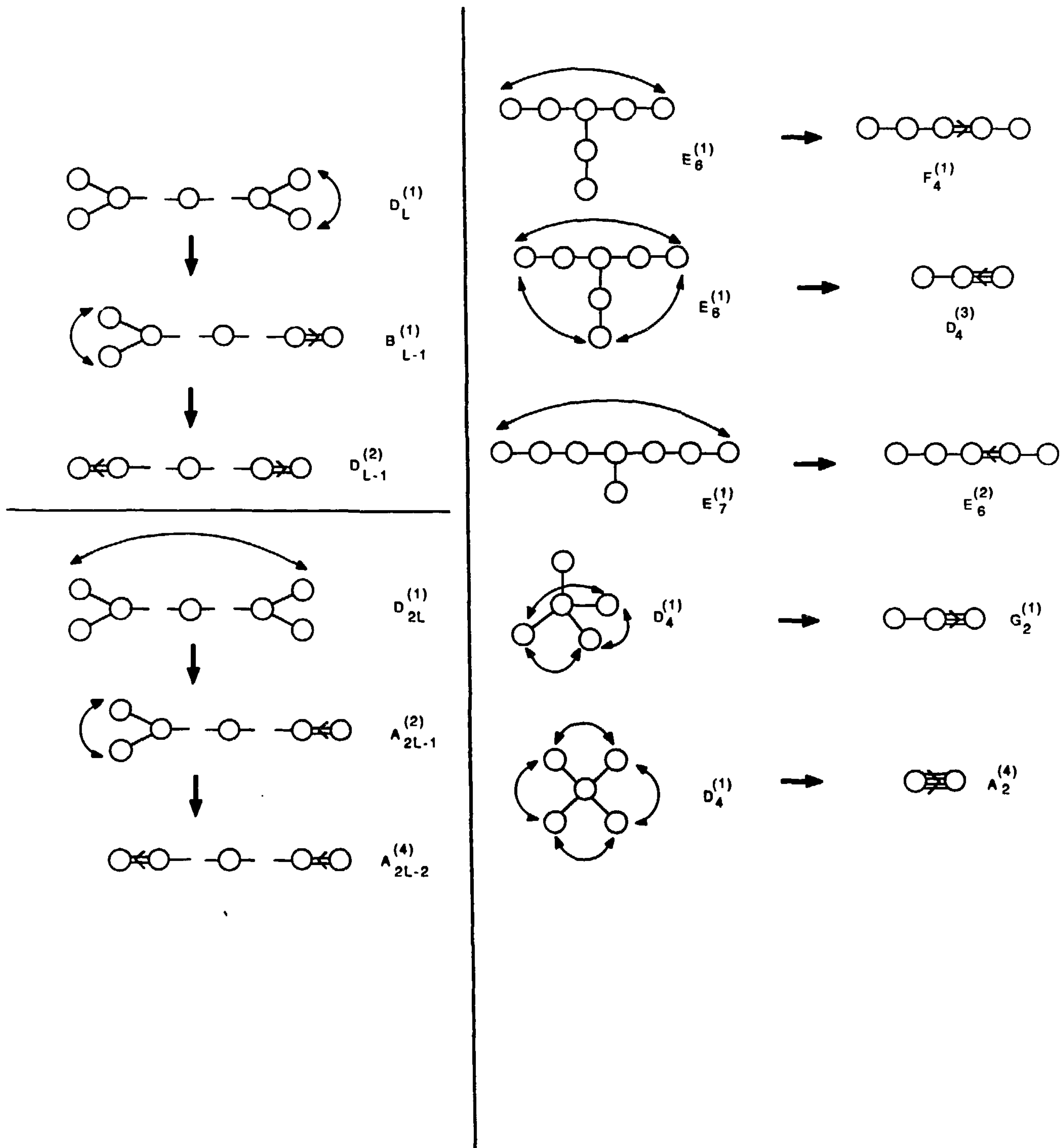
where $\mathcal{G}_i$ represents the set of generators at level $\mathbf{Z} + i/4$ $(i \text{ mod } 4)$.
For the cocycle operator, one can construct it as a cocycle operator on the rescaled root lattice $\frac{1}{\sqrt{2}}\Lambda_R(B_l)$, since the invariant horizontal algebra of $A_{2l}^{(4)}$ is B_l. This construction is very similar to those explained in the $A_{2l-1}^{(2)}$ case where the cocycle operator on the lattice $\frac{1}{\sqrt{2}}\Lambda_R(D_l)$ was constructed.
Remark : remind that the twisted algebra $A_{2l}^{(4)}$ is actually isomorphic to $A_{2l}^{(2)}$ [14].

6. CONCLUSION

In the method we have presented to construct the vertex operators for twisted affine algebras $\mathcal{G}^{(m)}$ $(m \neq 1)$, the property of a twisted algebra to show up as a subalgebra of an untwisted one is widely used. One could wonder on the relevance of such $\mathcal{G}^{(1)}$ subalgebras associated to outer automorphisms of $\mathcal{G}^{(1)}$ in physics and in particular in string theories where the notion of twist, relative to inner and outer $\mathcal{G}$ automorphisms, appears today as a basic tool (cf. orbifold compactification). Let us remark that maximal simple affine subalgebras of $\mathcal{G}^{(1)}$ with $\mathcal{G}$ compact can be classify in three different classes : the regular (resp. singular) subalgebras $\mathcal{H}^{(1)}$ with $\mathcal{H}$ being a regular (resp. singular) maximal $\mathcal{G}$ subalgebra, and the twisted subalgebras $\mathcal{G}^{(m)}$. This last class of subalgebras - when it exists - is directly due to the affine structure of $\mathcal{G}^{(1)}$ and has no counterpart at the finite level.

Concerning the above vertex construction itself, let us emphasize on its property of being conceptually simple, its fundamental feature standing in the adjunction of (in general dependent) auxiliary (para)fermionic fields to short root and also to affine short root operators. The folding of EDD allows a rather elegant approach, more tedious may appear the cocycle construction which has to be done in each case separately. Finally let us note that the same type of method can be used to construct vertex operators for untwisted and twisted affine superalgebras [15].

Table 2 : Folding schemes for affine and twisted algebras



REFERENCES

[1] I.B. Frenkel, V.G. Kac, Invent. Math. 62, (1980)
G. Segal, Comm. Math. Phys. 80, (1981), 301

[2] P. Goddard, W. Nahm, D.I. Olive, A. Schwimmer, Comm. Math. Phys 107, (1986), 179

[3] D. Bernard, J. Thierry-Mieg, Comm. Math. Phys. 111, (1987), 181

[4] V.G. Kac, D.H. Peterson, "Proceedings of the conference on anomaly, geometry and topology in Argonne", 1985, Ed. A. White (World Scientific)

[5] J. Lepowsky, Proc. Nat. Acad. Sci. USA 82, (1985), 8295

[6] P. Sorba, B. Torresani, preprint CPT 87/P.203 and LAPP TH 196/87 to appear in Int. Journ. of Mod. Phys. A, and references therein

[7] D. Gross, J. Harvey, E. Martinec, R. Rohm, Nucl. Phys. B 256, (1985), 253 and B 267, (1986), 75

[8] H. Kawai, D.C. Lewellen, S.H.H. Tye, Nucl. Phys. B 288, (1987), 1
I. Antoniadis, C.P. Bachas, C. Kounnas, Nucl. Phys. B 289, (1987), 87

[9] V.G. Kac , Advances in Math. 30, (1978), 85
V.G. Kac "Infinite dimensionnal Lie algebras" , Cambridge : Cambridge University Press, 1985

[10] D. Olive, N. Turok, Nucl. Phys. B 215, (1983), 47

[11] N. Bourbaki, "Groupes et algèbres de Lie", chap. 4,5,6, Ed. Hermann, Paris, 1968

[12] D. Bernard, Nucl. Phys. B 288, (1987), 628

[13] D. Altschüler, J. Lacki, Ph. Zaugg, preprint Université de Genève UGVA-DPT 1987/11-555

[14] A. Feingold , I.B. Frenkel , Advances in Math. 56, (1985), 117

[15] L. Frappat, A. Sciarrino, P. Sorba, in preparation