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Algebraic Topological Entropy

Thomas Hudetz
Institut für Theoretische Physik
Universität Wien
Boltzmanngasse 5, A-1090 Wien

Abstract

As a "by-product" of the Connes-Narnhofer-Thirring theory of dynamical entropy for (originally *non*-Abelian) nuclear C^*-algebras, the well-known variational principle for topological entropy is equivalently reformulated in purely *algebraically defined* terms for (separable) *Abelian* C^*-algebras. This "algebraic variational principle" should not only nicely illustrate the "feed-back" of methods developed for quantum dynamical systems to the classical theory, but it could also be *proved directly* by "algebraic" methods and could thus further simplify the original proof of the variational principle (at least "in principle").

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1 Introduction

The notion of *topological entropy* had been introduced in 1965 by Adler, Konheim and McAndrew [1] as a conjugacy invariant of continuous transformations on compact topological spaces, by close analogy with the Kolmogorov-Sinai [14] entropy invariant of measure preserving transformations on probability spaces, and thus leading [1] to the conjecture (relating topological and measure-theoretic entropy, see section 2) which was called *variational principle* for "physical" reasons later on (cf. [21], e.g.). The first part of this conjecture (that topological entropy *bounds* measure-theoretic entropy) was first proved by Goodwyn [10] in 1968, and this inequality (2.6,i) was then sharpened to the variational *equality* (2.6,ii) by Dinaburg [6] and Goodman [9] in 1970.

Although these original proofs have been simplified *technically* due to the work of Bowen (and others), resulting in the proofs to be found in the text-books on ergodic theory (see [5,23] and the references therein), there still remained the *conceptual* difficulty that comparing topological with measure-theoretic entropy involved a comparison of open covers (respectively, of spanning or separated sets in Bowen's equivalent definition of topological entropy) with "disjoint" Borel partitions of the compact metric space considered. This difficulty was partly overcome by Palm [20], who defined the entropy for *dynamical lattices* as a common generalization of topological and measure-theoretic entropy, such that also the latter is equivalently defined in terms of *not* necessarily *disjoint* Borel covers. This lattice-theoretic reformulation allowed a simple proof at least of Goodwyn's theorem (2.6,i); but still the proof involved a comparison of *all* Borel covers (as used in the redefinition of measure-theoretic entropy) with their "generating" subset of *open* covers, by means of a generalized Kolmogorov-Sinai theorem.

In this contribution, the (Borel-) measure-theoretic entropy of a homeomorphism $T : X \to X$ on a compact metric space X will be shown (4.1) to coincide with the *Connes-Narnhofer-Thirring* (CNT) *entropy* ([4], see section 3) of the *-automorphism induced by T on the (separable) Abelian C^*-algebra $C(X)$ of (complex-valued) continuous functions on X, what can be used as equivalent *definition* (3.6) because of the Gelfand *-isomorphism from *any* (separable) Abelian C^*-algebra $\mathcal{A} \ni 1$ to some $C(X)$ as above. This CNT entropy of a *-automorphism acting on $\mathcal{A} \cong C(X)$ is defined (4.2) in terms of (unity-preserving) positive linear maps from finite-dimensional Abelian C^*-algebras into $\mathcal{A}$, corresponding to finite partitions of unity $1 \in C(X)$ into positive continuous functions on X; and each positive continuous partition of $1 \in C(X)$ in turn (uniquely) determines an open cover of X. This correspondence will be used (4.3) for equivalently (4.5) redefining also the *topological* entropy of $T : X \to X$ in terms of these positive linear maps from finite-dimensional Abelian algebras into $\mathcal{A} \cong C(X)$ (with the *-automorphism induced by T), such that the variational principle (2.6) can be reformulated in purely *algebraically defined* terms for (separable) Abelian C^*-algebras $\mathcal{A}$. Furthermore, this "algebraic variational principle" (4.6) does *not* involve any more a comparison of *different* objects as arguments of the two different entropy functionals, compared for a fixed *-automorphism of $\mathcal{A}$. In spite of this *conceptual* simplification, however, the *direct* proof of the algebraic variational principle (4.6) has been hindered up to now[1] by the *technical* difficulties encountered in explicitly *calculating* the CNT-entropy for a given finite partition of unity $1 \in \mathcal{A}$ into positive elements from $\mathcal{A} \cong C(X)$. A first step in this direction will be made as an "appendix" to the present contribution, but the further steps have to be left to future investigations.

2 The Topological Variational Principle

Throughout this contribution (as in [5]), by a *topological dynamical system* (t.d.s.) (X,T) we shall understand a compact *metric* space X together with a *homeomorphism* $T : X \to X$. By definition, any open cover of X possesses a *finite* subcover; and we can restrict ourselves to the latter from the very beginning (cf. [5], p. 83), denoting by $\mathcal{O}(X)$ or simply $\mathcal{O}$ the set of finite open covers of X. Using the notation of [23], we recall the original definition [1] of topological entropy:

Definition (2.1): Let (X,T) be a t.d.s. We define the *join* of $\alpha, \beta \in \mathcal{O}(X)$ by $\alpha \vee \beta = \{A \cap B | A \in \alpha, B \in \beta\}$ and the *action* of T on $\alpha \in \mathcal{O}$ by $T^{-1}\alpha = \{T^{-1}A | A \in \alpha\}$, and we use the short notation $\alpha_T^n = \bigvee_{i=0}^{n-1} T^{-i}\alpha$ $(n \in N)$.

(i) We denote by $N(\alpha)$ the number of sets in a *minimal* subcover of $\alpha \in \mathcal{O}$ (with minimal cardinality). The *entropy* of $\alpha \in \mathcal{O}$ is defined by $H(\alpha) = \log N(\alpha)$.

(ii) The "entropy" of T w.r.t. $\alpha \in \mathcal{O}$ is defined by $h(T,\alpha) = \lim_{n\to\infty} \frac{1}{n} H(\alpha_T^n)$.

(iii) The *topological entropy* of T is defined by $h(T) = \sup_{\alpha\in\mathcal{O}} h(T,\alpha)$.

The basic properties of topological entropy are its *conjugacy invariance* $h(T) = h(\Phi \circ T \circ \Phi^{-1})$ for a t.d.s. (X,T) with a homeomorphism $\Phi : X \to X'$ (X' compact metric), and its *scaling* property $h(T^k) = |k| \cdot h(T)$ $\forall$ $k \in \mathbf{Z}$. - The analogous dependence on T is shown also by the Kolmogorov-Sinai [14] *measure-theoretic* entropy $h_\mu(T)$ of T as Borel measurable transformation, but in addition $h_\mu(T)$ depends on the T-invariant Borel probability measure $\mu = \mu \circ T^{-1}$ on X, chosen from the non-empty convex set $M(X,T)$ of all such measures. The original measure-theoretic definition (cf. [23]) of $h_\mu(T)$ is given in terms of the set $\mathcal{P}(X,\mu)$ of finite Borel partitions (μ-mod 0) of X (or the corresponding finite σ-algebras of Borel sets); and because of the obvious one-to-one correspondence (cf. the proof of (2.3) below) between $\mathcal{P}(X,\mu)$ and the set $\mathcal{F}(X,\mu)$ of all finite-dimensional *-subalgebras (with unity $\mathbf{1}$) of the Abelian W^*-algebra $\mathcal{M}_\mu \equiv L^\infty(X,\mu)$, we can equivalently reformulate the definition in *algebraic* terms (cf. [13,12]):

Definition (2.2): Let (X,T) be a t.d.s. and $\mu \in M(X,T)$. We define a *-*automorphism* τ_μ resp. a (faithful, normal) *state* ω_μ on $\mathcal{M}_\mu \equiv L^\infty(X,\mu)$ by $\tau_\mu(f) = f \circ T$, resp. $\omega_\mu(f) = \int_X f(x)d\mu(x)$ $\forall$ $f \in \mathcal{M}_\mu$; such that $\omega_\mu \circ \tau_\mu = \omega_\mu$. We denote by $\mathcal{F}$ the set of all finite-dimensional unital *-subalgebras of $\mathcal{M}_\mu$, and $\mathcal{A} \vee \mathcal{B}$ $(\in \mathcal{F})$ denotes the subalgebra *generated* by $\mathcal{A}, \mathcal{B} \in \mathcal{F}$.

[1] Deadline for the proceedings: November 15, 1989.

(i) For $\mathcal{A} \in \mathcal{F}$, $\dim \mathcal{A} = n \in \mathbf{N}$, with (uniquely determined, cf. [22], p. 50/51) minimal projectors $P_i \in \mathcal{A}$ $(i = 1, \ldots, n)$, the *entropy* of $\mathcal{A}$ is defined by

$$H_{\omega_\mu}(\mathcal{A}) = \sum_{i=1}^{n} \eta(\omega_\mu(P_i)),$$

where $\eta(x) \equiv -x \log x \;\forall\, x \in [0,1]$ with $\eta(0) \equiv 0$.

(ii) The "entropy" of τ_μ w.r.t. $\mathcal{A} \in \mathcal{F}$ is defined by

$$h_{\omega_\mu}(\tau_\mu, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} H_{\omega_\mu}(\mathcal{A} \vee \tau_\mu \mathcal{A} \vee \ldots \vee \tau_\mu^{n-1} \mathcal{A}).$$

(iii) The "Kolmogorov-Sinai" *entropy* of τ_μ is defined by

$$h_{\omega_\mu}(\tau_\mu) = \sup_{\mathcal{A} \in \mathcal{F}} h_{\omega_\mu}(\tau_\mu, \mathcal{A}).$$

Proposition (2.3): Let (X,T) be a t.d.s. and $\mu \in M(X,T)$, then the entropy (2.2,iii) coincides with the (original) *measure-theoretic* entropy $h_\mu(T)$ of T w.r.t. μ (cf. [23,5]): $h_{\omega_\mu}(\tau_\mu) = h_\mu(T)$.

Proof: Obvious from Def. (2.2) because of the one-to-one correspondence between $\xi \in \mathcal{P}(X,\mu)$ and $\mathcal{A}(\xi) \in \mathcal{F}(X,\mu)$: To $\xi = \{A_1, \ldots, A_n\}$ there corresponds a finite partition of unity $1 = \sum_{i=1}^n P_i$ in $\mathcal{M}_\mu \equiv L^\infty(X,\mu)$ given by the (μ-equivalence classes of) characteristic functions $\chi_{A_i} \equiv P_i$; and we denote by $\mathcal{A}(\xi) \subset \mathcal{M}_\mu$ the n-dimensional *-subalgebra generated by the P_i $(i = 1, \ldots, n)$, which in turn give the uniquely determined minimal projectors $P_i \in \mathcal{A}(\xi)$.

Definition (2.4): The *entropy map* of a t.d.s. (X,T) is the map $h_T : M(X,T) \to [0,\infty]$ defined by $h_T(\mu) = h_{\omega_\mu}(\tau_\mu) \;\forall\, \mu \in M(X,T)$, with Def. (2.2,iii) for the r.h.s.

Remarks (2.5):

1. Together with Prop. (2.3) above, this definition of the entropy map coincides with that of [23], simply expressing the dependence of $h_\mu(T)$ on $\mu \in M(X,T)$. "Strictly" speaking, however, the value of the entropy map (2.4) at $\mu \in M(X,T)$ has to be computed for the Abelian W^*-algebra $\mathcal{M}_\mu \equiv L^\infty(X,\mu)$ with *-automorphism τ_μ (both dependent on μ); and this "abstract nonsense" shows that the W^*-algebraic reformulation (2.2) is not well suited for defining an entropy map, the reason being that in general $M(X,T)$ is not contained in any dual $\mathcal{M}_\mu^*$ (cf. [22], p. 117) for $\mu \in M(X,T)$.

2. For the (Abelian) C^*-algebra $\mathcal{A} = C(X)$, however, $M(X,T)$ is identified with a weak-*-compact, convex (non-empty) subset of the unit ball in $\mathcal{A}^*$ (cf. [5,23], e.g.) by the bijection between the measure $\mu \in M(X,T)$ and the integral state $\omega_\mu \in \mathcal{A}^*$

as in (2.2); and the *-automorphism τ on $\mathcal{A}$, induced by T as in (2.2), is *independent* of μ. $\mathcal{A}$ would thus be taylor-made for defining an "algebraic" entropy map; but for a compact metric space X there will in general exist no finite-dimensional *-subalgebras of $\mathcal{A} = C(X)$ at all, such that the set $\mathcal{F}$ in Def. (2.2) and thus the definition itself will be empty.

3. Nevertheless, because of Prop. (2.3) also the entropy map h_T from (2.4) is *affine* on $M(X,T)$; and $h_T : M(X,T) \to [0,\infty)$ is weak-*-upper-semicontinuous for an ("asymptotically entropy-") expansive homeomorphism T (cf. [5,23]). We can also use h_T for finally recalling the *variational principle* for topological entropy:

Theorem (2.6): Let (X,T) be a t.d.s., then we have with Def. (2.1) resp. (2.4):

(i) $h(T) \geq h_T(\mu)\ \forall\, \mu \in M(X,T)$ (Goodwyn [10]), and even

(ii) $h(T) = \sup_{\mu \in M(X,T)} h_T(\mu)$ (Goodman [9]).

3 The Connes-Narnhofer-Thirring Entropy

This third section could also be read as a continuation of section 2 in the recent review [13], to which we refer for a motivation (cf. also [12]) of the following definition given by Connes, Narnhofer and Thirring [4]:

Definition (3.1): Let $\mathcal{A} \ni 1$ be a C^*-algebra with unity 1 and with a state ω on $\mathcal{A}$. We denote by $CP_1(\mathcal{A})$ or simply CP_1 the set of all *completely positive, unital* (c.p.u.) linear maps from finite-dimensional unital C^*-algebras into $\mathcal{A}$.

(i) For $\gamma_1, \ldots, \gamma_n \in CP_1(\mathcal{A})$ with $\gamma_k : A_k \to \mathcal{A}$ $(k = 1, \ldots, n)$, an *Abelian model* is given by a c.p.u. continuous linear map $P : \mathcal{A} \to \mathcal{B} \equiv \bigvee_{k=1}^{n} \mathcal{B}_k$ from $\mathcal{A}$ onto a finite-dimensional Abelian C^*-algebra $\mathcal{B} \ni 1$ with subalgebras $\mathcal{B}_k \ni 1$ generating $\mathcal{B}$, and with a state μ on $\mathcal{B}$ such that the dual P^* of P maps μ to $\omega = P^*(\mu) \equiv \mu \circ P$.

(ii) For an Abelian model as in (i), we denote by $Q^k_{i_k}$ $(i_k = 1, \ldots, \dim \mathcal{B}_k)$ the minimal projectors in $\mathcal{B}_k$ and by $E_k : \mathcal{B} \to \mathcal{B}_k$ the c.p.u. canonical *conditional expectation* onto $\mathcal{B}_k$ with restricted state $\mu_k \equiv \mu_{|\mathcal{B}_k}$ (such that $\mu_k \circ E_k = \mu$), and we define the c.p.u. "model map" $\rho_k : A_k \to \mathcal{B}_k$ by $\rho_k = E_k \circ P \circ \gamma_k$ $(k = 1, \ldots, n)$.

(iii) The *entropy of* $\gamma_1, \ldots, \gamma_n \in CP_1(\mathcal{A})$ is defined by

$$H_\omega(\gamma_1, \ldots, \gamma_n) = \sup_{\{\mu \circ P = \omega\}} [S(\mu) - \sum_{k=1}^{n} S(\mu_k) + \sum_{k=1}^{n} \sum_{i_k} \mu_k(Q^k_{i_k}) S(\mu_k \circ \rho_k | Q^k_{i_k} \circ \rho_k)],$$

where the supremum is taken over all Abelian models for $\gamma_1, \ldots, \gamma_n$. Here $S(\cdot)$ resp. $S(\cdot|\cdot)$ denotes the *entropy* resp. *relative entropy* functional of states on (Abelian, resp. non-Abelian) *finite*-dimensional (C^*-) algebras (cf. [16], also [12,13]); and in the second arguments of the relative entropy terms we have identified $\mathcal{B}_k^*$ with $\mathcal{B}_k$ defining the states $Q^k_{i_k}$ on $\mathcal{B}_k$ by $Q^k_{i_k}(Q^k_{j_k}) = \delta_{i_k j_k}$ $(i_k, j_k = 1, \ldots, \dim \mathcal{B}_k)$.

Remarks (3.2):

1. For the definition of completely positive maps resp. conditional expectations we refer e.g. to [22] (cf. also [4,12]). As remarked in [4], for any finite-dimensional C^*-algebra $\mathcal{A}_k$ there exists a matrix algebra $M_{d_k}(\mathbf{C})$ containing $\mathcal{A}_k$ as a subalgebra, together with a conditional expectation (i.e. norm-one projection) $\theta_k : M_{d_k} \to \mathcal{A}_k$; and because of the *invariance* of the *relative* entropy w.r.t. extending *both* states by θ_k (cf. [4], also [19]) we have $H_\omega(\gamma_1, \ldots, \gamma_n) = H_\omega(\gamma_1 \circ \theta_1, \ldots, \gamma_n \circ \theta_n)$ with c.p.u. maps $\gamma_k \circ \theta_k : M_{d_k} \to \mathcal{A}$. Thus we may assume $\mathcal{A}_k = M_{d_k}$. By the results of Choi and Effros (cf. [2,7]) there is a natural bijection between the set of c.p.u. maps $\gamma : M_d(\mathbf{C}) \to \mathcal{A}$ and the set $\{A = (a_{ij}) \in M_d(\mathcal{A})^+ | \sum_{i=1}^d a_{ii} = 1 \in \mathcal{A}\} \neq \emptyset$; while in general $\mathcal{A}$ does not contain any *finite*-dimensional *sub*algebras, see (2.5,2).

2. An Abelian model (i) for $n = 1$, given by a c.p.u. *continuous* linear map $P : \mathcal{A} \to \mathcal{B}$ with a state μ on $\mathcal{B}$ such that $\mu \circ P = \omega$ on $\mathcal{A}$, corresponds with the minimal projectors $Q_i \in \mathcal{B}$ $(i = 1, \ldots, d \equiv \dim \mathcal{B})$ to a *decomposition* $\omega = \sum_{i=1}^d \omega_i$ into positive linear (hence *continuous*) functionals ω_i on $\mathcal{A}$, uniquely determined by: $P(A) = \sum_{j=1}^d \omega_j(A/\omega_j(1))Q_j \ \forall A \in \mathcal{A}$, such that $\omega_i(1) = \mu(Q_i)$ for $i = 1, \ldots, d$. With $Q_i \circ P(A) = \omega_i(A)/\omega_i(1)$ we get an equivalent *definition* for the entropy (iii) of $\gamma_1 \in CP_1(\mathcal{A})$:

$$H_\omega(\gamma_1) = \sup_{\{\sum_i \omega_i = \omega\}} \left[\sum_{i=1}^d \omega_i(1) S(\omega \circ \gamma_1 | \frac{\omega_i \circ \gamma_1}{\omega_i(1)}) \right],$$

where the supremum is taken over all finite decompositions of ω into positive linear functionals ω_i on $\mathcal{A}$ (as above). In particular, for the c.p.u. *inclusion* $i_{\mathcal{A}_1}$ of a finite-dimensional unital *sub*-*-algebra $\mathcal{A}_1 \subset \mathcal{A}$, this expression for $H_\omega(i_{\mathcal{A}_1}) \equiv H_\omega(\mathcal{A}_1)$ coincides with that given by Narnhofer and Thirring [16] in a "side-remark"; and for $\gamma_1 \in CP_1(\mathcal{A})$ where $\gamma_1 : \mathcal{A}_1 \to \mathcal{A}$ with *Abelian* C^*-algebras $\mathcal{A}_1$ resp. $\mathcal{A}$, we shall explicitly *calculate* $H_\omega(\gamma_1)$ in the "appendix".

3. For $n > 1$ as in (i) and (ii), the minimal projectors in $\mathcal{B} \equiv \bigvee_{k=1}^n \mathcal{B}_k$ are given by $Q_{(i_1,\ldots,i_n)} = Q_{i_1}^1 \cdot \ldots \cdot Q_{i_n}^n$ for $(i_1, \ldots, i_n) \equiv I_n \in \mathbf{N}^n$, and the Abelian model corresponds to a uniquely determined *multi*-index decomposition $\omega = \sum_{I_n} \omega_{I_n}$ with $\omega_{I_n}(1) = \mu(Q_{I_n})$. We denote the n single-index partial sums by

$$\omega_{i_k}^k = \sum_{I_n, i_k \text{ fixed}} \omega_{I_n},$$

such that $\omega_{i_k}^k(1) = \mu_k(Q_{i_k}^k)$ and $Q_{i_k}^k \circ E_k \circ P(A) = \omega_{i_k}^k(A)/\omega_{i_k}^k(1) \equiv \hat{\omega}_{i_k}^k(A)$. For the finite-dimensional Abelian algebra $\mathcal{B}$, the definition of the entropy S as in (iii) coincides with Def. (2.2,i) by $S(\mu) = H_\mu(\mathcal{B})$ resp. $S(\mu_k) = H_{\mu_k}(\mathcal{B}_k)$, and we get for the entropy of $\gamma_1, \ldots, \gamma_n \in CP_1(\mathcal{A})$:

$$H_\omega(\gamma_1, \ldots, \gamma_n) = \sup_{\{\sum_{I_n} \omega_{I_n} = \omega\}} [\sum_{I_n} \eta(\omega_{I_n}(1)) - \sum_{k=1}^n \sum_{i_k} \eta(\omega_{i_k}^k(1)) +$$

$$+\sum_{k=1}^{n}\sum_{i_k}\omega_{i_k}^{k}(1)S(\omega\circ\gamma_k|\hat{\omega}_{i_k}^{k}\circ\gamma_k)],$$

where the supremum is taken over all finite multi-index decompositions of ω (as above). In particular, for the c.p.u. *inclusions* $i_{\mathcal{A}_k}$ of *sub*algebras $\mathcal{A}_k \subset \mathcal{A}$ ($k = 1,\ldots,n$), this expression for $H_\omega(i_{\mathcal{A}_1},\ldots,i_{\mathcal{A}_n}) \equiv H_\omega(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ coincides with that given by Connes [3]; and for an *Abelian* W^*-algebra $\mathcal{A}$ this expression again coincides with the classical expression (2.2,i): $H_\omega(\mathcal{A}_1,\ldots,\mathcal{A}_n) = H_\omega(\mathcal{A}_1 \vee \ldots \vee \mathcal{A}_n)$, cf. [4].

Definition (3.3): Let $\mathcal{A} \ni 1$ be a C^*-algebra with a *-automorphism $\theta \in \mathrm{Aut}(\mathcal{A})$ and with an invariant state $\omega = \omega \circ \theta$ on $\mathcal{A}$.

(i) The *entropy* of θ w.r.t. $\gamma \in CP_1(\mathcal{A})$ is defined by

$$h_\omega(\theta,\gamma) = \lim_{n\to\infty}\frac{1}{n}H_\omega(\gamma,\theta\circ\gamma,\ldots,\theta^{n-1}\circ\gamma)$$

with Def. (3.1,iii) for the r.h.s.

(ii) The *Connes-Narnhofer-Thirring* (CNT) *entropy* of θ is defined by

$$h_\omega(\theta) = \sup_{\gamma\in CP_1} h_\omega(\theta,\gamma).$$

By (3.2,1) it is sufficient to take the supremum over all c.p.u. maps $\gamma : M_d(\mathbf{C}) \to \mathcal{A}$ $\forall\, d \in \mathbf{N}$.

Again, the basic property of the CNT-entropy is its conjugacy invariance $h_\omega(\theta) = h_{\omega\circ\sigma}(\sigma^{-1}\circ\theta\circ\sigma)$ for $\sigma \in \mathrm{Aut}(\mathcal{A})$; but the most *important* property is the following analogue of a *weak* version of the "Kolmogorov-Sinai" theorem for the measure-theoretic entropy (cf. [23], p. 99 and [5], p. 65):

Theorem (3.4): Let $\mathcal{A} \ni 1$ be a *nuclear* C^*-algebra, i.e. the *C^*-algebraic tensor product* $\mathcal{A}\otimes\mathcal{B}$ with any C^*-algebra $\mathcal{B}$ is uniquely determined (cf. [7,22]).

(i) $\mathcal{A}$ has the following "approximation property": There exist c.p.u. continuous linear maps $\sigma_n : \mathcal{A} \to \mathcal{A}_n$, $\tau_n : \mathcal{A}_n \to \mathcal{A}$ with finite-dimensional C^*-algebras $\mathcal{A}_n$ ($\forall n \in \mathrm{N}$) such that $\lim_{n\to\infty}\|\tau_n\circ\sigma_n(A) - A\| = 0$, $\forall\, A \in \mathcal{A}$ (Choi and Effros [2], cf. also [7]).

(ii) The CNT entropy of $\theta \in \mathrm{Aut}(\mathcal{A})$ with invariant state $\omega = \omega\circ\theta$ is given by $h_\omega(\theta) = \lim_{n\to\infty} h_\omega(\theta,\tau_n)$, with $\tau_n \in CP_1(\mathcal{A})$ as in (i); [4].

As a corollary of (ii), again the *scaling* property $h_\omega(\theta^k) = |k|\cdot h_\omega(\vartheta)\ \forall\, k \in \mathbf{Z}$ of the CNT entropy (originally proved for *von Neumann* algebras $\mathcal{A}$ in [4]) was proved also for *nuclear* C^*-algebras $\mathcal{A}$ in [11], which was no surprise because of the following result of [4]:

Theorem (3.5): Let $\mathcal{A} \ni 1$ be a nuclear C^*-algebra with $\theta \in \mathrm{Aut}(\mathcal{A})$ and with state $\omega = \omega \circ \theta$ on $\mathcal{A}$, and we denote by $\bar{\theta}$ resp. $\bar{\omega}$ the natural extensions of θ resp. ω to the von Neumann algebra $\mathcal{M}_\omega = \pi_\omega(\mathcal{A})''$ generated by the GNS-representation $\pi_\omega(\mathcal{A})$ of $\mathcal{A}$ constructed with ω (cf. [22], e.g.). Then the CNT entropy of $\bar{\theta} \in \mathrm{Aut}(\mathcal{M}_\omega)$ with invariant state $\bar{\omega} = \bar{\omega} \circ \bar{\theta}$ coincides with the original one: $h_{\bar{\omega}}(\bar{\theta}) = h_\omega(\theta)$.

In spite of this "representation-independence", the CNT entropy $h_\omega(\theta)$ of course *depends* on the θ-invariant state ω chosen on $\mathcal{A}$:

Definition (3.6): Let $\mathcal{A} \ni 1$ be a nuclear C^*-algebra with $\theta \in \mathrm{Aut}(\mathcal{A})$, and we denote by $E^\theta_\mathcal{A} \subset \mathcal{A}^*$ the weak-*-compact, convex (nonempty) set of all θ-invariant states $\omega = \omega\circ\theta$ on $\mathcal{A}$. The *entropy map* of the C^*-dynamical system $(\mathcal{A}, \theta)$ is the map $\bar{h}_\theta : E^\theta_\mathcal{A} \to [0,\infty]$ defined by $\bar{h}_\theta(\omega) = h_\omega(\theta)\ \forall\, \omega \in E^\theta_\mathcal{A}$ with Def. (3.3,ii) for the r.h.s.

Remarks (3.7):

1. It should be remarked that in (3.5) the von Neumann algebra $\mathcal{M}_\omega$ is *not* nuclear as a C^*-algebra in general; e.g. $\mathcal{B}(\mathcal{H})$ for $\dim\mathcal{H} = \infty$ does not have the approximation property (3.4,i). By the results of Choi and Effros (cf. [2,7]), the *nuclear* C^*-algebras $\mathcal{A}$ are exactly those for which the bidual $\mathcal{A}^{**}$ (or equivalently, the envelopping von Neumann algebra in the universal representation of $\mathcal{A}$, cf. [22]) and thus also $\mathcal{M}_\omega \equiv \pi_\omega(\mathcal{A})''\ \forall\, \omega \in E^\theta_\mathcal{A},\ \theta \in \mathrm{Aut}(\mathcal{A})$, is *injective*. But as remarked in [4], for nuclear $\mathcal{A}$ each $\mathcal{M}_\omega$ is even generated by an ascending *sequence* of finite-dimensional (von Neumann) subalgebras $\mathcal{M}_n \subset \mathcal{M}_{n+1} \subset \mathcal{M}_\omega$ ($\forall n \in \mathbb{N}$) with weakly dense union $(\bigcup_{n=1}^\infty \mathcal{M}_n)'' = \mathcal{M}_\omega$, although this is not the case for a *general* injective von Neumann algebra (such as $\mathcal{A}^{**}$, cf. [8]); and the CNT entropy of $\bar{\theta} \in \mathrm{Aut}(\mathcal{M}_\omega)$ with invariant state $\bar{\omega}$ as in (3.5) is given by $h_{\bar{\omega}}(\bar{\theta}) = \lim_{n\to\infty} h_{\bar{\omega}}(\bar{\theta}, \mathcal{M}_n)$ with Def. (3.3,i) for the r.h.s. [3,4].

2. The definition (3.6) of the entropy map seems even more "tautological" than Def. (2.4), showing that the C^*-dynamical system $(\mathcal{A}, \theta)$ is "taylor-made" for defining the map $\bar{h}_\theta$, which has been generally proved in [4] to be *concave* on $E^\theta_\mathcal{A}$ and is *affine* at least for all *asymptotically Abelian* automorphisms θ (cf. [18]). In [17] it was shown that the map $E^\theta_\mathcal{A} \ni \omega \mapsto h_\omega(\theta, \gamma) \in [0,\infty]$, with Def. (3.3,i) for fixed $\gamma \in CP_1(\mathcal{A})$, is weak-*-uppersemicontinuous; but because of the *supremum* in Def. (3.3,ii) nothing is known as yet about the possible weak-*-uppersemicontinuity of the entropy map $\bar{h}_\theta : E^\theta_\mathcal{A} \to [0,\infty)$ for some $\theta \in \mathrm{Aut}(\mathcal{A})$ of a general nuclear C^*-algebra $\mathcal{A}$. A sufficient condition would be the existence of a *generator* $\gamma \in CP_1(\mathcal{A})$ such that $h_\omega(\theta,\gamma) = h_\omega(\theta)\ \forall\, \omega \in E^\theta_\mathcal{A}$ (cf. [5], pp. 107, 160; cf. also [15]).

4 The Algebraic Variational Principle

Let (X,T) be a t.d.s. (as in section 2), then the Abelian C^*-algebra $\mathcal{A} = C(X)$ is *separable* (cf. e.g. [23], p. 17), and the Gelfand space (or *spectrum*) of $\mathcal{A} \ni 1$ is again homeomorphic to the compact metric space X (cf. [22], p. 18). Thus the *-automorphism

τ of $\mathcal{A}$ induced by T (as in (2.5,2)) uniquely determines T again, and the set $M(X,T)$ is identified by (2.5,2) with the set $E^{\tau}_{\mathcal{A}} \subset \mathcal{A}^*$ as in (3.6). We call $(\mathcal{A},\tau)$ the C^*-dynamical system *associated* to the t.d.s. (X,T) by this one-to-one correspondence. Note that the entropy map (3.6) of $(\mathcal{A},\tau)$ is well-defined, since *any* Abelian C^*-algebra $\mathcal{B}$ is *nuclear* (as defined in (3.4); cf. [22], p. 215).

Proposition (4.1): Let (X,T) be a t.d.s. with associated C^*-dynamical system $(\mathcal{A},\tau)$, then we have for the entropy maps (2.4) of (X,T) resp. (3.6) of $(\mathcal{A},\tau)$: $h_T(\mu) = \bar{h}_\tau(\omega_\mu)$ $\forall\ \mu \in M(X,T)$, with the induced state $\omega_\mu \in E^{\tau}_{\mathcal{A}}$ as in (2.5,2).

Proof: Let $\mu \in M(X,T)$ with W^*-algebra $\mathcal{M}_\mu \equiv L^\infty(X,\mu)$ from (2.2), and we denote by $\bar{\tau}_\mu$ resp. $\bar{\omega}_\mu$ the *-automorphism resp. state induced on $\mathcal{M}_\mu$ by T resp. μ as in (2.2). By Def. (2.4), we have for the l.h.s. above $h_T(\mu) = h_{\bar{\omega}_\mu}(\bar{\tau}_\mu)$ with Def. (2.2,iii) for the r.h.s. – The GNS representation $\pi_{\omega_\mu}(\mathcal{A})$ of $\mathcal{A}$, constructed with ω_μ as above, is realized as the multiplication representation of $C(X)$ on the Hilbert space $\mathcal{H}_\mu \equiv L^2(X,\mu)$, and $\mathcal{M}_\mu = \pi_{\omega_\mu}(\mathcal{A})''$ acts by multiplication as maximally Abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_\mu)$ (cf. [22], p. 103). Furthermore, the maps $\bar{\tau}_\mu$ resp. $\bar{\omega}_\mu$ induced on $\mathcal{M}_\mu = \pi_{\omega_\mu}(\mathcal{A})''$ are the natural extensions of the maps τ resp. ω_μ induced on $\mathcal{A}$ (by T resp. μ). By (3.7,1) together with (3.2,3), $h_{\bar{\omega}_\mu}(\bar{\tau}_\mu)$ from (2.2,iii) coincides with the CNT entropy (3.3,ii) for the Abelian von Neumann algebra $\mathcal{M}_\mu$, and for the latter (3.5) gives $h_{\bar{\omega}_\mu}(\bar{\tau}_\mu) = h_{\omega_\mu}(\tau) \equiv \bar{h}_\tau(\omega_\mu)$, with Def. (3.6).

Proposition (4.2): Let $\mathcal{A} \ni 1$ be an Abelian C^*-algebra with $\theta \in \mathrm{Aut}(\mathcal{A})$ and $\omega \in E^{\theta}_{\mathcal{A}}$. We denote by $P_1(\mathcal{A})$ or simply P_1 the set of all *positive* unital linear maps from finite-dimensional unital *Abelian* C^*-algebras into $\mathcal{A}$. Then the CNT entropy (3.3,ii) is given by $h_\omega(\theta) = \sup_{\gamma\in P_1} h_\omega(\theta,\gamma)$, with Def. (3.3,i) for the r.h.s.

Proof: Since $\mathcal{A}$ is Abelian, $CP_1(\mathcal{A})$ is the set of all *positive* unital linear maps from finite-dimensional unital C^*-algebras into $\mathcal{A}$ (cf. e.g. [22], p. 194). But $\mathcal{A}$ is always nuclear (as mentioned above), and in the approximation property (3.4,i) the finite-dimensional C^*-algebras $\mathcal{A}_n$ can be chosen to be *Abelian* as $\mathcal{A}$ is (cf. [2,7]; in fact, the approximating maps $\tau_n \in P_1(\mathcal{A})$ and $\sigma_n : \mathcal{A} \to \mathcal{A}_n$ can be easily constructed explicitly), such that by (3.4,ii) we have $h_\omega(\theta) = \lim_{n\to\infty} h_\omega(\theta,\tau_n)$ with $\tau_n \in P_1(\mathcal{A})$. By the same argument as in the proof of Prop. (2.2) in [11] we can conclude $\lim_{n\to\infty} h_\omega(\theta,\tau_n) = \sup_{\gamma\in P_1} h_\omega(\theta,\gamma)$.

Using the notation of [22, p. 4], we denote by $G(\mathcal{A})^+$ the set of all *invertible* positive elements A in the C^*-algebra $\mathcal{A} \ni 1$, or equivalently the set of all *strictly positive* elements $A \in \mathcal{A}^+$ with $\varphi(A) > 0$ for any nonzero positive linear functional $\varphi \in \mathcal{A}^*$ (cf. [22], p. 31). For the *Abelian* C^*-algebra $\mathcal{A} = C(X)$ as above this is equivalent to $A \in \mathcal{A}^+$ being a *strictly* positive continuous function on the compact space X; and in this case we denote by $P(\mathcal{A})$ the set of all positive linear maps γ from finite-dimensional *Abelian* C^*-algebras (with unity 1) into $\mathcal{A}$ with $\gamma(1) \in G(\mathcal{A})$, such that $P(\mathcal{A}) \supset P_1(\mathcal{A})$ as in (4.2).

Definition (4.3): Let $\mathcal{A} \ni 1$ be an Abelian C^*-algebra. For $\gamma_1, \gamma_2 \in P(\mathcal{A})$ with $\gamma_k : \mathcal{A}_k \to \mathcal{A}$, we define the *join* $\gamma_1 \vee \gamma_2 : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}$ by the (linear) extension of $(\gamma_1 \vee \gamma_2)(A_1 \otimes A_2) = \gamma_1(A_1) \cdot \gamma_2(A_2)\ \forall\ A_k \in \mathcal{A}_k\ (k = 1,2)$; such that $\gamma_1 \vee \gamma_2 \in P(\mathcal{A})$, resp. $\gamma_1 \vee \gamma_2 \in P_1(\mathcal{A})$ for $\gamma_1, \gamma_2 \in P_1(\mathcal{A})$. For $\gamma \in P_1(\mathcal{A})$ with $\gamma : \mathcal{B} \to \mathcal{A}$, we call $\gamma' \in P(\mathcal{A})$ *submap* of γ if $\gamma' : \mathcal{B}' \to \mathcal{A}$ with a unital subalgebra $\mathcal{B}' \subset \mathcal{B}$ generated (as linear span) by a *subset* of the minimal projectors in $\mathcal{B}$ (with identity $1' \in \mathcal{B}'$ *different* from $1 \in \mathcal{B}$ in general), such that $\gamma' = \gamma_{|\mathcal{B}'}$ (but still $\gamma'(1') \in G(\mathcal{A})$, by assumption).

(i) A submap $\gamma' : \mathcal{B}' \to \mathcal{A}$ of $\gamma \in P_1(\mathcal{A})$ is called *minimal*, if the (linear) dimension $\dim \mathcal{B}'$ is minimal in the set of all submaps of γ. We denote by $N(\gamma) = \dim \mathcal{B}'$ this dimension for a minimal submap $\gamma' : \mathcal{B}' \to \mathcal{A}$ of $\gamma \in P_1(\mathcal{A})$. The *entropy* of $\gamma \in P_1(\mathcal{A})$ is defined by $H(\gamma) = \log N(\gamma)$.

(ii) The "entropy" of $\tau \in \mathrm{Aut}(\mathcal{A})$ w.r.t. $\gamma \in P_1(\mathcal{A})$ is defined by

$$\bar{h}(\tau, \gamma) = \lim_{n \to \infty} \frac{1}{n} H(\gamma \vee (\tau \circ \gamma) \vee \ldots \vee (\tau^{n-1} \circ \gamma)).$$

The limit exists because of Lemma (4.4) below.

(iii) The "topological" *entropy* of $\tau \in \mathrm{Aut}(\mathcal{A})$ is defined by $\bar{h}(\tau) = \sup_{\gamma \in P_1} \bar{h}(\tau, \gamma)$.

Lemma (4.4): Let $\mathcal{A} \ni 1$ be an Abelian C^*-algebra, then the entropy functional (4.3,i) on $P_1(\mathcal{A})$ has the properties of subadditivity (i) and invariance (ii):

(i) $0 \le H(\gamma_1 \vee \gamma_2) \le H(\gamma_1) + H(\gamma_2)\ \forall\ \gamma_1, \gamma_2 \in P_1(\mathcal{A})$,

(ii) $H(\tau \circ \gamma) = H(\gamma)\ \forall\ \tau \in \mathrm{Aut}(\mathcal{A}), \gamma \in P_1(\mathcal{A})$.

Proof:

(i) Let γ_1' resp. γ_2' be minimal submaps of γ_1 resp. γ_2, then $\gamma_1' \vee \gamma_2'$ is a submap of $\gamma_1 \vee \gamma_2$, i.e. $N(\gamma_1 \vee \gamma_2) \le N(\gamma_1) \cdot N(\gamma_2)$.

(ii) For any C^*-algebra $\mathcal{A} \ni 1$, we have $\tau(G(\mathcal{A})) = G(\mathcal{A})\ \forall\ \tau \in \mathrm{Aut}(\mathcal{A})$.

Theorem (4.5): Let (X, T) be a t.d.s. with associated C^*-dynamical system $(\mathcal{A}, \tau)$, then we have for the (topological) entropies (2.1,iii) of (X, T) resp. (4.3,iii) of $(\mathcal{A}, \tau)$: $h(T) = \bar{h}(\tau)$.

Proof: We prove (i) $h(T) \ge \bar{h}(\tau)$ and (ii) $h(T) \le \bar{h}(\tau)$.

(i) Let $\gamma \in P_1(\mathcal{A})$ with $\gamma : \mathcal{B} \to \mathcal{A}$, then γ corresponds to a finite partition of unity $1 = \sum_i f_i$ in $C(X)$ with $f_i(x) \ge 0\ (\forall\, x \in X)$, given by $\gamma(Q_i) \equiv f_i$ with the minimal projectors $Q_i \in \mathcal{B}\ (i = 1, \ldots, \dim \mathcal{B})$; and we denote by $\alpha(\gamma) \in \mathcal{O}(X)$ the finite open cover of X uniquely determined by $\alpha(\gamma) = \{A_i | i = 1, \ldots, \dim \mathcal{B}\}$, where A_i is given by the open interior of the closed $\mathrm{supp}(f_i) \subseteq X$ (cf. also the proof of (2.3)!) Then obviously by Def. (4.3,i) for a *minimal submap* γ' of $\gamma \in P_1(\mathcal{A})$, with $\gamma'(1') \in G(\mathcal{A})$, $\alpha(\gamma')$ defined as above is a *minimal subcover* of $\alpha(\gamma)$ as in Def. (2.1,i). Furthermore, by Def. (2.5,2) resp. (2.2) of the induced automorphism τ together with Def. (2.1) of the *action* by T on $\mathcal{O}(X)$, we have $\alpha(\tau \circ \gamma) = T^{-1}\alpha(\gamma)$; and by Def. (4.3) resp. (2.1) of the *join* in $P(\mathcal{A})$ resp. $\mathcal{O}(X)$ we have:

$$\alpha(\bigvee_{k=0}^{n-1} \tau^k \circ \gamma) = \bigvee_{k=0}^{n-1} \alpha(\tau^k \circ \gamma) = \bigvee_{k=0}^{n-1} T^{-k}\alpha(\gamma) \equiv \alpha_T^n(\gamma) \quad \forall n \in \mathbf{N}.$$

Together we see from Def. (4.3) resp. (2.1): $\forall\, \gamma \in P_1(\mathcal{A})\ \exists\ \alpha(\gamma) \in \mathcal{O}(X)$ with $\bar{h}(\tau,\gamma) = h(T,\alpha(\gamma))$; i.e. $\bar{h}(\tau) \leq h(T)$.

(ii) Conversely, let $\beta \in \mathcal{O}(X)$ be a finite open cover $\beta = \{A_1, \ldots, A_n\}$, then there exists a finite partition of unity $1 = \sum_{i=1}^n f_i$ in $C(X)$ with $\mathrm{supp}(f_i) = \bar{A}_i$ $(i = 1, \ldots, n)$; and we define a map $\gamma_\beta \in P_1(\mathcal{A})$ with $\gamma_\beta : \mathbf{C}^n \to \mathcal{A} = C(X)$ by linear extension of $\gamma_\beta(e_i) = f_i$, where e_i denote the minimal projectors in the C^*-algebra $\mathbf{C}^n$ (with component-wise multiplication) given by $(e_i)_j = \delta_{ij}$ $(i,j = 1, \ldots, n)$. Then together with the map $\alpha : P(\mathcal{A}) \to \mathcal{O}(X)$ from (i) we have $\alpha(\gamma_\beta) = \beta$; and by (i) we can conclude: $\forall\, \beta \in \mathcal{O}\ \exists\ \gamma_\beta \in P_1$ with $\bar{h}(\tau, \gamma_\beta) = h(T, \beta)$; i.e. $\bar{h}(\tau) \geq h(T)$.

Corollary (4.6): Let (X,T) be a t.d.s. with associated C^*-dynamical system $(\mathcal{A},\tau)$, then we have for the latter with Def. (4.3) resp. (3.6) the *algebraic variational principle*:

(i) $\bar{h}(\tau) \geq \bar{h}_\tau(\omega)\ \forall\, \omega \in E_{\mathcal{A}}^\tau$, and even

(ii) $\bar{h}(\tau) = \sup_{\omega \in E_{\mathcal{A}}^\tau} \bar{h}_\tau(\omega)$.

Proof: Obvious from (2.6) together with (4.1) and (4.5).

By definition (4.3), the l.h.s. is given by

$$\bar{h}(\tau) = \sup_{\gamma \in P_1} \lim_{n \to \infty} \frac{1}{n} H(\gamma \vee (\tau \circ \gamma) \vee \ldots \vee (\tau^{n-1} \circ \gamma));$$

whereas the r.h.s. of (i) is defined by (3.6) and (3.3), and together with (4.2) is given by

$$\bar{h}_\tau(\omega) = \sup_{\gamma \in P_1} \lim_{n \to \infty} \frac{1}{n} H_\omega(\gamma, \tau \circ \gamma, \ldots, \tau^{n-1} \circ \gamma)$$

with Def. (3.1,iii) resp. (3.2,3) for the r.h.s., which could thus be compared with Def. (4.3,i) for *fixed* $\gamma \in P_1(\mathcal{A})$ as argument of *both* entropy functionals. This could "in principle" lead to a *direct* proof at least of (i), as also in [20] (as mentioned in the introduction; cf. the appendix). Actually, also (4.2) still involves a comparison of the set $CP_1(\mathcal{A})$ of *all positive* unital linear maps from finite-dimensional unital C^*-algebras into $\mathcal{A}$ (as used in Def. (3.3,ii) for Abelian $\mathcal{A}$) with the "generating" subset $P_1(\mathcal{A})$, by means of the generalized Kolmogorov-Sinai theorem (3.4,ii). But we could use Def. (4.3) of the *join*

$\gamma_1 \vee \gamma_2$ also for $\gamma_1, \gamma_2 \in CP_1(\mathcal{A})$ without change (cf. [22], p. 218), such that we could equivalently define also the "topological" entropy $\bar{h}(\tau)$ as a supremum over $\gamma \in CP_1(\mathcal{A})$ in (4.3,ii), and thus we would not need (4.2) for a *direct* comparison of $\bar{h}(\tau)$ with $\bar{h}_\tau(\omega)$. – To extend Def. (4.3) not only to maps $\gamma \in CP_1(\mathcal{A})$, but also for a *non-Abelian* C^*-algebra $\mathcal{A}$ with $\tau \in \mathrm{Aut}(\mathcal{A})$ (cf. also [19]), remains a further challenge for the future.

Appendix

Let X be a compact metric space with Borel measure μ, and let $\mathcal{A} \equiv C(X)$ be the associated Abelian (separable) C^*-algebra with induced state ω on $\mathcal{A} \ni \mathbf{1}$ given by $\omega(f) = \int_X f(x)d\mu(x)\ \forall\ f \in \mathcal{A}$. As in (4.2), we denote by $P_1(\mathcal{A})$ the set of all *positive* unital linear maps from finite-dimensional unital *Abelian* C^*-algebras into $\mathcal{A}$.

Theorem: Let $\gamma_1 \in P_1(\mathcal{A})$ with $\gamma_1 : \mathcal{A}_1 \to \mathcal{A}$, and we denote the images of the minimal projectors $e_i \in \mathcal{A}_1$ by $\gamma_1(e_i) \equiv f_i \in \mathcal{A}^+$ ($i = 1, \ldots, d_1 \equiv \dim \mathcal{A}_1$), such that $\sum_{i=1}^{d_1} f_i = \mathbf{1} \in \mathcal{A}$. Then the entropy (3.2,2) of $\gamma_1 \in P_1$ is given by the following equivalent expressions:

$$H_\omega(\gamma_1) = S(\omega \circ \gamma_1) + \sum_{j=1}^{d_1} \int_X f_j(x) \log f_j(x) d\mu(x) = \sum_{j=1}^{d_1} \int_X f_j(x) \log \frac{f_j(x)}{\omega(f_j)} d\mu(x),$$

where $S(\omega \circ \gamma_1)$ denotes the entropy as in (3.1,iii).

Proof: We choose a refining sequence $\{\xi_k\}$ of finite measurable partitions $\xi_k \in \mathcal{P}(X, \mu)$ of X into 2^k sets $\xi_k = \{A^k_{i_k} | i_k = 1, \ldots, 2^k\}$ of *equal* measure $\mu(A^k_{i_k}) = 2^{-k}$ and such that $A^k_{i_k} = A^{(k+1)}_{2i_k} \cup A^{(k+1)}_{(2i_k-1)}\ \forall\ k \in \mathbf{N}$. We define a sequence of decompositions $\omega = \sum_{i_k} \omega^k_{i_k}$ into 2^k positive linear functionals $\{\omega^k_{i_k} | i_k = 1, \ldots, 2^k\}$ on $\mathcal{A}$ by $\omega^k_{i_k}(f) = \int_{A^k_{i_k}} f(x)d\mu(x)$ $\forall\ f \in \mathcal{A}$; such that $\omega^k_{i_k}(\mathbf{1}) = 2^{-k}$ and $\omega^k_{i_k} = \omega^{(k+1)}_{2i_k} + \omega^{(k+1)}_{(2i_k-1)}\ \forall\ k \in \mathbf{N}$. Using these decompositions in Def. (3.2,2) for $H_\omega(\gamma_1)$, we have for the second arguments of the relative entropy terms ($\forall\ k \in \mathbf{N}$):

$$\frac{\omega^k_{i_k} \circ \gamma_1}{\omega^k_{i_k}(\mathbf{1})} = 2^k \cdot \omega^k_{i_k} \circ \gamma_1 = \frac{1}{2}(\hat{\omega}^{(k+1)}_{2i_k} + \hat{\omega}^{(k+1)}_{(2i_k-1)}) \circ \gamma_1,$$

where we denoted the normalized states as in (3.2,3). This gives by the *strict convexity* of the relative entropy in the second argument (cf. [4]):

$$\sum_{i_k} \omega^k_{i_k}(\mathbf{1}) \cdot S(\omega \circ \gamma_1 | \frac{\omega^k_{i_k} \cdot \gamma_1}{\omega^k_{i_k}(\mathbf{1})}) < \sum_{i_{(k+1)}} \omega^{(k+1)}_{i_{(k+1)}}(\mathbf{1}) \cdot S(\omega \circ \gamma_1 | \hat{\omega}^{(k+1)}_{i_{(k+1)}} \circ \gamma_1),$$

such that the supremum as in (3.2,2) taken only over *this* sequence of decompositions equals the *limit* of these expressions as $k \to \infty$. But an *arbitrary* decomposition $\omega = \sum_j \omega_j$

is given by $\omega_j(f) = \omega(p_j \cdot f)$ with $p_j \in \mathcal{A}^+$, $\sum_j p_j = 1 \in \mathcal{A}$, and can be refined to $\omega = \sum_{i_k,j} \omega^k_{i_k j}$ with $\omega^k_{i_k j}(f) \equiv \omega^k_{i_k}(p_j \cdot f)\ \forall\ f \in \mathcal{A}_j$, such that $\forall\ \varepsilon > 0\ \exists\ N(\varepsilon)$ and $\forall\ k > N(\varepsilon)\ \exists\ \lambda^k_{i_k j} > 0$, $\sum_j \lambda^k_{i_k j} = 1$, with $\|\omega^k_{i_k j} - \lambda^k_{i_k j} \cdot \omega^k_{i_k}\| < \varepsilon\ \forall\ i_k = 1, \ldots, 2^k; \forall\ j$. Again by the strict convexity of the relative entropy, the decomposition $\{\omega^k_{i_k j}\}$ is better than $\{\omega_j\}$ for $H_\omega(\gamma_1)$ and becomes best as $k \to \infty$. As in the proof [18] of the strong continuity of $\omega \mapsto H_\omega(\gamma_1)$, we can replace $\{\omega^k_{i_k j}\}$ by $\{\lambda^k_{i_k j} \cdot \omega^k_{i_k}\}$ as $k \to \infty$; but because of the *strict* convexity of the relative entropy, $\{\lambda^k_{i_k j} \cdot \omega^k_{i_k}\}$ gives the same value for the above expression as $\{\omega^k_{i_k}\}$, and thus the limit $k \to \infty$ for this latter sequence equals already the supremum over *all* decompositions. – With the explicit expression for the relative entropy terms of states on the *Abelian* algebra $\mathcal{A}_1$ (cf. [16], also [12]) we get:

$$H_\omega(\gamma_1) = S(\omega \circ \gamma_1) + \sum_{j=1}^{d_1} \lim_{k \to \infty} \sum_{i_k} \left[\frac{\omega^k_{i_k}(f_j)}{\omega^k_{i_k}(1)} \log \frac{\omega^k_{i_k}(f_j)}{\omega_{i_k}(1)} \right] \omega^k_{i_k}(1),$$

and the first expression for $H_\omega(\gamma_1)$ results from standard integration theory. The second expression is then obvious from the explicit form of $S(\omega \circ \gamma_1)$, cf. (3.2,3).

Remark: Unfortunately, this theorem cannot be extended for n maps $\gamma_1, \ldots, \gamma_n \in P_1(\mathcal{A})$ with $\gamma_k : \mathcal{A}_k \to \mathcal{A}$, for *two* (related) reasons: First, the two additional entropy sums in the expression (3.2,3) for $H_\omega(\gamma_1, \ldots, \gamma_n)$ *spoil* the joint convexity of the *relative* entropy terms. Those two sums cancel each other only for a multi-index decomposition $\omega = \sum_{I_n} \omega_{I_n}$ (with $I_n \equiv (i_1, \ldots, i_n)$) given by a multi-index *equipartition* $\xi = \{A_{(i_1,\ldots,i_n)}\}$ of X (as in the proof above) with $i_k = 1, \ldots, d$ ($\forall\, k = 1, \ldots, n$), such that $\omega_{I_n}(1) = d^{-n}$ resp. $\omega^k_{i_k}(1) = d^{-1}$ for the partial sums $\omega^k_{i_k}$ (*different* from $\omega^k_{i_k}$ in the proof above!) and thus $\omega_{I_n}(1) = \prod_{k=1}^n \omega^k_{i_k}(1)$. But still then, and secondly, these partial sums $\omega^k_{i_k}$ in the remaining relative entropy terms, integrating the functions $\gamma_k(e^k_{j_k}) \equiv f^k_{j_k} \in C(X)$ on the *disconnected* sets $\bigcup_{I_n, i_k \text{ fixed}} A_{I_n}$ (with the minimal projectors $e^k_{j_k} \in \mathcal{A}_k$; $k = 1, \ldots, n$), pick up "non-local" contributions from metrically *separated* regions of X for arbitrarily large d. – Instead of an explicit expression for $H_\omega(\gamma_1, \ldots, \gamma_n)$, we can only give the following *upper bound* (cf. [4,12]):

$$H_\omega(\gamma_1, \ldots, \gamma_n) \le H_\omega(\bigvee_{k=1}^n \gamma_k) = S(\omega \circ \bigvee_{k=1}^n \gamma_k) + \sum_{k=1}^n \sum_{j_k} \int_X f^k_{j_k}(x) \log f^k_{j_k}(x) d\mu(x)$$

with the *join* (4.3) in $P_1(\mathcal{A})$, and by the above theorem for the r.h.s.

References

[1] Adler, R.L., Konheim, A.G. and McAndrew, M.H., Trans. Amer. Math. Soc. 114 (1965), 309-319.

[2] Choi, M.-D. and Effros, E.G., Amer. J. Math. 100 (1978), 61-79.

[3] Connes, A., C.R. Acad. Sci. Paris 301 I (1985), 1-6.

[4] Connes, A., Narnhofer, H. and Thirring, W., Comm. Math. Phys. 112 (1987), 691-719; Connes, A., Narnhofer, H. and Thirring, W., in H. Mitter and L. Pittner (eds.), *Recent Developments in Mathematical Physics*, Springer, Berlin (1987), p. 102–136.

[5] Denker, M., Grillenberger, C. and Sigmund, K., *Ergodic Theory on Compact Spaces*, Springer Lecture Notes in Math. 527, Berlin (1976).

[6] Dinaburg, E.I., Soviet Math. Dokl. 11 (1970), 13-16.

[7] Effros, E.G., in R.V. Kadison (ed.), *C^*-Algebras and Applications to Physics*, Springer Lecture Notes in Math. 650, Berlin (1978), p.1–40.

[8] Elliott, G.A., Math. Scand. 39 (1976), 91-101;
Elliott, G.A., Canad. Math. Bull. 21 (1978), 415-418.

[9] Goodman, T.N.T., Bull. London Math. Soc. 3 (1971), 176-180.

[10] Goodwyn, L.W., Proc. Amer. Math. Soc. 23 (1969), 679-688.

[11] Hudetz, T., Lett. Math. Phys. 16 (1988), 151-161.

[12] Hudetz, T., *Dynamical Entropy: From Classical to Quantum Theory*, thesis (in German, unpublished), Univ. Vienna (1989).

[13] Hudetz, T., *Non-Linear Entropy Functionals and a Characteristic Invariant of Symmetry Group Actions on Infinite Quantum Systems*, to appear in the proceedings of "International Conference on Selected Topics in Quantum Field Theory and Mathematical Physics", J. Niederle (ed.), World Scientific, 1989.

[14] Kolmogorov, A.N., Dokl. Akad. Nauk. SSSR 119 (1958), 861-864;
Sinai, Ya.G., Dokl. Akad. Nauk. SSSR 124 (1959), 768-771.

[15] Narnhofer, H., *Beispiele von Algebren mit gleicher dynamischer Entropie*, Vienna preprint UWThPh-1988-42, to appear in honour of A. Uhlmann, Leipzig, 1989.

[16] Narnhofer, H. and Thirring, W., Fizika 17 (1985), 257-265.

[17] Narnhofer, H. and Thirring, W., Lett. Math. Phys. 14 (1987), 89-96.

[18] Narnhofer, H. and Thirring, W., Lett. Math. Phys. 15 (1988), 261-273.

[19] Narnhofer, H. and Thirring, W., *Algebraic K-Systems*, Vienna preprint UWThPh-1989-17, to appear.

[20] Palm, G., Z. Wahrscheinlichkeitstheorie verw. Geb. 36 (1976), 27-45; Palm, G., Soc. Math. de France, Astérisque 40 (1976), p. 159-165.

[21] Ruelle, D., *Thermodynamic Formalism*, Enc. Math. Appl. 5, Addison-Wesley, Reading (1978).

[22] Takesaki, M., *Theory of Operator Algebras I*, Springer, New York (1979).

[23] Walters, P., *An Introduction to Ergodic Theory*, Springer Graduate Texts in Math. 79, New York (1982).