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QUANTUM ISING MODEL ON HIERARCHICAL STRUCTURES *

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ABSTRACT

A quantum Ising chain with both the exchange couplings and the transverse fields arranged in a hierarchical way is considered. Exact analytical results for the critical line and energy gap are obtained. It is shown that when $R_1 \neq R_2$, where R_1 and R_2 are the hierarchical parameters for the exchange couplings and the transverse fields, respectively, the system undergoes a phase transition in a different universality class from the pure quantum Ising chain with $R_1 = R_2 = 1$. On the other hand, when $R_1 = R_2 = R$, there exists a critical value R_c dependent on the furcating number of the hierarchy. In case of $R > R_c$, the system is shown to exhibit an Ising-like critical point with the critical behaviour the same as in the pure case, while for $R < R_c$ the system belongs to another universality class.

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It is known that hierarchical structures appear in different physical contexts ranging from molecular diffusion on complex macromolecules[1] to anomalous relaxation in spin glasses [2-3] and computing architectures [4]. Recently, several authors have studied the problem of transport [5-8] and the electronic and vibrational properties [9-13] of hierarchical systems. In the case of diffusion, it has been shown that a hierarchical arrangement of energy barriers can give rise to anomalous behavior. Furthermore, a dynamical phase transition from ordinary to anomalous diffusion is found as the hierarchical parameter R is varied. In the electronic or vibrational problems, the eigenspectrum is found to be a zero-measure Cantor set and the eigenfuctions self-similar and critical. In this letter, we investigate a quantum Ising model where the the strengths of the couplings are distributed in a hierarchical way, as shown schematically in figure 1. The quantum mechanical phase transition of this system is supposed to be equivalent to the critical behavior of some two-dimensional classical layered Ising model [14]. To our knowledge this is the first analytical investigation of a model with non-trivial phase transition where the interaction is hierarchically modulated.

The model is given by the Hamiltonian:

$$H = -\sum_i J(i)\sigma_i^x\sigma_{i+1}^x - \sum_i h(i)\sigma_i^z \tag{1}$$

where σ_i^x and σ_i^z are the Pauli matrices at site i and the couplings $J(i)$ and the transverse fields $h(i)$ are given by (in bifurcating hierarchical way)

$$J(i) = J_0 R_1(i) = \begin{cases} J_0 & i = 2j+1 \\ J_0 R_1^k & i = 2^k(2j+1) \end{cases} \tag{2}$$

and

$$h(i) = R_2(i) = \begin{cases} 1 & i = 2j+1 \\ R_2^k & i = 2^k(2j+1) \end{cases} \tag{3}$$

Here R_1 and R_2 are hierarchical parameter chosen to be positive for simplicity. Note that $R_2 = 1$ corresponds to the case of the uniform transverse field and $R_1 = R_2 = 1$ corresponds to the pure quantum Ising model [15,16].

To obtain the critical line $J_{0c} = J_{0c}(R_1, R_2)$ and the energy gap $\Delta E = E_1 - E_0$ between the first excited and the ground-state energies, we proceed in the well-known fashion [17] known as Jordan-Wigner transformation. Thus the Hamiltonian can be expressed in terms of fermionic operators c_i's:

$$H = c^+ Ac + \frac{1}{2}(c^+ Bc^+ + h.c.) \tag{4}$$

where $c = (c_1, c_2, \ldots, c_{2^N})$ and the matrices A and B are given by:

$$\begin{aligned} A_{i,j} &= -J(j)\delta_{i,j+1} - 2h(i)\delta_{i,j} - J(i)\delta_{i,j-1} \\ B_{i,j} &= J(j)\delta_{i,j+1} - J(i)\delta_{i,j-1} \end{aligned} \tag{5}$$

In this paper we work in the so-called "c-cyclic" problem [17,18] with periodic boundary condition for (4). In fact, the system of period $p_N = 2^N$ is obtained by setting all couplings $J(i)$ and transverse fields $h(i)$ in (2) and (3) with $k > N-1$ to be of the following forms:

$$\begin{cases} J(i) = J_0 R_1^{N+1}, \\ h(i) = R_2^{N+1}. \end{cases} \qquad i = 2^k(2j+1) \text{ with } k \geq N \tag{6}$$

Thus the real hierarchical system corresponds to $N \to \infty$. Since we are interested in properties of the infinite system, the chosen boundary conditions are not important [18]. The quantum-mechanical transition is known to be driven by the soft mode of the Hamiltonian (4), which is given by $(A_c - B_c)\phi_0 = 0$ and $(A_c + B_c)\psi_0 = 0$, where A_c and B_c are the matrices A and B calculated at the critical line $J_{0c} = J_{0c}(R_1, R_2)$. The solutions to these equations are easily obtained as follows:

$$\begin{aligned} \phi_{0,j} &= (-1)^{j-1}\phi_1 \prod_{i=1}^{j-1} \frac{J_c(i)}{h(i+1)} \\ \psi_{0,j} &= (-1)^{j-1}\psi_1 \prod_{i=1}^{j-1} \frac{h(i)}{J_c(i)} \end{aligned} \tag{7}$$

where ϕ_1 and ψ_1 are normalised constants and $J_c(i)$ is given by (2) with $J_0 = J_{0c}$. The periodic boundary condition $\phi_{0,2^N+1} = \phi_{0,1}$ gives $\prod_{i=1}^{2^N} \frac{J_c(i)}{h(i+1)} = 1$, and thus one obtains the critical line immediately

$$J_{0,c} = \frac{R_2}{R_1} \equiv x \tag{8}$$

Now we turn to the calculation of the energy gap $\Delta E = E_1 - E_0$. For general bilinear fermionic forms like (4), which possess a zero mode, a perturbative study shows that the energy gap is given by [17,18]

$$\Delta E = 2r \frac{|\phi_0 H' \psi_0|}{|\phi_0|\ |\psi_0|} + O(r^2) \tag{9}$$

where $r = |J_0 - J_{0,c}|/J_{0,c}$ and the matrix $H'_{i,j} = J_c(i)\delta_{i,j-1}$. To evaluate this expression, we define:

$$x(i) \equiv \frac{R_2(i)}{R_1(i)} = \begin{cases} 1 & i = 2j+1 \\ x^k & i = 2^k(2j+1) \end{cases} \tag{10}$$

Thus

$$\begin{aligned} |\phi_0|^2 &= \sum_{j=1}^{2^N} \phi_{0,j}^2 \\ &= \phi_1^2 \sum_{j=0}^{2^{N-1}} \prod_{i=1}^{j} \frac{x^2 R_1^2(i)}{R_2^2(i+1)} \\ &= \phi_1^2 \sum_{j=0}^{2^{N-1}} \frac{1}{R_2^2(j+1)} \prod_{i=1}^{j} \frac{x^2}{x^2(i)} \\ &= \phi_1^2 \left[(1+x^2)^{N-1} + \frac{(1+x^2)^{N-2}}{R_1^2} + \frac{(1+x^2)^{N-3}}{R_1^4} + \ldots \right. \\ &\qquad \left. + \frac{(1+x^2)}{R_1^{2(N-2)}} + \frac{1}{R_1^{2(N-1)}} + \frac{1}{x^2 R_1^{2(N+1)}} \right] \\ &= \phi_1^2 (1+x^2)^N A_0^2 \end{aligned}$$

with

$$A_0^2 = \frac{R_1^2[1 - (yR_1^2)^{-N}]}{yR_1^2 - 1} + \frac{y}{x^2(yR_1^2)^{N+1}}$$

where we have used (7)-(8) and denoted $1 + x^2$ by y. In a similar way, we obtain

$$|\psi_0|^2 = \psi_1^2 (1 + x^{-2})^N.$$

With these results and $|\phi_0 H' \psi_0| = 2^N \phi_1 \psi_1$ we have

$$\Delta E = \frac{2r}{A_0} \left(\frac{2}{x + x^{-1}} \right)^N + O(r^2) \tag{11}$$

Thus when $x \neq 1$, i. e. the strengths of the couplings and the transverse fields are distributed hierarchically with different hirarchical parameter, with $x + x^{-1} > 2$, we have $\Delta E \sim O(r^2)$ for real hierarchical system corresponding to $N \to \infty$, which implies a correlation length exponent $\nu \neq 1$ and the transition belongs to a different universality class from that of the pure

Ising model [15,16]. On the other hand, when $x = 1$, the strengths of the couplings and the transverse fields are arranged in an identically hierarchical way, with $x + x^{-1} = 2$, and $R_1 = R_2 = R$, Eq.(11) gives

$$\Delta E = \begin{cases} 2r\frac{\sqrt{2R^2-1}}{R} + O(r^2) & R^2 > 1/2 \\ O(r^2) & R^2 < 1/2 \end{cases} \tag{12}$$

So we observe a critical value of the hierarchical parameter $R_c^2 = \frac{1}{2}$. In the case of $R > R_c$, Eq.(12) implies a correlation length exponent $\nu = 1$ and the transition belongs to the pure Ising universality class(at least for the thermal sector), whereas for $R < R_c$, with $\nu \neq 1$ from (12), the transition will fall into another universality class. This is quite different from the quasiperiodic quantum Ising model, where the correlation length exponent $\nu = 1$ for all quasiperiodic parameter [18,19].

Moreover, it can be proved that the dispersion relation for the first excited state for the hierarchical system with $x = 1$ and $R^2 > \frac{1}{2}$ is $\Lambda_k = C(R)(r^2 + k^2)^{\frac{1}{2}}$ where $C(R) = 2\sqrt{2R^2-1}/R$ is the sound velocity for the hierarchical quantum Ising model.

The Hamiltonian (4) could be diagonalised by a canonical transformation [17,19]. The resulting diagonal Hamiltonian assumes the following form

$$H = \sum_k \Lambda_k \left(\eta_k^+ \eta_k - \frac{1}{2}\right) \tag{13}$$

where η_k^+ and η_k are the fermion creation and annihilation operators, respectively. The energy of the modes Λ_k may be obtained from the solution of the following eigenvalue problem

$$G\Phi_k = \frac{\Lambda_k^2}{4}\Phi_k \tag{14}$$

with

$$G_{i,j} = [J^2(i-1) + h^2(i)]\delta_{i,j} + h(i)J(i)\delta_{i,j-1} + h(i-1)J(i-1)\delta_{i,j+1} \tag{15}$$

Here we also work in 'c-cyclic' problem and periodic boundary condition.

It is easy to see that at critical point $J_{0,c} = x = 1$,

$$\Phi_{0,j} = C_0 \sin(\pi j + \theta) \tag{16}$$

is the solution of (14) with zero eigenvalue. For the low excited modes near the critical point, we may choose the eigenvectors in the following form

$$\Phi_{k,j} = C \sin\varphi_j \sin\gamma_j \tag{17}$$

where

$$\varphi_j = (\pi - k)j + k\beta_j + \theta \qquad \gamma_j = r(\alpha_j - j) + \varsigma \tag{18}$$

with $r \ll 1$, $k \ll 1$. The parameters α_j and β_j are position dependent, but independent of k and r in leading order. Inserting (17) into (14) and retaining terms up to the second order of k and r, we have:

$$\begin{aligned} \Psi_{k,j} &\equiv (G\Phi_k)_j \\ &= \Phi_{k,j}\{r(A_{j-1} - A_j)\cot\gamma_j + k(B_{j-1} - B_j)\cot\varphi_j \\ &\quad + \frac{1}{2}r^2(a(j-1)A_{j-1} + a(j)A_j) + \frac{1}{2}k^2(b(j-1)B_{j-1} + b(j)B_j) \\ &\quad - rk(a(j-1)B_{j-1} + a(j)B_j)\cot\varphi_j\cot\gamma_j\} + O_3 \end{aligned} \tag{19}$$

where

$$\begin{aligned} a(j) &= \alpha_{j+1} - \alpha_j - 1 \quad b(j) = \beta_{j+1} - \beta_j - 1 \\ A_j &= a(j)R^2(j) \qquad B_j = b(j)R^2(j) \end{aligned} \tag{20}$$

Note that in (20) we have dropped the subscript for R since $R_1(i) = R_2(i)$ for $x = 1$. From (19) we observe that if we could choose parameters α_j and β_j to fulfil the following conditions:

$$A_j = A = \text{const.} \qquad B_j = B = \text{const.} \tag{21}$$

Φ_k in (17) gives an eigenvector of G to linear order with zero eigenvalue. Paying attention to the definition of the parameters $a(j)$, which assume the form of the difference between α_{j+1} and α_j, we observe that $a(j)$ may have only $(N+1)$ distinct values: $a_0, a_1, \ldots, a_N$ — depending on the value of the jth coupling. From Eq.(20) we have

$$\begin{aligned} 2^{-N}\textstyle\sum_j(a(j)+1) &= 2^{-1}(a_0+1) + 2^{-2}(a_1+1) + \ldots + 2^{-N}(a_{N-1}+1) \\ &\quad + 2^{-N}(a_N+1) \\ &= 0. \end{aligned} \tag{22}$$

On the other hand, the condition (21) requires

$$a_0 = a_1 R^2 = a_2 R^4 = \ldots = a_{N-1}R^{2(N-1)} = a_N R^{2(N+1)} \tag{23}$$

Combining (22) and (23) we obtain

$$A^{-1} = a_0^{-1} = -\left\{\frac{R^2\left[1-(2R^2)^{-N}\right]}{2R^2-1} + 2\left(\frac{1}{2R^2}\right)^{N+1}\right\}\left(1-2^{-N}\right)^{-1} \quad (24)$$

In a similar way, we have

$$B = A \quad (25)$$

With these eigenvectors Φ_k up to linear order, we could calculate the eigenvalues up to the second order by:

$$\Lambda_k^2/4 = \left(\sum_j \Psi_{k,j}\Phi_{k,j}\right)\left(\sum_j \Phi_{k,j}\Phi_{k,j}\right)^{-1} \quad (26)$$

This expression is easy to evaluate with the result

$$\Lambda_k^2/4 = -A(r^2+k^2) + O_3 \quad (27)$$

where use has been made of Eq.(19) and the relations:

$$\left(\sum_j \Phi_{k,j}\Phi_{k,j}X_j\right)\left(\sum_j \Phi_{k,j}\Phi_{k,j}\right)^{-1} = -1 + O_1 \quad (28)$$

with $X_j = a_j, a_{j-1}, b_j$ or b_{j-1} and

$$\left(\sum_j \sin(2\varphi_j)\sin(2\gamma_j)\right)\left(\sum_j \sin^2(\varphi_j)\sin^2(\gamma_j)\right)^{-1} = O_1 \quad (29)$$

Thus, the dispersion relation for the first excited state is

$$\Lambda_k = C(R)(r^2+k^2)^{1/2} \quad (30)$$

with

$$C(R) = 2\lim_{N\to\infty}\sqrt{-A} = \frac{2\sqrt{2R^2-1}}{R} \quad (31)$$

being the sound velocity. The dispersion relation (30) is the same as for the pure quantum Ising model ($R = 1$) [15,16], thus the low-energy excitations will produce a transition in the same universality class (at least for the thermal sector). Further, from (30) we note that at the critical point the

dispersion relation is linear $\Lambda_k(J_0 = J_{0,c}) = C(R)k$ and the energy gap vanishes linearly:

$$E_1 - E_0 = C(R)r \quad (32)$$

which gives again (12). Thus the correlation length exponent is $\nu = 1$ and futhermore the specific heat has a usual logarithmic singularity at $J_0 = J_{0,c}$.

The above discussion was concentrated on the bifurcating hierarchical system, i. e. the system with a regular uniformly bifurcating hierarchical arrays of couplings and the transverse fields. For general m-furcating hierarchical structure (see figure 2 for a trifurcating hierarchical structure), system with a period $p_N = m^N$ is obtained similarly by setting all couplings $J(i)$ and transverse fields $h(i)$ of the following forms:

$$J(i) = \begin{cases} J_0 R_1^k & i = m^k(mj+l), l = 1,2,\ldots,m-1 \quad k < N \\ J_0 R_1^{N+\frac{1}{m-1}} & i = m^k(mj+l), l = 1,2,\ldots,m-1 \quad k \geq N \end{cases} \quad (33)$$

$$h(i) = \begin{cases} R_2^k & i = m^k(mj+l), l = 1,2,\ldots,m-1 \quad k < N \\ R_2^{N+\frac{1}{m-1}} & i = m^k(mj+l), l = 1,2,\ldots,m-1 \quad k \geq N \end{cases} \quad (34)$$

and the real hierarchical system corresponds to $N \to \infty$.

In a similar way, one has the critical line

$$J_{0,c} = \left(\frac{R_2}{R_1}\right)^{\frac{1}{m-1}} \quad (35)$$

and the energy gap ΔE

$$\Delta E = 2r\left(\frac{m}{y_m}\right)^N D^{-\frac{1}{2}} \quad (36)$$

where

$$y_m = \sum_{n=0}^{m-1} x^{\frac{2n}{m-1}-1} \quad (37)$$

$$D = \frac{(z-x^2)R_1^2|1-(zR_1^2)^{-N}|}{zR_1^2-1} + \frac{1}{(xR_1)^{\frac{2}{m-1}}(zR_1^2)^N} \quad (38)$$

with

$$z = \sum_{n=0}^{m-1} x^{\frac{2n}{m-1}}$$

Similarly to bifurcating hierarchical structure, when $x \neq 1$, one has $\Delta E \sim O(\tau^2)$, which implies the correlation length exponent $\nu \neq 1$ and the model does not belong to the pure quantum Ising universility class. On the other hand, in case of $x = 1$, there exists a critical value for $R_1 = R_2 = R$. The the m-furcating hirarchical systems with $R^2 > 1/m = R_c^2$ belong to the Ising universility class, while $R^2 < 1/m$ models belong other universality class. Further, when $R > R_c$, it can be proved that the dispersion relation for the first excited state is

$$\Lambda_k = C_m(R)(\tau^2 + k^2)^{\frac{1}{2}} \tag{39}$$

where

$$C_m(R) = \frac{2}{R}\sqrt{\frac{mR^2 - 1}{m - 1}} \tag{40}$$

is the sound velocity for the m-furcating hierarchical quantum Ising model. The dispersion relation is the same as for the pure Ising model, so the low-energy excitation produce a transition in the same universility class. The correlation length exponent is $\nu = 1$ and the specific heat has a logarithmic singularity at the critical point $J_{0,c} = 1$.

To summarise, we have studied a quantum Ising model on a family of hierarchical structures and obtained analytical results for the critical line, energy gap between the first excited and ground-state energies and the dispersion relation for the low excited states. It has been shown that the Ising-like phase transition is preservevd only when both transverse fields and exchange couplings are distributed hierarchically with the same hierarchical parameter $R > R_c = \sqrt{1/m}$, for a m-furcating hierarchical structures. In this case the low-lying excitations at the critical point have a conformal tower structures. On the other hand, when $R < R_c$, the soft-mode driving phase transition falls into a different universality class.

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Figure captions

Figure 1. Bifurcating hierarchical structure. The lengths of the vertical segments represent the strengths of the exchange couplings and the diameter of the filled circles represent the strengths of the transverse fields.

Figure 2. Trifurcating hierarchical structure. The lengths of the vertical segments represent the strengths of the exchange couplings and the diameters of the filled circles represent the strengths of the transverse fields.

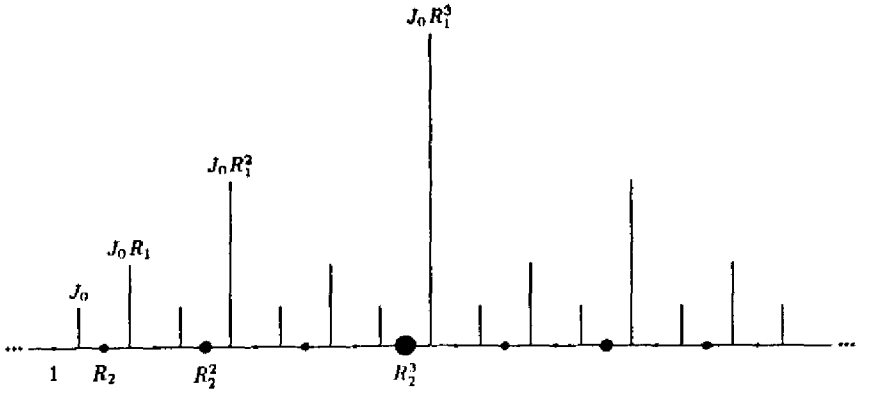

Figure 1

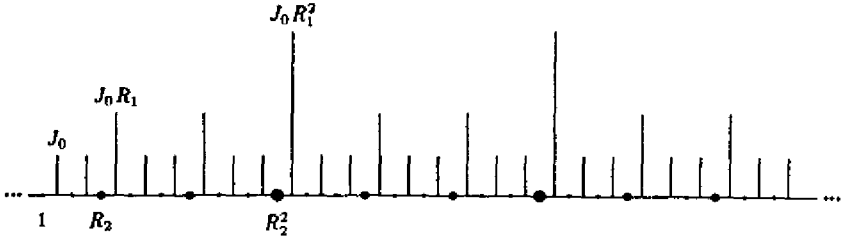

Figure 2