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FINITE CLUSTER RENORMALIZATION AND NEW TWO STEP RENORMALIZATION GROUP FOR ISING MODEL *

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ABSTRACT

New types of renormalization group theory using the generalized Callen identities are exploited in the study of the Ising model. Another type of two-step renormalization is proposed. Critical couplings and critical exponents y_T and y_H are calculated by these methods for square and simple cubic lattices, using different size clusters.

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INTRODUCTION

Different renormalization group technics have been developed during the last fifteen years. Some of the generally used methods for the Ising models are the Migdal-Kadanoff (M.K) transformations (Migdal 1976; Kadanoff 1976), the decimation methods (Barber 1975; Nelson and Fisher 1975; Kadanoff and Houghton 1975; Young and Stinchcombe 1976), the mean field renormalization group (M.F.R.G) (Indekeu et al. 1982) and recently proposed finite cluster renormalization method (F.C.R) (Benayad et al. 1988 (a) and (b)).

In the spirit of the phenomenological renormalization (Nightingale 1976) these two last methods are based on the comparison of systems of different sizes. This yields a single recursion relation and it is not possible to determine the complete renormalization flow in the parameter space of the Hamiltonian. However, any improvement in the results can be achieved by using large size cells. Furthermore, the results of the F.C.R are very satisfactory and are more accurate than those obtained from the M.F.R.G. method. Indeed, if one considers two finite systems (clusters) with N' and N spins, respectively (N' < N), one computes the order parameter, i.e. the average magnetization, for both clusters by finite cluster approximation (Boccara 1983) which leads to better results than those obtained from the mean field approximation.

In this work we apply, at first, a F.C.R to Ising models in the presence of magentic field on d-dimensional hypercubic lattices with a scaling factor given by $b_2 = (N/N')^{1/d}$. We calculate the reduced critical coupling K_c, non-trivial fixed point, and the critical exponents y_T and y_H for various values of N' and N and we compare them with the other values given by M.F.R.G.

Secondly, we use a two-step renormalization group method at the same model. In this method we first apply a M.K method, with a scaling factor $b_1 = 2$, obtaining two separate recursion relations for the renormalized coupling and renormalized magnetic field. This renormalized lattice is then treated within the F.C.R from which a single approximation recursion relation is obtained for the final renormalized parameters in terms of the original lattice parameters. We give the numerical values of K_c, y_T and y_H with this method and with the values given by M.K.M.F.R.G (Evangelista and Saxena 1985).

FINITE CLUSTER RENORMALIZATION

The effective Hamiltonian which describes an Ising model in the presence of magnetic field on d-dimensional hypercubic lattices is given by

$$-\beta\mathcal{H} = K \sum_{\langle i,j\rangle} \sigma_i\sigma_j + h \sum_i \sigma_i , \qquad \beta = \frac{1}{kT}$$

where $\sigma_i = \pm 1$ are the Ising spins. The summation runs over all nearest neighbour pairs. K is the coupling and $h = \beta H$ is the reduced magnetic field.

We restrict ourselves to the smallest possible systems. Then we consider, for example, the one and two spin clusters ($N' = 1$, $N = 2$). For fixed values of all spins outside the clusters, the magnetization per spin in each case is given by

$$\langle \sigma_0 \rangle_c = \frac{\mathrm{Tr}_c\, \sigma_0\, e^{-\beta\mathcal{H}}}{\mathrm{Tr}_c\, e^{-\beta\mathcal{H}}} \tag{1}$$

$$\frac{1}{2}\langle \sigma_1 + \sigma_2 \rangle_c = \frac{1}{2}\, \frac{\mathrm{Tr}_c\, (\sigma_1 + \sigma_2)\, e^{-\beta\mathcal{H}}}{\mathrm{Tr}_c\, e^{-\beta\mathcal{H}}} \tag{2}$$

The c under the symbol Tr indicates that the traces are performed over the spins of the clusters denoted by σ_i ($i = 0$ and $i = 1,2$, respectively). Eqs.(1) and (2) give

$$\langle \sigma_0 \rangle_c = \tanh\left[K \sum_{i=1}^{z} \sigma_i + h\right]$$

$$\frac{1}{2}\langle \sigma_1 + \sigma_2 \rangle_c = \frac{\sinh\left[K\left(\sum_{\substack{\langle i,1\rangle \\ i\neq 2}} \sigma_i + \sum_{\substack{\langle j,2\rangle \\ j\neq 1}} \sigma_j\right) + 2h\right]}{\cosh\left[K\left(\sum_{\substack{\langle i,1\rangle \\ i\neq 2}} \sigma_i + \sum_{\substack{\langle j,2\rangle \\ j\neq 1}} \sigma_j\right) + 2h\right] + e^{-2K} \cosh\left[K\left(\sum_{\substack{\langle i,1\rangle \\ i\neq 2}} \sigma_i - \sum_{\substack{\langle j,2\rangle \\ j\neq 1}} \sigma_j\right)\right]} \tag{4}$$

In relation (3) the summation runs over all nearest neighbours (n.n) of spin σ_0. In relation (4), the symbols $\sum_{\langle i,1\rangle} \sigma_i$ $\left(\sum_{\langle j,2\rangle} \sigma_j\right)$ means that the summation is over all n.n of σ_1 except σ_2 (over all n.n of σ_2 except σ_1).

Relations (3) and (4) are exact. The right-hand sides of these relations depend upon boundary spins and magnetic field. Relation (3) has been used by Callen (Callen 1963) to derive the identity

$$m_1 = \left\langle \tanh\left(K \sum_{i=1}^{z} \sigma_i + h\right) \right\rangle \tag{5}$$

where the average on the right-hand side is over all spins configurations and m_1 is the exact magnetization per spin. Similarly, a generalized Callen's identity can be derived from (4).

Neglecting correlations between different spins we perform the thermal average using the approximate probability distribution

$$P_{F.C.A}(\{\sigma_i\}) = \prod_i \left\{ \frac{1+m}{2}\, \delta(\sigma_i - 1) + \frac{1-m}{2}\, \delta(\sigma_i + 1) \right\} \tag{6}$$

This distribution gives better results than the mean field approximation (Boccara 1983) which corresponds to the distribution

$$P_{M.F.A.}(\{\sigma_i\}) = \prod_i \delta(\sigma_i - m) \tag{7}$$

To average any function of the random variables σ_i distributed according to (6), it is easier to use the following theorem (Boccara 1983):

> The set of all bounded real fucntions of $\sigma_1, \sigma_2, \ldots, \sigma_z$ is a 2^z dimensional euclidean space; the set $\{1, \sigma_1, \sigma_2, \ldots, \sigma_z, \sigma_1\sigma_2, \ldots, \sigma_z, \sigma_1\sigma_2, \ldots, \sigma_z\}$ which contains all the products of different spins, is an orthonormal basis for the inner product defined by
>
> $$\langle f_1 | f_2 \rangle = \frac{1}{2^z} \mathop{\mathrm{Tr}}_{\sigma_1, \sigma_2, \ldots, \sigma_z} f_1(\sigma_1, \sigma_2, \ldots, \sigma_z)\, f_2(\sigma_1, \sigma_2, \ldots, \sigma_z)$$

Using this theorem and making the thermal average of σ we can have the equation of state by F.C.A for the first cluster ($N' = 1$)

$$m_1 = A_0^{(z)}(K', h') + A_1^{(z)}(K', h')\, m_1' + A_2^{(z)}(K', h')\, m_1'^2 + \ldots \tag{8}$$

For the second cluster we can have the following equation of state:

$$m_2 = B_0^{(z)}(K,h) + B_1^{(z)}(K,h)\, m_2' + B_2^{(z)}(K,h)\, m_2'^2 + \dots \qquad (9)$$

where $A_i^{(z)}$ and $B_i^{(z)}$ are the coefficients of square ($z = 4$) or cubic ($z = 6$) lattices.

In the spirit of the phenomenological renormalization, to obtain the renormalization recursion relation, we use the scaling of the magnetization and of the magnetic field (Indekeu et al. 1982; Indekeu et al. 1984; Benayad et al. 1988). For small mean values of the boundary spins and magnetic field (8) and (9) lead to the following equations:

$$A_0^{(z)}(K') = B_0^{(z)}(K)\, b_2^{-2y_H+d} \qquad (10a)$$

$$A_1^{(z)}(K') = B_1^{(z)}(K) \qquad (10b)$$

where $b_2 = (N/N')^{1/d}$, d is the dimensionality and $A_0^{(z)}(k')$, $B_0^{(z)}(k)$, $A_1^{(z)}(k')$ and $B_1^{(z)}(k)$ are given in the appendix.

Considering clusters of larger sizes, higher order approximate recursion relations can be derived. The numerical results concerning square and simple cubic lattices are given in Tables 1 and 2, together with those obtained from M.F.R.G and the exact results for comparison.

TWO STEP RENORMALIZATION (M.K.F.C.R)

In this section we show that the above described method (F.C.R) can be combined with Migdal-Kadanoff, still using small clusters to obtain in a straightforward way very good results for the critical coupling and critical exponents.

In the first step we apply the M.K transformation to the d-dimensional hypercubic Ising model described by the above Hamiltonian. This leads to the following recursion relations, with a scale factor $b_1 = 2$, for the normalized coupling K' and magnetic field h':

$$K' = b_1^{d-1}\, K(\tilde{K},h) \qquad (11a)$$

$$h' = b_1^{d-1}\, \tilde{h}(K,h) \qquad (11b)$$

where

$$\tilde{K} = \frac{1}{4} \ln \frac{\cosh(2K+h)\cosh(2K-h)}{\mathrm{ch}^2 h} \qquad (12a)$$

$$\tilde{h} = h + \frac{1}{2} \ln \frac{\cosh(2K+h)}{\cosh(2K-h)} \qquad (12b)$$

For a small value of the magnetic field h and for a first order the relations (12a) and (12b) are given by

$$K' \simeq \frac{2^{d-1}}{4} \ln \frac{1+\cosh 4K}{2} \qquad (13a)$$

$$h' \simeq 2^{d-1}(1+\mathrm{tank}\,2K)h \qquad (13b)$$

In the second step we apply the F.C.R to the renormalized lattice defined by the parameters (K',h'). The scale factor in this case is given by

$$b_2 = (N/N')^{1/d}$$

where N and N' are the number of spins in the two clusters and $N' < N$. For $N' = 1$ and $N = 2$ we can have the recursion relations for the normalized parameters (K'',h'') in terms of (K',h')

$$A_0^{(z)}(K'') = B_0^{(z)}(K')\, b^{-2y_H+d} \qquad (14a)$$

$$A_1^{(z)}(K'') = B_1^{(z)}(K') \qquad (14b)$$

We used Eqs.(14a) and (14b) to calculate the critical couplings, and exponents y_T and y_H for 2D and 3D Ising models. We also considered the four spin cluster. The result of our calculations for square and simple cubic lattices for various scaling parameters $b = b_1 b_2$ are presented in Tables I and II, respectively.

In conclusion, we have presented firstly the results given by the F.C.R and compared them with those of M.F.R.G. Secondly, we have proposed a new method (M.K.F.C.R). For both methods our results are very satisfactory and are more accurate, even for small size clusters than those obtained from the M.F.R.G or M.K.M.F.R.G methods.

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Appendix :

7 4 square lattice.

$$A_0^{(4)}(K') = \frac{1}{8}\left(\frac{1}{\cosh^2 4K'} + \frac{4}{\cosh^2 2K'} + 3\right)$$

$$A_1^{(4)}(K') = \frac{1}{8}\left(\tanh 4K' + 2\tanh 2K'\right)$$

$$B_0^{(4)}(K) = \frac{1}{16}\left(\frac{1+e^{-2K}\cosh 6K}{(\cosh 6K+e^{-2K})^2} + 6\,\frac{1+e^{-2K}\cosh 2K\,\cosh 4K}{(\cosh 4K+e^{-2K}\cosh 2K)^2}\right.$$

$$+\,6\,\frac{1+e^{-2K}\cosh 4K\,\cosh 2K}{(\cosh 2K+e^{-2K}\cosh 4K)^2} + 9\,\frac{1+e^{-2K}\cosh 2K}{(\cosh 2K+e^{-2K})^2}$$

$$\left. + \frac{1}{1+e^{-2K}\cosh 6K} + \frac{9}{1+e^{-2K}\cosh 2K}\right)$$

$$B_1^{(4)}(K) = \frac{1}{32}\left(\frac{\sinh 6K}{\cosh 6K+e^{-2K}} + 4\,\frac{\sinh 4K}{\cosh 4K+e^{-2K}\cosh 2K} + 2\,\frac{\sinh 2K}{\cosh 2K+e^{-2K}\cosh 4K}\right.$$

$$\left. + 3\,\frac{\sinh 2K}{\cosh 2K+e^{-2K}}\right)$$

Z = 6 simple cubic lattice

$$A_0^{(6)}(K') = \frac{1}{64}\left(\frac{2}{\cosh^2 6K'} + \frac{12}{\cosh^2 4K'} + \frac{30}{\cosh^2 2K'} + 20\right)$$

$$A_1^{(6)}(K') = \frac{1}{64}\left(2\tanh 6K' + 8\tanh 4K' + 10\tanh 2K'\right)$$

$$B_0^{(6)}(K) = \frac{1}{1024}\left(\frac{4(1+e^{-2K}\cosh 10K)}{(\cosh 10K+e^{-2K})^2} + 10\,\frac{4(1+e^{-2K}\cosh 2K\cosh 8K)}{(\cosh 8K+e^{-2K}\cosh 2K)^2}\right.$$

$$+ 20\,\frac{4(1+e^{-2K}\cosh 4K\cosh 6K)}{(\cosh 6K+e^{-2K}\cosh 4K)^2}$$

$$+ 25\,\frac{4(1+e^{-2K}\cosh 6K)}{(\cosh 6K+e^{-2K})^2} + 20\,\frac{4(1+e^{-2K}\cosh 6K\cosh 4K)}{(\cosh 4K+e^{-2K}\cosh 6K)^2}$$

$$+ 100\,\frac{4(1+e^{-2K}\cosh 4K\cosh 2K)}{(\cosh 4K+e^{-2K}\cosh 2K)^2} + 10\,\frac{4(1+e^{-2K}\cosh 8K\cosh 2K)}{(\cosh 2K+e^{-2K}\cosh 8K)^2}$$

$$+ 100\,\frac{4(1+e^{-2K}\cosh 4K\cosh 2K)}{(\cosh 2K+e^{-2K}\cosh 4K)^2} + 100\,\frac{4(1+e^{-2K}\cosh 2K)}{(\cosh 2K+e^{-2K})^2}$$

$$\left. + 2\,\frac{2}{1+e^{-2K}\cosh 10K} + 50\,\frac{2}{1+e^{-2K}\cosh 6K} + 200\,\frac{2}{1+e^{-2K}\cosh 2K}\right)$$

$$B_1^{(6)}(K) = \frac{1}{1024}\left(\frac{2\sinh 10K}{\cosh 10K+e^{-2K}} + 8\,\frac{2\sinh 8K}{\cosh 8K+e^{-2K}\cosh 2K} + 12\,\frac{2\sinh 6K}{\cosh 6K+e^{-2K}\cosh 4K}\right)$$

$$+ 15\,\frac{2\sinh 6K}{\cosh 6K+e^{-2K}} + 8\,\frac{2\sinh 4K}{\cosh 4K+e^{-2K}\cosh 6K} + 40\,\frac{2\sinh 4K}{\cosh 4K+e^{-2K}\cosh 2K}$$

$$\left. + 2\,\frac{2\sinh 2K}{\cosh 2K+e^{-2K}\cosh 8K} + 20\,\frac{2\sinh 2K}{\cosh 2K+e^{-2K}\cosh 4K} + 20\,\frac{2\sinh 2K}{\cosh 2K+e^{-2K}}\right)$$

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Table I

	N'	N	K_c	y_T	y_H
F.C.R	1	2	0.358	0.72	1.41
	1	4	0.371	0.85	1.49
	2	4	0.379	0.99	1.58
M.F.R	1	2	0.347	0.60	1.41
	1	4	0.361	0.69	1.50
	2	4	0.370	0.78	1.58
M.K.F.C.R	1	2	0.526	0.92	1.74
	1	4	0.499	0.95	1.70
M.K.M.F.R	1	2	0.538	0.75	1.75
	1	4	0.506	0.78	1.70
Exact			0.441*	1.00*	1.87*

Critical coupling K_c and critical exponents y_T and y_H for the square lattice.

* Exact

Table II

	N'	N	K_c	y_T	y_H
F.C.R	1	2	0.206	0.75	1.89
M.F.R	1	2	0.203	0.65	1.89
M.K.F.C.R	1	2	0.252	0.93	2.55
M.K.M.F.R	1	2	0.252	0.89	2.55
Exact			0.222*	1.59**	2.48**

Critical coupling K_c and critical exponents y_T and y_H for the simple cubic model

* Domb (1974)

** Le Guillon and Zinn-Justin (1980)