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DYNAMICS OF MASSLESS HIGHER SPINS IN THE SECOND ORDER IN CURVATURES *

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ABSTRACT

The consistent equations of motion of interacting massless fields of all spins $s = 0, 1/2, 1 \ldots \infty$ are constructed explicitly to the second order of the expansion in powers of the higher spin strengths.

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1. INTRODUCTION

Recently, it was argued [1],[2], that equations of motion of interacting massless fields of all spins $s = 0, 1/2, 1 \ldots \infty$ in 3 + 1 dimensions can be formulated in terms of some infinite-dimensional free differential algebra (FDA). In this approach the general coordinate invariance is explicit and the higher spin gauge invariance is a simple consequence of the formal consistency of the higher spin equations *). The formalism of Refs.[1],[2] suggests that one can look for higher spin interactions by expanding them in powers of the higher spin curvatures which generalize the Weyl tensor in gravity ($s = 2$). This expansion procedure turns out to be much more efficient than the ordinary expansion in powers of interactions used in Ref. [5] where the nonlinear higher spin action was proposed which contained gravity and was shown to be consistent to the lowest order in the interactions. In Refs. [1],[2], the higher spin FDA was constructed explicitly to the first nontrivial order in powers of the higher spin Weyl tensors that leads to the higher spin equations of motion containing a lot of interactions beyond the cubic interactions of Ref.[5].

The approach of Ref.[1] was supported in a highly nontrivial way by the results of Refs.[7], [8] where it was shown that the sets of massless fields which give rise to consistent equations of motion in Ref. [1] are in a one to one correspondence with the massless unitary representations of those infinite-dimensional higher spin superalgebras of Refs. [4],[9],[10],[11],[8] which obey the admissibility condition of Refs. [7],[8]. However the problem of finding consistent higher spin equations in the closed form (i.e. in all orders) remains to be solved and seems to be difficult enough. In order to make sure in the most direct way that it admits some solution at all, it seems reasonable to analyse the higher spin FDA in the second order. This question is addressed in the present paper. Our main result consists of the derivation of the explicit form of all those terms which ensure consistency of the nonlinear higher spin equations of motion to the second order in the higher spin Weyl tensors (Weyl 0-forms).

*) As is well-known [3], all free spin $s \geq 1$ massless fields are abelian gauge fields. The requirement of the higher spin gauge invariance serves as the main principle which governs the structure of the higher spin interactions, and it is very convenient to reduce it to some Frobenius-type consistency conditions. Also it is worth mentioning that the gauge invariant higher-spin-gravitational interactions involve [4],[5] both positive and negative powers of the cosmological constant. As a result, it is impossible to formulate the higher-spin-gravitational dynamics in the framework of some expansion procedure near the flat space if the higher spin gauge symmetries are assumed to remain unbroken, although this is possible for the AdS background. (In Refs. [4,5,1,2] and in the present paper, the AdS background metrics arises implicitly as some vacuum solution of the higher spin equations of motion with vanishing higher spin curvature 2-forms). It is this property which enables one to overcome the "no-go" statements of Refs. [6] (see also deWit and Freedman in Ref. [3]) where it was argued that no gauge invariant higher-spin-gravitational interaction can exist at all but was assumed implicitly that it is possible to expand higher-spin-gravitational interactions near the flat background.

2. THE FORMALISM

According to Refs. [1],[2], we describe massless fields of all spins in terms of the generating functions $\omega(\bar{y}_\alpha, \bar{y}_{\dot\alpha}, \kappa, \bar\kappa|x)$ and $C(\bar{y}_\alpha, \bar{y}_{\dot\alpha}, \kappa, \bar\kappa|x)$ which are respectively 1- and 0- forms in the 3+1-dimensional space–time with the coordinates x_ν $(\nu = 0 - 3)$, i.e. $\omega(\ldots) = \omega_\nu(\ldots)dx^\nu$. In what follows, we often use shorthand notations in which the dependence on the coordinates x_ν is implicit. The exterior differential $d = dx^\nu \frac{\partial}{\partial x^\nu}$ is in fact the only operator which acts on the space–time coordinates in our formalism. The auxiliary variables y_α and $\bar{y}_{\dot\alpha}$ are mutually conjugated commuting two–component spinors $[y_\alpha, y_\beta] = 0, [\bar{y}_{\dot\alpha}, \bar{y}_{\dot\beta}] = 0, [y_\alpha, \bar{y}_{\dot\beta}] = 0, \bar{y}_{\dot\alpha} = (y_\alpha)^\dagger$, $\alpha, \beta = 1, 2, \dot\alpha, \dot\beta = 1, 2$. The variables κ and $\bar\kappa$ behave as the Klein operators for the variables y_α and $\bar{y}_{\dot\alpha}$ respectively. This means that, by definition, they obey the conditions

$$\kappa^2 = \bar\kappa^2 = 1, \quad [\kappa, \bar\kappa] = 0 \tag{1a}$$

$$\{\kappa, y_\alpha\} = 0, [\kappa, \bar{y}_{\dot\alpha}] = 0, \ \{\bar\kappa, \bar{y}_{\dot\alpha}\} = 0, [\bar\kappa, y_\alpha] = 0 \tag{1b}$$

($\{,\} = 0$, and $[\,,\,]$ denote the anticommutator and commutator respectively). The coefficients of the expansion of ω and C in powers of the auxiliary variables $y, \bar{y}, \kappa$ and $\bar\kappa$ are identified with the ordinary space–time fields,

$$\omega(y, \bar{y}, \kappa, \bar\kappa|x) = \sum_{\substack{A,B=0,1\\ n,m=0\ldots\infty}} \frac{1}{2\, i n! m!}(\kappa)^A(\bar\kappa)^B y_{\alpha_1}\ldots y_{\alpha_n}\bar{y}_{\beta_1}\ldots\bar{y}_{\beta_m} \tag{2a}$$
$$\omega^{AB\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_m}(x),$$

$$C(y, \bar{y}, \kappa, \bar\kappa|x) = \sum_{\substack{A,B=0,1\\ n,m=0\ldots\infty}} \frac{1}{2\, i n! m!}(\kappa)^A(\bar\kappa)^B y_{\alpha_1}\ldots y_{\alpha_n}\bar{y}_{\beta_1}\ldots\bar{y}_{\beta_m} \tag{2b}$$
$$C^{AB\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_m}(x)$$

where the multispinors of odd (even) ranks are assumed to be (anti)commuting. A massless spin-s field is described [11],[1] by the set of 1–forms $\omega^{AA\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_m}(x)$ with $n+m = 2(s-1)$, and the sets of 0–forms $C^{A+1A\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_m}(x)$ with $m-n = 2s$ and $C^{AA+1\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_m}(x)$ with $n-m = 2s$. The physical massless fields coincide with the 1–forms $\omega^{AA\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_m}$ with $|n-m| \le 1$ for spins $s \ge 1$ and with 0–forms $C^{AA+1\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_m}$ with $n+m \le 1$ for $s = 0$ or $1/2$. All other fields in the above sets are auxiliary variables in the sense that they can be expressed in terms of (derivatives of) the physical fields [11],[1]. As for the fields ω^{AA+1} and C^{AA}, they are of auxiliary type. These fields do not possess their own degrees of freedom [12],[1] and in fact can be disregarded at the level of equations of motion.

In accordance with the approach of Refs. [1],[2], the equations of motion of interacting massless fields are assumed to be of the structure

$$R(\omega, C) = d\omega + \omega\wedge\omega + \omega\wedge\omega C + \omega\wedge\omega CC + O(C^3) = 0 \tag{3}$$

$$D(\omega, C) = dC + (\omega C - C\omega) + (\omega C - C\omega)C + (\omega C - C\omega)C^2 + O(C^4) = 0 \tag{4}$$

$(d = dx^\nu \frac{\partial}{\partial x^\nu})$. The most important condition imposed on Eqs. (3),(4) is their formal consistency, i.e. the equations $d^2\omega = 0$ and $d^2 C = 0$ should automatically hold as consequence of Eqs. (3) and (4). In accordance with the terminology of Refs. [13], we say in this case that Eqs. (3) and (4) define some FDA *).

Because of using the language of exterior algebra, the equations of motion (3) and (4) are explicitly general coordinate invariant. Simultaneously, the consistency of Eqs. (3) and (4), ensures [1] that these equations are invariant under the higher spin gauge symmetries. As a result, the consistency of the equations (3) and (4) is in fact the only essential requirement that fixes their explicit form in the highest orders.

Let us rewrite the left–hand–sides of Eqs.(3) and (4) in the form

$$R(\omega, C) = \sum_{n=0}^{\infty} R_n(\omega, C), \quad \mathcal{D}(\omega, C) = \sum_{n=0}^{\infty}\mathcal{D}_n(\omega, C) \tag{5}$$

where

$$R_n(\omega, \lambda C) = \lambda^n R_n(\omega, C), \mathcal{D}_n(\omega, \lambda C) = \lambda^{n+1}\mathcal{D}_n(\omega, C) \tag{6}$$

(the lowest term of the expansion of $\mathcal{D}(\omega, C)$ is linear in C). For example, $R_0 = d\omega + \omega\wedge\omega$, $R_1 = \omega\wedge\omega C$. Generally, the zero–order parts $R_0(\omega)$ and $D_0(\omega, C)$ describe, respectively, a curvature 2–form, which corresponds to some Lie superalgebra with the gauge potentials ω, and the covariant derivative in a representation of this superalgebra realized by the 0–forms C. In Ref.[1] it was suggested that this zero–order superalgebra coincides with the superalgebra of higher spin and auxiliary fields, $shsa(1)$, proposed in Ref.[10]. As for the 0–forms, these were argued in Ref.1 to belong to the adjoint representation of $shsa(1)$. The main result of Refs.[1],[2] consists of the derivation of the explicit forms of $R_1(\omega, C)$ and $\mathcal{D}_1(\omega, C)$ which describe some nontrivial deformation of the original FDA based on R_0 and $\mathcal{D}_0$ and, what is in fact most important, lead to the correct equations of motion of free massless fields of all spins at the linearized level. As observed in Ref. [1], in the case under investigation it is reasonable to assume that the following relation holds in all orders:

$$\mathcal{D}(\omega, C) = -C^i\frac{\delta}{\delta\omega^i}R(\omega, C) \tag{7}$$

(it is essential here that C belongs to the adjoint representation of $shsa(1)$ as is manifested by the zero–order relation $\mathcal{D}_0(\omega, C) = dC + \omega C - C\omega$).

The zero–order curvatures R_0 of $shsa(1)$ can be reduced to the form

$$R_0(\omega)(\Phi_0) = d\omega(\Phi_0) +$$

*) Strictly speaking, it is often assumed that FDA's are based on p–forms with $p > 0$ while in our scheme 0–forms C are also allowed. Obviously the 0–forms C serve as a source of the nonpolynomiality of Eqs.(3) and (4).

$$+\int d^4Y_1 d^4Y_2\,\omega(\Phi_1)\wedge\omega(\Phi_2)\exp[-i\sum_{\substack{n,m=0\\n>m}}^{2}(-1)^{n+m}(Y_m,Y_n)] \quad (8)$$

where we use the shorthand notations

$$\Phi_n = (Y_n;\kappa;\bar\kappa),\quad Y_n = (y_n^\alpha;\bar y_n^{\dot\alpha}) \quad (9)$$

The label n distinguishes between different variables (note that the Klein operators κ and $\bar\kappa$ do not carry the index n, i.e. they coincide for different variables). The symplectic form $(Y_m,\overline{Y}_n)$ is defined as follows:

$$(Y_m,Y_n) = y_{m\alpha}y_n^\alpha + \bar y_{m\dot\alpha}\bar y_n^{\dot\alpha} \quad (10)$$

In Eq.(8), the integration over y_α and $\bar y_{\dot\alpha}$ is carried out as if they would be independent real variables. Let us note that the superalgebra $shsa(1)$ is isomorphic [10] to the algebra of polynomials which depend on the operators $\hat y_\alpha$ and $\hat{\bar y}_{\dot\alpha}$ obeying the Heisenberg commutation relations

$$[\hat y_\alpha,\hat y_\beta] = 2i\epsilon_{\alpha\beta},[\hat{\bar y}_{\dot\alpha},\hat{\bar y}_{\dot\beta}] = 2i\epsilon_{\dot\alpha\dot\beta},\ [\hat y_\alpha,\hat{\bar y}_{\dot\beta}] = 0 \quad (11)$$

($\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \epsilon_{12} = 1$), and on the Klein operators κ and $\bar\kappa$ obeying the relations (1) with $\hat y$ and $\hat{\bar y}$ in place of y and $\bar y$. This fact follows from the observation that the second term on the r.h.s. of Eq.(8) coincides in its form with the product of the Weyl symbols of operators in the integral version of the theory of symbols of operators by Berezin [14] (see also Ref. [9]).

The deformation R_1 which leads to the correct equations of motion of massless fields of all spins [1], can be reduced to the form [2]

$$R_1(\omega,C)(\Phi_0) = \int d^4Y_1d^4Y_2d^4Y_3d^4Ud^4S\,\omega(\Phi_1)\wedge\omega(\Phi_2)C(\Phi_3)$$
$$\times\exp(-i[\sum_{\substack{n,m=0\\n>m}}^{3}(-1)^{n+m}(Y_m,Y_n) + \sum_{n=0}^{3}(-1)^n(Y_n,S) + 2(U,S)])\times$$
$$\times(\mu\hat\kappa\delta(\bar u)\Delta(y_0+u,s+u+y_2-y_3,y_3-u)$$
$$+\bar\mu\hat{\bar\kappa}\delta(u)\Delta(\bar y_0+\bar u,\bar s+\bar u+\bar y_2-\bar y_3,\bar y_3-\bar u)) \quad (12)$$

where $S = (s_\alpha;\bar s_{\dot\alpha})$ and $U = (u_\alpha;\bar u_{\dot\alpha})$ are spinorial integration variables analogous to Y_{1-3}, μ is an arbitrary complex parameter, $\bar\mu$ is its complex conjugate and $\Delta(a,c,b)$ is the function of the three two-component spinors a_α, b_α and c_α defined by means of the contour integral

$$\Delta(a,b,c) = \frac{1}{2\pi i}\oint_{t_{abc}} d\ell n z \quad (13)$$

The points of the complex plane in Eq. (13) are identified with the two-component spinors *) while t_{abc} denotes the triangle with vertices at the points a_α, b_α and c_α ($\alpha = 1,2$).

*) Any two-component spinor a_α is realized here as the complex number $\alpha = \alpha_1 + i\alpha_2$. It is essential that just in the same way as in Eq.(8), all two-component spinors are assumed to be real when calculating the integrals in Eq. (12) while the complexification of the variables y_α and $\bar y_{\dot\beta}$ should be carried out after completing integrations over spinorial variables. This latter complexification has nothing to do with the complex variables in Eq.(13).

The function $\Delta(a,b,c)$ possesses the following simple properties which follow from its definition (13)

$$\Delta(a,b,c) = -\Delta(b,a,c);\ \Delta(a,b,c) = \Delta(b,c,a) \quad (14a)$$

$$\Delta(-a,-b,-c) = \Delta(a,b,c) \quad (14b)$$

$$\Delta(a,b,c) = \Delta(d,b,c) + \Delta(a,d,c) + \Delta(a,b,d) \quad (15)$$

(for more detail see Ref. [2]). The relation (15) was called in Ref. [2] triangle identity. It expresses the simple fact that there exist two ways for splitting any quadrangle in pairs of triangles, □ = ⧄ = ⧅. Using Eqs.(14), (15), it is not difficult to show [2] that $R_0(8)$ and $R_1(12)$ define some FDA (5) which is consistent to the first order of the expansion in powers of C.

We conclude this section by the following useful formula [2] which makes it possible to connect the integral formulation of Ref. [2] with the differential formulation of Ref. [1]:

$$\Delta(a_1,a_2,a_3) = \int d^3\beta\,\nu(\beta)(a_{1\alpha}a_2^\alpha + a_{2\alpha}a_3^\alpha - a_{1\alpha}a_3^\alpha)\delta(\sum_{n=1}^{3}a_n\beta_n) \quad (16a)$$

where $\beta_{1,2,3}$ are some real integration variables and

$$\nu(\beta) = \theta(\beta_1)\theta(\beta_2)\theta(\beta_3)\delta(\sum_{n=1}^{3}\beta_n - 1) \quad (16b)$$

3. HIGHER SPIN FDA IN THE SECOND ORDER

The second order part, R_2, of the higher spin FDA turns out to be of the form

$$R_2(\omega,C)(\Phi_0) = \int d^4Y_1\ldots d^4Y_4\exp[-i\sum_{\substack{n>m\\n,m=0}}^{4}(-1)^{n+m}(Y_m,Y_n)]$$
$$\omega(\Phi_1)\wedge\omega(\Phi_2)C(\Phi_3)C(\Phi_4)\times$$
$$\times[\mu^2R^{20}(y_0\ldots y_4) + \bar\mu^2R^{02}(\bar y_0\ldots\bar y_4) + \mu\bar\mu\kappa\bar\kappa R^{11}(Y_0\ldots Y_4)] \quad (17)$$

Here

$$R^{20}(y_0\ldots y_4) = (4\pi)^2\int d^2vd^2qd^2sd^2t\exp -i[2(v_\alpha q^\alpha) - 4(s_\alpha t^\alpha)]$$
$$[\Delta(p_1,p_2,p_3)(\Delta(p_1+s,s-p_3,s-p_5) - \Delta(s+p_1,s-p_3,s-p_4))]$$
$$+\Delta(p_4,p_2,p_3)\Delta(s+p_1,s+p_2,s-p_4) + \chi(p_3,p_1,s,p_4) - \chi(p_3,p_2,s,p_4) - \chi(p_2,p_1,s,p_4)] \quad (18)$$

where

$$p_1 = y_0 + v - \frac{1}{2}\sum_{n=0}^{4}(-1)^ny_n + t - s,\ p_2 = q + \frac{1}{2}\sum_{n=2}^{4}(-1)^ny_n - \frac{1}{2}\sum_{n=0}^{1}(-1)^ny_n + v + t - s$$

$$p_3 = -q - \sum_{n=3}^{4}(-1)^n y_n - v + \frac{1}{2}\sum_{n=0}^{4}(-1)^n y_n + s + t,\; p_4 = t - \frac{1}{2}q,\; p_5 = t + v + y_4 + s - \frac{1}{2}\sum_{n=0}^{4}(-1)^n y_n \tag{19}$$

and $\chi(a_1, \ldots a_4)$ is some arbitrary function.

The part $R^{02}(\bar y_0 \ldots \bar y_n)$ can be obtained from R^{20} by replacing all undotted spinors by the dotted ones (however, an arbitrary function $\bar\chi(\bar a_1, \ldots \bar a_4)$ in R^{02} can differ from the function χ in R^{20}).

The part R^{11} reads

$$R^{11}(Y_0, \ldots Y_4) = \int d^4X d^4U d^4V \exp -i\{2\sum_{n=0}^{4}(-1)^n(Y_n, X) + (V, U)\}$$

$$[\tfrac{1}{2}\Delta(y_0 - x, y_2 - y_3 + y_4 + x + v - u, u + y_0 - y_1 + y_2 - x)\times$$

$$\times(\Delta(\bar y_0 - \bar x, \bar u + \bar y_0 - \bar y_1 + \bar y_2 - \bar x, \bar y_4 + \bar x) +$$

$$+\frac{1}{2}\Delta(\bar y_0 - \bar x, \bar y_2 - \bar y_3 + \bar y_4 + \bar x + \bar v - \bar u, \bar u + \bar y_0 - \bar y_1 + \bar y_2 - \bar x)) +$$

$$(y_{n\alpha} \leftrightarrow \bar y_{n\dot\alpha}; x_\alpha \leftrightarrow \bar x_{\dot\alpha}; u_\alpha \leftrightarrow \bar u_{\dot\alpha}; v_\alpha \leftrightarrow \bar v_{\dot\alpha})]$$

$$+\psi(Y_0 - X, X - Y_0 + Y_1, U, V) - \psi(Y_0 - X, Y_3 - Y_4 - X, U, V)$$

$$+\psi(Y_2 - Y_3 + Y_4 + X, Y_3 - Y_4 - X, U, V)] \tag{20}$$

(the ambiguity in the functions χ in Eq.(18) and ψ in Eq.(20) originates from the field redefinitions $\omega \to \omega + \omega CC$ and therefore does not affect the physical content of the higher spin equations of motion).

The main result of the paper is that the curvatures and covariant derivatives (5) defined by the relations (7), (8), (12), (17)–(20) give rise to the FDA which is consistent to the second order. Via Eqs. (3), (4), this FDA gives rise to the equations of motion of massless fields which are consistent (gauge invariant) to the second order in the Weyl 0–forms C. The explicit verification of the consistency is straightforward but cumbersome. It is essential that just in the same way as in the first order [1],[2], it suffices to use the associativity of the product law of ω's and C's in Eqs. (12) and (17) when verifying the consistency in the second order. In other words, one can assume that $\omega(\Phi)$ and $C(\Phi)$ take on their values in an arbitrary associative algebra (e.g. $Mat_n(\mathbb{C})$). As emphasized in Refs. [1],[8], this property offers the way for constructing various extended–type higher spin theories with nontrivial internal symmetries. Therefore, equations of motion of these theories are shown to admit consistent extensions to the second order in the Weyl 0–forms as well.

The second order part of the expression for $d^2\omega$ resulting from the differentiation of Eq. (3) has the folowing structure:

$$d^2\omega = -\omega\wedge[\omega\wedge\omega C^2] + [\omega\wedge\omega C^2]\wedge\omega + [\omega\wedge\omega C]\wedge\omega C - \omega\wedge[\omega\wedge\omega C]C$$

$$+\omega\wedge\omega\wedge[(\omega C - C\omega)C] + [\omega\wedge\omega]\wedge\omega CC - \omega\wedge[\omega\wedge\omega]CC$$

$$+\omega\wedge\omega\wedge[\omega C - C\omega]C + \omega\wedge\omega C\wedge[\omega C - C\omega] \tag{21}$$

where the terms in the square brackets originate from the insertion of the expressions for $d\omega$ and dC which follow from Eqs. (3), (4). (For example, in the second order in C, we have $(d\omega)\wedge\omega C = -[\omega\wedge\omega C]\wedge\omega C)$. The consistency requirement implies that the r.h.s. of Eq. (21) should vanish identically. There are three types of terms on the r.h.s. of Eq. (21): ω^3C^2–type terms, $\omega^2C\omega C$–type terms and $\omega^2C^2\omega$–type terms. It turns out that they cancel separately, and it immediately follows from this observation that, indeed, the assumption that all quantities take on their values in an arbitrary associative algebra does not spoil the consistency.

The verification of the cancellation of the $\omega^2C\omega C$– and $\omega^2C^2\omega$–type terms is relatively simple, while the most cumbersome part of the task consists of the proof that the ω^3C^2 terms cancel. This verification is carried out with the aid of some changes of the integration variables which reduce the arguments of the functions $\Delta(\ldots)$ to the form in which the cancellations become obvious consequences of the properties (14), (15). Due to the lack of the space, it is impossible to explain here this procedure in detail. As an illustration, we mention the useful trick which consists of making the change of variables $t \to t + \alpha s$ that preserves the form of the exponential in Eq. (18) and induces the transformation $p_i \to p_i + \alpha s$ (cf. Eq. (19)). As a result, one derives the useful relations

$$\int d^2v d^2q d^2s d^2t \exp -i[2v_\alpha q^\alpha - 4s_\alpha t^\alpha]\times$$

$$\times[\Delta(p_{i_1}, p_{i_2}, p_{i_3})\Delta(s + p_{j_1}, s + p_{j_2}, s + p_{j_3}) - \Delta(s - p_{i_1}, s - p_{i_2}, s - p_{i_3})\times$$

$$\times\Delta(p_{j_1}, p_{j_2}; p_{j_3}] = 0\ . \tag{22}$$

In this paper, we use the form of the higher–spin FDA in which the curvatures R are arranged in such a way that all 1–forms ω stand from the left while all 0–forms C stand from the right (see Eq. (3)). With the aid of field redefinitions $\omega \to \omega + \omega C + C\omega + \omega C^2 + C\omega C + \ldots$, $C \to C^1 + C^2 + C^3 \ldots$, one can construct other forms of the higher–spin FDA. All of them are physically equivalent at least perturbatively. The reason for using the "left" form (representative) of the higher spin FDA is that in this case the calculations and final expressions are simpler than for other possible orderings *) of ω and C. The very fact that the left form exists at all is highly intriguing and suggestive. However this form is inconvenient for the verification of the invariance under hermitian conjugation and transformations induced by antiautomorphisms because these operations reverse the order of product factors. The forms of the higher spin FDA in which these operations act linearly will be discussed elsewhere (to the first order, these forms are constructed in Refs.[1],[2]).

Another point we would like to mention here is that just in the same way as in the first order [1], the higher–spin FDA in the second order contains an arbitrary complex deformation

*) Of course, one can equally well use the "right" form in which all 1–forms stand from the right. It can be obtained from the left form with the aid of the antiautomorphism of $shsa(1)$ defined in Refs. [10],[1].

parameter μ (cf. Eqs. (12), (17)). Its absolute value $|\mu|$ can be compensated by the field redefinition $C \to \frac{1}{|\mu|}C$, while the phase $\frac{\mu}{|\mu|}$ survives as the essential deformation parameter because the fields ω and C are assumed to be resticted by some reality conditions which are not invariant under the multiplication by complex numbers (for more detail see Ref.[1]). This implies that the analysis of the higher–spin equations of motion in the second order does not fix the value of this arbitrary phase parameter which in fact parametrizes the class of inequivalent equations. In Ref. [8], it was conjectured that these equations can correspond to unitary higher spin theories only for some specific values of this phase parameter.

4. REGULARITY

The highly important property of the second order corrections (17)–(20) is that they are well-defined on the higher spin fields identified with the coefficients in the expansions (2). In Ref. [2], this property was called regularity. It means that if ω and C are polynomial functions of their spinorial arguments $Y_n (n > 0)$ screened in Φ_n according to Eq. (9), then the r.h.s. of Eqs. (12) and (19) are required to be polynomial in Y_0. Let us emphasize that the regularity requirement imposes stringent restrictions on the coefficient functions R^{20}, R^{11} and R^{02} and is necessary for physical reasons as providing the possibility to interpret the formal manipulations with generating functions in terms of the physical fields (perhaps in a slightly weaker form in which polynomials are allowed to give rise to power series). Note that beyond the class of regular deformations, the expressions for R_1 and R_2 can be derived from R_0 by means of some formal change of variables $\omega \to \omega' = \omega + \omega C + \omega C^2$, $C \to C' = C + C^2$. However, no regular change of variables exists which induces the deformation R_1 (12) and makes sense for power series (2), i.e. the deformation (12) is nontrivial in the class of regular deformations and field redefinitions [1].

The regularity of the first order deformation (12) was shown in Ref. [2], where it was also noted that even small modifications of the arguments of the functions $\Delta(\ldots)$ on the r.h.s. of Eq. (12) make the deformation (12) irregular. The situation with the second–order terms is analogous. Below, we describe the class of regular second order terms which is sufficient for the practical analysis. In the sector of R^{20}, this class is characterized by the following

Lemma Every $R^{20}(y_{0-4})$ of the form

$$R^{20}(y_0 - 4) = \int d^2v d^2q d^2s d^2t \exp(-i[2v_\alpha q^\alpha - 4s_\alpha t^\alpha]) \Delta(p_{i_1}, p_{i_2}, p_{i_3}) \Delta(s + \rho_1 p_{j_1}, s + \rho_2 p_{j_2}, s + \rho_3 p_{j_3}) \qquad (23)$$

gives rise to the regular terms R_2(17) if ρ_{1-3} are arbitrary real numbers while p_{i_n} and p_{j_m} are some of variables (18) which belong either to the set $\{p_1, p_2, p_3, p_4\}$ or to the set $\{p_1, p_2, p_3, p_5\}$ (i.e. p_4 and p_5 are not allowed to emerge simultaneously – cf. Eq. (18)).

Let us consider the Fourier transform of the r.h.s. of Eq. (23) with respect to the spinorial variables $y_1 \ldots y_4$,

$$\tilde{R}^{20}(y_0, z_1 \ldots z_4) = \int d^8 y_{1-4} d^2v d^2q d^2s d^2t d^2\xi d^2\eta d^3\alpha\nu(\alpha) d^3\beta\nu(\beta)$$

$$\sigma(p) \exp -i[\sum_{\substack{n>m \\ n,m=0-4}} (-1)^{n+m}(y_m, y_n) + 2(v, q) - 4(s, t) + (\xi, \sum_{k=1}^{3} \alpha_k p_{i_k})$$

$$+(\eta, (s + \sum_{k=1}^{3} \beta_k \rho_k p_{j_k})) + \sum_{n=1}^{4}(y_n, z_n)] \qquad (24)$$

This expression is derived with the aid of Eqs.(16). The spinorial δ–functions originating from Eq. (16a) are rewritten here as integrals over the additional spinorial integration variables ξ_α and η_α. The real integation variables α_k and β_k with $k = 1, 2, 3$ originate from β_k in Eq. (16a). Also it follows from Eq. (16a) that

$$\sigma(p) = \sum_{\substack{n>m \\ n,m=1-3}} (-1)^{n+m}(p_{i_m}, p_{i_n}) \sum_{\substack{u>v \\ u,v=1-3}} (-1)^{u+v}(p_{j_v}, p_{j_u}) \rho_u \rho_v \qquad (25)$$

Note that in the term proportional to η in the exponential on the r.h.s. of Eq. (24), we have taken into account that $s\Sigma_{k=1}^{3}\beta_k = s$ because of the factor $\delta(\Sigma_{k=1}^{3}\beta_k - 1)$ in $\nu(\beta)$ (see Eq. (16b)).

Now one observes that, disregarding the inessential dependence on the variables $\kappa, \overline{\kappa}$ and $\bar{y}$ (in the sector of these variables the regularity is obvious) the part of R_2 induced by R^{20} of the form (23) reduces to the expression $\tilde{R}^{02}(y_0, i\frac{\partial}{\partial y_n})\, \omega(y_1)\omega(y_2)\omega(y_3)C(y_4)$. As a result, in order to prove the Lemma, it suffices to prove that $\tilde{R}^{02}(y_0, z_n)$ is some analytic function of its arguments *)

Since the quantities p_i (19) depend linearly on the spinorial integration variables, Eq. (24) reduces to some Gaussian integral over these variables. However, the terms proportional to ξ and η depend on the variables α_i and β_k. This dependence can spoil the regularity if the quadratic form of the Gaussian integral degenerates for some values of α_k and β_k belonging to the support

*) In addition, $\tilde{R}^{02}(y_0, z)$ is required to admit an expansion in powers of z_n with the coefficients which are polynomial in y_0. This requirement results from the convention that polynomials should give rise to polynomials and can be disregarded if the definition of the regularity allows polynomials to give rise to power series. The main motivation for chosing this strong (polynomial) definition of the regularity is that it automatically holds for the expressions under investigation as a consequence of the explicit $sp(2;\mathbb{C}) \sim sl(2,\mathbb{C})$ invariance which ensures that the final result depends on the invariant combinations $(y_{0\alpha} z_n^\alpha)$ and $(z_{n\alpha} z_m^\alpha)$ and therefore the power of y_0 cannot exceed the overall power in z_n (note that $(y_{0\alpha}, y_0^\alpha) \equiv 0$).

of $\nu(\alpha)\nu(\beta)$. Actually, after completing the integration over spinorial variables, one arrives at the expression of the form

$$\int d^3\alpha\nu(\alpha)d^3\beta\nu(\beta)\tilde{\sigma}\Big(q^i,\frac{\partial}{\partial q^j}\Big)\exp -i\Big[\sum_{i,j=0}^{5} B_{ij}(\alpha,\beta)(q^i,q^j)\Big] \tag{26}$$

where $q^i = (y_0, z_n)$. If the Gaussian integral in Eq.(24) degenerates for some values of α_k and β_k the integral over α_k and β_k in Eq. (26) can be ill–defined near these values. Our goal is to show that in the case under consideration this does not happen and $B_{ij}(\alpha,\beta)$ turns out to be polynomial in α and β. Because the domain of integration over α and β is compact, this automatically implies regularity.

Our strategy is to show that there exists such a change of spinorial integration variables with the unit Jacobian, say, $u = a(\alpha,\beta)u'$, that (i) the matrix $a(\alpha,\beta)$ is polynomial in α and β and (ii) in the variables u', the quadratic part of the Gaussian integral (24) does not depend on α and β and is non–degenerate. Since after performing this change of variables the dependence on α and β will be contained only in the terms which are linear in the spinorial integration variables and this dependence is polynomial due to (i), it immediately follows from (ii) that the result of the integration is of the form (26) with some polynomial matrix $B_{ij}(\alpha,\beta)$.

The fact that such a change of variables really exists follows from the observation that $\tilde{R}^{20}$ can be reduced to the form

$$\tilde{R}(y_0,z_n) = \int d^8 y_{1-4}\, d^2 v d^2 q d^2 s d^2 t d^2 \xi d^2 \eta d^5 \alpha\hat{\nu}(\alpha) d^5\beta\hat{\nu}(\beta)$$

$$\sigma(p)\exp -i[\,2(p_{4\alpha}p_5^{\alpha}) + \sum_{k=1}^{3}(p_{k\alpha}u_k^{\alpha}) + (\xi_\alpha\sum_{k=1}^{5}\alpha_k p_k^{\alpha}) + (\eta_\alpha(s^\alpha + \sum_{k=1}^{5}\beta_k\rho_k p_k^{\alpha})) + \ldots] \tag{27}$$

where

$$u_1 = 2t + y_2 - y_1 - y_4,\; u_2 = y_1 - y_2,\; u_3 = -y_4 - q \tag{28}$$

and dots denote those terms which are linear in the spinorial integration variables (i.e. contain either y_0 or z_k). Note that in Eq. (27) we introduced the formal integration over variables α_i and β_i with $i = 1\ldots5$ assuming that $\hat{\nu}(\alpha)$ and $\hat{\nu}(\beta)$ contain extra δ-functions compensating the integrations over four extra variables α andβ, i.e. it is assumed that $\hat{\nu}(\alpha)$ is of the form

$$\hat{\nu}(\alpha) = \prod_{i=1}^{5}\theta(\alpha_i)\delta(1-\sum_{i=1}^{5}\alpha_i)\delta(\alpha_{j_1})\delta(\alpha_{j_2}) \tag{29}$$

for some $j_{1,2}$ and similarly for $\hat{\nu}(\beta)$. It is important that the set of variables p_{1-5}, u_{1-3} constitutes the full set of independent variables which is equivalent to the original set y_{1-4}, v, q, s, t. In accordance with Eqs.(19), (28) these two sets of spinorial variables are related by some linear transfomation with constant coefficients. Also it is important that the following relation holds

$$s = \frac{1}{4}[\,u_1 + u_2 + u_3 + 2p_5 - 2p_4 - 2p_1\,] \tag{30}$$

Now we are ready to prove the Lemma. For definiteness, suppose that the factor $\sigma(p)$ and the terms proportional to ξ and η in Eq. (27) depend on p_{1-4} and do not depend on p_5 and u_{1-3} that corresponds to the case when the functions $\Delta(\ldots)$ in Eq. (23) depend on p_{1-4}. Making the change of variables

$$u_i \to u_i' + \alpha_i\xi,\;\; p_5 \to p_5' + \frac{1}{2}\alpha_4\xi,\;\; p_{1-4} \to p_{1-4},\; \xi \to \xi,\;\; \eta \to \eta \tag{31}$$

one compensates the terms in the exponential in Eq. (27) which are proportional to ξ. On the other hand in accordance with Eq. (30), s transforms as follows $s \to s' + \frac{1}{4}\sum_{i=1}^{4}\alpha_i\xi$. Using the fact that $\hat{\nu}(\alpha) \sim \delta(\alpha_5)\delta(1-\sum_{i=1}^{5}\alpha_i)$ because we consider the case when p_5 does not emerge among the arguments of the functions $\Delta(\ldots)$ in Eq. (23), we see that $s \to s' + \frac{1}{4}\xi$. As a result, disregarding primes, one gets

$$\tilde{R}(y_0,z_n) = \int d^3 y_{1-4}\, d^2 v d^2 q d^2 s d^2 t d^2 \xi d^2 \eta d^5 \alpha\hat{\nu}(\alpha) d^5\beta\hat{\nu}(\beta)$$

$$\sigma(p)\exp -i[\,2(p_4,p_5) + \sum_{k=1}^{3}(p_k,u_k) + (\eta, s + \frac{1}{4}\xi + \sum_{k=0}^{5}\beta_k\rho_k p_k) + \ldots] \tag{32}$$

Finally, making the change of variables $\xi \to \xi' - 4s - 4\sum_{k=0}^{5}\beta_k\rho_k p_k$, one reduces (32) to the desired form in which the quadratic form in the exponential is non–degenerate and does not depend on α_k and β_k. This is achieved with the aid of the changes of variables which possess the unit Jacobian and are polynomial in α_n and β_m. The case when p_4 is replaced by p_5 can be considered analogously. This completes the proof of the Lemma.

It is worth mentioning that the regular terms (23) give rise to some regular terms in the verification of the consistency (i.e. in the explicit form of Eq.(21)). The resulting class of regular terms described by the appropriate generalization of the Lemma is sufficient to prove the consistency of the higher spin FDA in the second order.

As for the proof of the regularity of R^{11}, it is close to the proof of the regularity of the first order deformation (12) given in Ref.[2]. Namely by making the change of variables $Y_n \to Y_n' - 2X$ for $n = 1-4$ and disregarding the dependence on κ and $\bar{\kappa}$ which is inessential for the proof of regularity, the corresponding part of R_2 can be reduced to the form

$$\int d^{16}Y_{1-4}d^4 X d^4 U d^4 V \exp -i[\sum_{\substack{n>m\\ n,m=0}}^{4}(-1)^{n+m}(Y_m,Y_n) + 2(Y_0,X) + VU]$$

$$\sum_n \Delta(a_n - x, b_n - x, c_n - x)\Delta(\bar{a}_n - \bar{x}, \bar{b}_n - \bar{x}, \bar{c}_n - \bar{x})\omega(Y_1 - 2X)$$

$$\omega(Y_2 - 2X)C(Y_3 - 2X)C(Y_4 - 2X) \tag{33}$$

with X - independent quantities $a_n, b_n, c_n, \bar{a}_n, \bar{b}_n$ and $\bar{c}_n$. Now one observes that since the quadratic form of the Gaussian integral (33) does not depend on X, it does not contain the dependence on the integration variables β_i induced by Eq. (16a) after completing the integration over X. Analogously to the case of R^{20}, this proves the regularity of the terms (33).

5. CONCLUSION

The existence of the consistent (gauge invariant) equations of motion of massless fields of all spins in the second order in curvatures gives the strong indication that there exists the closed system of equations of motion of interacting massless fields of all spins which include the gravitational field ($s = 2$). Although the approach used throughout the paper is efficient enough to operate with infinities of nonlinear terms, the explicit verification of the consistency of the higher spin equations in the second order in the Weyl 0–forms is cumbersome enough. So, some more refined methods should be developed to formulate the dynamics of the higher spin gauge fields in the closed form.

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