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THE FIBER BUNDLE FORMALISM FOR THE QUANTIZATION IN CURVED SPACES

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Abstract

We set up a geometrical formulation of the canonical quantisation of a free Klein-Gordon field on a gravitational background. We introduce the notion of the Bogolubov bundle as the principal fiber bundle over the space of all Cauchy surfaces belonging to some fixed foliation of space-time, with the Bogolubov group as the structure group, as a tool in considering local Bogolubov transformations. Sections of the associated complex structure bundle have the meaning of attaching Hilbert spaces to Cauchy surfaces. We single out, as physical, sections defined by the equation of parallel transport on the Bogolubov bundle. The connection is then subjected to a certain nonlinear differential equation. We find a particular solution, which happens to coincide with a formula given by L. Parker for Robertson-Walker space-times. Finally, we adopt the adiabatic hypothesis as the physical input to the formalism and fix in this way a free parameter in the connection. Concluding, we comment on a possible geometrical interpretation of the regularisation of stress-energy tensor and on generalisations of the formalism toward quantum gravity.

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1 Introduction

The explosive interest in quantum field theory in the past decade brought a huge number of papers whose authors developed several techniques allowing to compute quantities like e.g. the regularized stress-energy tensor or quanta production rate in a given curved background or even to handle the back-reaction problem. On the other hand, a different kind of papers was written at a very formal level (e.g. dealing with the GNS construction) which, although mathematically satisfactory, brings no direct computational facilities. One can thus have a feeling that there exist at least two approaches, using different languages, both incomplete and far from constituting what e.g. the Maxwell theory does for the electromagnetic phenomena.

Probably, the reason why many people gave up and started working on quantum gravity is that they believed more in constructing the full quantum theory at once, rather than via a semiclassical approach. After these attempts have not brought any brilliant success, one again tries to move step by step towards quantum gravity, dealing with auxiliary models like e.g. mini super-space and quantum cosmology.

In this paper, which logically follows [1], we choose an alternative way in the same direction, namely we analyze old results looking for some structure hidden behind them and trying to find out a better formalism for the semiclassical theory. We are aware that we have no criterion telling us what kind of formalism will be good for quantum gravity. Similarly, classical mechanics gives no preferences to the Hamilton-Jacobi equation, which plays so significant role in quantum mechanics. History of science teaches us only that each new formulation of an old theory may help to establish a new one, thus being not necessarily just a waste of time.

We present an attempt of applying the elegant, geometric language of fiber bundles to the theory of quantized fields in curved spaces. This approach is based on the supposition that the Bogolubov group plays, in the description of the interaction between a quantized field and gravity, a similar role to that of the groups of gauge transformations in Yang-Mills theories. If this is the case, one can expect the language of fiber bundles to be well suited to the problem and to provide us with a better understanding of it.

This investigation is restricted to Klein-Gordon fields, free in the sense that they couple only to gravity. We hope that involving other interactions to the description will be possible in the future and that it will not destroy the background we are preparing here. The extension of the formalism to higher spins does not seem essentially difficult and may, in our opinion, be done along the same lines as for scalar fields.

We adopt the well established adiabatic approach proposed by Parker [2] and interpret it geometrically using the concept of local Bogolubov transformations. Contrary to [1] we don't concentrate on the problem of quanta production, which for several reasons seems to be less physical (see e.g. the analysis given in [2]), but we rather make up our formalism so as to be able to give a geometrical interpretation to the regularization of the stress-energy tensor. A paper on the latter, based on the adiabatic regularization technique proposed by Parker and Fulling [3] is in preparation [4].

The paper is organized as follows. In section 2 we briefly recall the most important notions one needs for the symplectic formulation of the quantization of field theory, based mainly on the reviews given in [5], [6] and [7]. We summarize the ideas introduced in [1] and formulate more carefully certain definitions, paying attention to their mathematical meaning. Thus e.g. the creation-annihilation operator is defined in a distributional sense.

Section 3 is devoted to the definition of the Bogolubov bundle, a principal fiber bundle which occupies the central position in our formalism, and of the bundles associated to it: the Parker bundle and the complex structure bundle. We decided to introduce the name "Parker bundle", since this object was *implicitly* used by Leonard Parker in 1969 [2] in the context which, in a sense, motivated this investigation. We are aware that also various versions of the Hamiltonian diagonalization can be formulated in terms of this same structure. The reason for choosing the name is not, that they generally failed to *describe* the physical reality, but rather that, in *our opinion*, this object seems to be more transparent in the context of adiabatic methods

In section 4 we summarize the important geometrical notions of the connection and covariant derivatives on the bundles introduced previously. A slight loss of generality is due to the fact that the base-space considered in this investigation is one-dimensional. Although its topology also makes the bundle trivial, we don't use this feature and treat it as if it were not, in order to be able to deal also with more complicated cases in the future [1]. It is one-dimensionality of the base space rather than triviality of the bundle, that is mathematically more significant for this investigation. Our approach is based mainly on that, given by Daniel and Viallet [8], but see also [9], [10] and [11].

In section 5 we impose a constraint on the connection by attaching a special dynamical meaning to the parallel transport on the Bogolubov bundle. The constraint results in a nonlinear differential equation for the connection.

Section 6 is devoted to the so-called adiabatic approach which consists first, in using a comfortable parameterization of the frames for the Parker bundle, and second, in adopting the hypothesis of minimal quanta *production in* the adiabatic limit [3], as the proper physical input to the theory. We find a particular solution of the equation derived in the previous section. We call this solution Parker connection, for it coincides with the objects introduced in [2]. Then we use the adiabatic hypothesis to fix a free parameter in the connection and, consequently, to define the "dynamics" of the quantization in curved spaces (e.g. the family of adiabatic vacua).

We conclude in section 7 with a few remarks on a geometrical formulation of the regularization of stress-energy *tensor within* the framework developed here. We *also briefly discuss* a possible enlarging of the Bogolubov bundle to a bundle over the space of all Cauchy surfaces of space-time. Such a structure might be useful for quantizing gravity.

Our notation and conventions follow (unless specified) those of [1]. Contrary to [1], we do not distinguish between abstract operators and their matrices in a given frame, for it is, in our opinion, clear from the context, what kind of object we

[1] In section 7 we discuss a possible generalisation of the notion of the Bogolubov bundle, which avoids a pre-choice of the foliation of space-time.

actually mean. Some kind of hats would only make the formulae less transparent. Referring to [2], some of the conventions used there are slightly modified (signs and $2^{-1/2}$ factors) for the full consistency with the rest of our paper. The units are chosen so that $\hbar = c = 1$ and the signature of the space-time metric is $(-,+,+,+)$.

2 Preliminaries

As in [1], we restrict ourselves to considering only the cases in which the Klein-Gordon equation is separable. We choose in a globally hyperbolic space-time (M, g) a global time function $t : M \to I\!R$ defining a foliation of M. Let S be a manifold, whose points are the Cauchy surfaces $t(x^a) = const$. It has the topology $I\!R^1$. Our goal now is to make mathematically more precise the idea of attaching Hilbert spaces to Cauchy surfaces, which often appears in the literature. We will briefly describe the structures we need for this purpose. For details see [1] and references quoted there.

Let $\tilde{\mathcal{V}}$ be the space of all smooth, complex Klein-Gordon fields on M, whose Cauchy data induced on each Cauchy surface fall off sufficiently fast at spatial infinity, to make convergent all integrals throughout this paper, and let $\mathcal{V} \subset \tilde{\mathcal{V}}$ be the corresponding space of real fields. $\tilde{\mathcal{V}}$ is equipped with a scalar product

$$(\phi, \phi') := i \int_\Sigma \phi^* \overleftrightarrow{\nabla}_a \phi' d\Sigma^a . \tag{1}$$

This form is clearly antisymmetric on $\mathcal{V}$ and therefore defines a natural symplectic structure on it. The group B of all linear, bounded, real automorphisms of $\mathcal{V}$ which leave the symplectic structure invariant is called *Bogolubov group*. This notion is crucial for our further investigations.

A complex structure J on $\mathcal{V}$ (i.e. a linear map $J : \mathcal{V} \to \mathcal{V}$ with $J^2 = -1$) such that $-i(\phi, J\phi')$ is positive definite, defines a sesquilinear, positive definite inner product on $\mathcal{V}$

$$< \phi, \phi' > := \frac{1}{2}(\phi, \phi') - \frac{1}{2} i(\phi, J\phi'),$$

which, in turn, promotes $\mathcal{V}$ to a Hilbert space $\mathcal{H}$ (strictly speaking, $\mathcal{H}$ is the Cauchy completion of $\mathcal{V}$, with the scalar product introduced above).

To build a quantum field theory we further need a $*$-algebra $\mathcal{A}$ of self-adjoint field operators A on $\mathcal{V}$ and a Hilbert space of states. As usual, we will require the $*$-algebra to be linear in the elements of $\mathcal{V}$ and to fulfill the following commutation relations

$$[A(\phi), A(\phi')] = -(\phi, \phi') I ,$$

with $A \in \mathcal{A}$, $\phi, \phi' \in \mathcal{V}$ and I being the identity on $\mathcal{A}$.

This construction is equivalent to the well known canonical quantization of field theory. Indeed, a classical solution $\phi \in \mathcal{V}$ is uniquely determined by its Cauchy data

$$f := \phi|_\Sigma , \; g := n^a \nabla_a \phi|_\Sigma$$

on each hypersurface Σ from S, with n^a being the unite future-directed vector field, normal to Σ. Denoting $A(0, g)$ by $\Phi(g)$ and $A(f, 0)$ by $\Pi(f)$ respectively, we find that these quantities, linear in their arguments and self-adjoint, fulfill the canonical commutation relations. We call them (smeared out) field configuration and field momentum operators. Their algebra is naturally isomorphic to the above constructed $*$-algebra of $A(\phi)$'s.

The operators $A(\phi)$ are, on the other hand, just smeared out versions of the *creation-annihilation operators* heuristically introduced in [1]. The standard creation and annihilation operators $a^*(\phi)$ and $a(\phi)$ are obtained from $A(\phi)$ by means of the projectors $P_\pm$ in the following way

$$a^*(\phi) = P_+ A(\phi) := \frac{1}{2}(A(\phi) - iA(J\phi)),$$

$$a(\phi) = P_- A(\phi) := \frac{1}{2}(A(\phi) + iA(J\phi)).$$

Here J is the complex structure.

The action of the Bogolubov group on $\mathcal{V}$ causes a change in $a^*(\phi)$ and $a(\phi)$ which is due to the induced action of B either on the $*$-algebra of operators $A(\phi)$ or alternatively, on the space of complex structures. The first possibility was called in [1] Schrödinger, the second Heisenberg picture (for B acts on the space of all J's through its adjoint representation).

The space of states is obtained from $\mathcal{H}$ by the usual Fock construction

$$\mathcal{F} = \bigoplus_{k=0}^{\infty} \mathcal{H}_{sym}^{\otimes k}$$

and our definition of the inner product for $\mathcal{H}$ guarantees that the $*$-representation of $\mathcal{A}$ into the $*$-algebra of observables on $\mathcal{F}$ fulfills certain well-motivated conditions (the detailed construction of the $*$-representation is extensively discussed in [5]). All the freedom we still have in choosing the $*$-representation is the remaining freedom in the choice of the inner product for $\mathcal{H}$, i.e. the choice of the complex structure, since the product (1) is hypersurface independent for functions satisfying the Klein-Gordon equation. Obviously, for the same reason attaching Hilbert spaces (and $*$-representations) to Cauchy surfaces means just considering a map from S to the set of all complex structures.

When dealing with the fields $\phi \epsilon \mathcal{V}$, we will use a special kind of frames λU, where $\lambda(t)$ is a real, smooth function of time. The elements of these frames $\lambda u_{\mathbf{k}}$, orthonormal in the product (1), are from the space $\tilde{\mathcal{V}}$ rather than from $\mathcal{V}$, but their structure is such that, roughly speaking, the second half of a frame is the complex conjugate of the first half. In other words, U, a row matrix, looks like

$$U = \{(u_{\mathbf{k}}^*, u_{\mathbf{k}})\},$$

or, as pointed out in [1], it satisfies the condition $U\Im = U^*$, with "$*$" being the complex conjugate and $\Im$ the inversion operator

$$\Im := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Each element of $\mathcal{V}$ will be written in a given frame as

$$\phi = \lambda U A_\phi. \tag{2}$$

A_ϕ is, as usual, a column matrix of coefficients and the standard form of (2) reads

$$\phi = \lambda \sum_{\mathbf{k}} (a^*_{\mathbf{k}} u^*_{\mathbf{k}} + a_{\mathbf{k}} u_{\mathbf{k}}) .$$

The slight modification of the convention adopted in [1] is done in order to have after separating variables, for the modes $u_{\mathbf{k}}$

$$\frac{d^2 u_{\mathbf{k}}}{dt^2} + \omega^2_{\mathbf{k}}(t) u_{\mathbf{k}} = 0 , \tag{3}$$

an equation without first derivative. This requirement determines $\lambda(t)$.

We call $\omega_{\mathbf{k}}(t)$ the (time-dependent) *frequency* attached to the mode $u_{\mathbf{k}}$. Thus, according to our conventions, ω stands for the linear operator attaching to each mode its frequency (in the sense of eq. (3)). Using the matrix notation, (3) can be written as

$$\frac{d^2}{dt^2} U + U \omega^2(t) = 0 .$$

Inspection of this equation shows that ω must transform according to the adjoint representation of the Bogolubov group. Changing e.g. from U to $\tilde{U} = U\Omega$ with Ω being a (global) Bogolubov transformation, we find that

$$\tilde{\omega} = \Omega^{-1} \omega \Omega .$$

Here the matrix ω is diagonal in U, and $\tilde{\omega}$ in $\tilde{U}$ respectively. The action of the *operator* ω on ϕ is denoted by $\phi\omega$ and expressed as $\lambda U \omega A_\phi$ in a frame λU, ω being the appropriate *matrix* of the considered operator.

The matrix Ω has the form

$$\begin{pmatrix} \alpha^\dagger & \beta^T \\ \beta^\dagger & \alpha^T \end{pmatrix} ,$$

where α and β stand for matrices of usual Bogolubov coefficients. The relations to hold are

$$\tilde{\Omega}\Omega = I \quad , \quad \Omega^* = \Im \Omega \Im ,$$

with "$*$" denoting the complex conjugate and $\tilde{\Omega} := \mathbb{I} \Omega^\dagger \mathbb{I}$, where

$$\mathbb{I} := \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} .$$

3 The Bogolubov, and associated bundles

We start our approach with a couple of definitions, following exactly the geometrical treatment of Yang-Mills theories. So, the basic structure we need is a principal fiber bundle.

Definition 1 *The Bogolubov bundle $\mathcal{B}(S, B)$ is the principal fiber bundle with base space S and structure group B.*

The base space $S = \mathcal{B}/B$, and the canonical projection on $\mathcal{B}$, $\pi : \mathcal{B} \to S$, is defined by: $\pi(p) :=$ *equivalence class of* p, where $p \epsilon \mathcal{B}$.

A local [2] *cross section* of the Bogolubov bundle, $\sigma : S_\alpha \to \mathcal{B}$, such that $\pi(\sigma(t)) = t$ for all $t \epsilon S_\alpha$, an open subset of S, has the meaning of assigning elements of the Bogolubov group to Cauchy surfaces.

A principal fiber bundle is a very abstract structure. Its fibers are naturally diffeomorphic to the structure group itself. For our purpose we need, however, more concrete objects, involving spaces on which the Bogolubov group acts, e.g. the space of solutions or the space of complex structures. Roughly speaking, we want to erect their copies on each Cauchy surface in (M, g). This idea is mathematically realized by means of the appropriate associated fiber bundles. We introduce now the necessary definitions.

Definition 2 *The Parker bundle* $\mathcal{P}(S, B, \mathcal{V}', \mathcal{B}) = \mathcal{B} \times \mathcal{V}'/B$ *is the bundle associated to* $\mathcal{B}$ *with the typical fiber* $\mathcal{V}'$.

The typical fiber $\mathcal{V}'$ is the space of (well-behaved, as those from $\mathcal{V}$) real solutions of another, "auxiliary" Klein-Gordon equation, which has nothing to do with the physical space-time and just separates to

$$\frac{d^2}{dt^2}U + U\omega'^2(t) = 0\,,$$

with $\omega'(t)$ generally different from $\omega(t)$, the time-dependent frequency arising from the "original" Klein-Gordon equation in the considered space-time. The space-dependent parts of the modes of both equations are supposed to be identical, so that we effectively consider $\mathcal{V}'$ to be the space of solutions whose frequency is "shifted"[3] from $\omega(t)$ to $\omega'(t)$. Each section of the Parker bundle $\varphi : S \to \mathcal{P}$ defines a function $\varphi(t)$. There is a class of sections $\{\phi\} \subset \{\varphi\}$, called *dynamical,* such that the corresponding functions $\phi \epsilon \mathcal{V}$, i.e. solve the "original", physical Klein-Gordon equation. They will be specified later.

Let $\mathcal{J}'$ be the set of all complex structures J' on $\mathcal{V}'$. We have:

Definition 3 *The complex structure bundle* $\mathcal{C}(S, B, \mathcal{J}', \mathcal{B}) = \mathcal{B} \times \mathcal{J}'/B$ *is the bundle associated to* $\mathcal{B}$ *with the typical fiber* $\mathcal{J}'$.

Notice that there is an important difference between the Parker bundle and the complex structure bundle: the Bogolubov group acts on the fiber of the latter through the adjoint representation.

4 Basic geometrical notions

Again, as in Yang-Mills theories, we wish to define dynamics in terms of geometry and we therefore need to know the horizontal direction on our bundles. This is, of course, not given a priori but appears as an extra structure to be specified by means of the appropriate connection form. In what follows, we briefly recall its definition and the construction of related objects.

[2] As is well known from differential geometry, only trivial principal bundles admit global sections.

[3] This is essential for adiabatic methods.

Let $\mathcal{A}(B)$ be the Lie algebra of the Bogolubov group, *the Bogolubov algebra*, and $\rho \epsilon \mathcal{A}(B)$. We associate to ρ a fundamental vector field $\Lambda(\rho)$ on B. Let $p \epsilon B$ and $T_p(B)$ be the vector space, tangent to B at p. We represent $T_p(B)$ as a direct sum of two linear subspaces

$$T_p(B) = Q_p \oplus B_p,$$

where B_p is the space of all vectors tangent to the fiber through p at p and Q_p, its linear supplementary, is called horizontal subspace at p. It is determined by means of the connection form, i.e. the Bogolubov algebra valued 1-form ξ^B on B such that

(i) $\xi^B(\Lambda(\rho)) = \rho$,

(ii) $(R_\Omega^* \xi^B)(X) = Ad_{\Omega^{-1}} \xi^B(X)$,

where $X \epsilon T(B)$ and R_Ω^* is the diffeomorphism (pull back) generated by the right action of the Bogolubov group element Ω. Then the horizontal subspace Q_p is defined as the kernel of ξ^B

$$Q_p := ker(\xi^B).$$

The connection form ξ^B may also be expressed in terms of local forms ξ_α defined in an open subset S_α of the base space S. Given any local section of the Bogolubov bundle $\sigma_\alpha : S_\alpha \rightarrow B$, we can define a Bogolubov algebra valued 1-form on S_α

$$\xi_\alpha := \sigma_\alpha^* \xi^B .$$

Unless required by the context (like just below), we will omit the index α and think of ξ as of a local form on S. We will use the symbol ξ_t for the local, Bogolubov algebra valued *function* on S, such that

$$\xi = \xi_t dt .$$

ξ_t is therefore the analog of a Yang-Mills potential matrix $A_a^i \tau^i$, with τ^i being gauge group generators.

Let σ_β be another local section defined on S_β and ξ_β the corresponding local form. Then ξ_α and ξ_β must satisfy the compatibility condition on $S_\alpha \cap S_\beta$

$$\xi_\beta = Ad_{\psi_{\alpha\beta}^{-1}} \cdot \xi_\alpha + \psi_{\alpha\beta}^{-1} d\psi_{\alpha\beta}, \tag{4}$$

where $\psi_{\alpha\beta}$ is the transition function from σ_α to σ_β and d denotes the exterior derivative on the base space S.

Of course, if the bundle is trivial we can take $S_\alpha = S_\beta = S$ and consider global sections as well. Then the compatibility condition (4) should be interpreted just as the transformation law for the connection form on S under a change from one global section to another. To illustrate this, choose two sections σ_1 and σ_2 and let $\Omega : S \rightarrow B$ be a function transforming σ_1 into σ_2

$$\sigma_2 = \sigma_1 \cdot \Omega .$$

The sections define two forms ξ_1 and ξ_2 on S

$$\xi_1 = \sigma_1^*(\xi^B) \quad , \quad \xi_2 = \sigma_2^*(\xi^B)$$

related to each other by

$$\xi_2 = Ad_{\Omega^{-1}}\xi_1 + \Omega^{-1}d\Omega\,.$$

Having introduced the connection on the principal fiber bundle $\mathcal{B}$, we are able to define covariant derivatives on $\mathcal{B}$ and on all bundles associated to it. We proceed as follows. Let X be a vector field on S. We define the vector field $\dot{X}$ on $\mathcal{B}$ to be the *horizontal lift* of X iff

(i) $\xi^{\mathcal{B}}(\dot{X}_p) = 0$.

(ii) $\pi_*(\dot{X}_p) = X_{\pi(p)}$

for all $p \in \mathcal{B}$. The condition (i) means, of course, that the field $\dot{X}$ is horizontal.

Now we define the *covariant derivative* $\mathcal{D}_t$ on $\mathcal{B}$ as the horizontal lift of the derivative ∇_t on S

$$\mathcal{D}_t := \dot{\nabla}_t\,.$$

In local coordinates in S we have

$$\dot{\nabla}_t|_p = \sigma_*\nabla_t - \Lambda(\xi_t)$$

with $p = \sigma(t)$.

As mentioned above, the connection on a principal fiber bundle allows one to construct covariant derivatives also on the associated bundles. One simply uses the fact, that their sections are in one-to-one correspondence with the functions f from the principal fiber bundle to the typical fibers of associated bundles, provided f have well defined transformation properties under the action of the structure group [8]. The construction proceeds as follows. Let $\mathcal{E}$ be a bundle associated to $\mathcal{B}$ with typical fiber E and $\Gamma(\mathcal{E})$ the set of all sections of $\mathcal{E}$. Given any $\varphi \in \Gamma(\mathcal{E})$, we associate to it a function $f : \mathcal{B} \to E$ and define $f' := \mathcal{D}_t f$. This f' also takes values in E and has the same transformation properties as f, so it again defines a section of $\mathcal{E}$ which we denote by $D_t\varphi$. The map

$$D_t : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$$

is called *covariant derivative on* $\mathcal{E}$.

The action of D_t on sections of $\mathcal{E}$ depends on how the structure group acts on the fiber E. So in the case of the Parker bundle $\mathcal{P}$ (right action) we have

$$D_t = \nabla_t - \cdot\,\xi_t\,,$$

while for the complex structure bundle $\mathcal{C}$ (adjoint action)

$$D_t = \nabla_t - [\cdot, \xi_t]\,.$$

It seems natural to identify the derivative ∇_t in S with the Lie derivative $\mathcal{L}_{Nn}$ in the space-time (M, g). Here n^a is the unite, future pointing vector field, normal to to the foliation, and $N(t)$ the lapse function defined by $N\mathcal{L}_n t = 1$. One often uses coordinates in which $Nn^a = (\partial/\partial t)^a$

5 The choice of connection

We suppose now each dynamical section $\phi : S \to \mathcal{P}$ of $\mathcal{P}$ to be induced by a *horizontal* section $\Omega : S \to \mathcal{B}$ of $\mathcal{B}$. This means that Ω has to satisfy the equation of parallel transport on $\mathcal{B}$

$$\mathcal{D}_t\Omega = 0\,, \tag{5}$$

i.e. $\nabla_t\Omega = \Omega\xi_t$. The requirement that the section $\phi(t) = \phi'(t)\Omega(t)$ of $\mathcal{P}$ be dynamical, is expressed by the condition

$$\lambda\nabla_t^2\lambda^{-1}\phi + \phi\omega^2(t) = 0\,, \tag{6}$$

obtained after separating variables from the "original" Klein-Gordon equation. Thus (5) and (6) determine the connection on $\mathcal{B}$. They can be written in a compact way as

$$\lambda D_t^2\lambda^{-1}\phi + \phi\, Ad_{\Omega^{-1}(t)}\omega'^2(t) = 0\,, \tag{7}$$

for ϕ' is by definition a solution of the "shifted" equation

$$\lambda\nabla_t^2\lambda^{-1}\phi' + \phi'\omega'^2(t) = 0\,.$$

One can also easily prove the following relation

$$(\phi, \lambda D_t\lambda^{-1}\phi) = (\phi', \lambda\nabla_t\lambda^{-1}\phi')\,,$$

whose general form allows one to think of a possible extension of the formalism also to the cases in which the field equation is non-separable.

The equation (7) yields the following nonlinear differential equation for the connection

$$2(\lambda\nabla_t\lambda^{-1}\phi)\xi_t + \phi\nabla_t\xi_t - \phi\xi_t^2 = \phi[\,Ad_{\Omega^{-1}(t)}\omega'^2 - \omega^2\,]\,, \tag{8}$$

that must be satisfied for all ϕ's. One can see now that, in a way, ξ_t measures the difference between $\omega(t)$ and $\omega'(t)$ and should vanish if those two frequencies are identical (which means that the functions from $\mathcal{V}'$ are solutions of the "original" Klein-Gordon equation as well). In order to compare $\omega(t)$ with $\omega'(t)$, one must, however, first transform one of them in such a way that the resulting expression, e.g. $Ad_{\Omega^{-1}}\omega'^2 - \omega^2$, be diagonal either in U or in U'. This automatically happened in (8).

The sections of the complex structure bundle, induced by the horizontal sections of $\mathcal{B}$, define the local functions

$$J(t) = Ad_{\Omega^{-1}(t)}J(t_0)\,, \tag{9}$$

with $\Omega(t_0) = I$ and t_0 fixed. Differentiating (9) over t and using (5) together with the Bogolubov algebra property, $\bar{\xi} = -\xi$, we find that $\nabla_t J = [J, \xi_t]$, or, in other words

$$D_t J = 0\,. \tag{10}$$

This equation will be used later.

6 The adiabatic approach

The starting point for the adiabatic approach to the quantization in curved spaces is the WKB-like parameterization introduced in [2]. One first chooses a frame $\lambda U'$ for $\mathcal{V}'$ and writes formally the time-dependence of its elements, $u'_{\mathbf{k}}$, as

$$\chi'_{\mathbf{k}}(t) = (2W_{\mathbf{k}}(t))^{-1/2} e^{-i\int^t W_{\mathbf{k}}(\tau)d\tau}, \tag{11}$$

where the functions $W_{\mathbf{k}}(t)$ are to be specified later. The obvious identity

$$\frac{d^2\chi'_{\mathbf{k}}}{dt^2} + (W_{\mathbf{k}}^2 - W_{\mathbf{k}}^{1/2}\frac{d^2}{dt^2}W_{\mathbf{k}}^{-1/2})\chi'_{\mathbf{k}} = 0$$

encourages us to write

$$\omega_{\mathbf{k}}'^2 = W_{\mathbf{k}}^2 - W_{\mathbf{k}}^{1/2}\frac{d^2}{dt^2}W_{\mathbf{k}}^{-1/2}. \tag{12}$$

We propose now, for the matrix of connection in $U = U'\Omega(t)$

$$\xi_t = \frac{i}{2}\Omega^{-1}(t)\begin{pmatrix} W^{-1} & W^{-1}e^{-2i\int^t W d\tau} \\ -W^{-1}e^{2i\int^t W d\tau} & -W^{-1}\end{pmatrix}[\Omega(t)\omega^2 - \omega'^2\Omega(t)], \tag{13}$$

where W stands for the diagonal matrix [4] with entries $W_{\mathbf{k}}$ and $\Omega(t)$ satisfies the equation of parallel transport (5). One can easily check that (13) indeed takes values in the Bogolubov algebra and satisfies equation (8). We call the above formula (13) *Parker connection*, since, adopting the notation used in [2]

$$\Omega\omega^2\Omega^{-1} - \omega'^2 = 2WS,$$

one obtains from the equation of parallel transport (5) the following formulae for the matrix components of $\Omega(t)$ in Robertson-Walker spaces

$$\alpha_{\mathbf{k}}(t) = 1 - i\int_{t_0}^t d\tau S_{\mathbf{k}}(\tau)[\alpha_{\mathbf{k}}(\tau) + \beta_{\mathbf{k}}(\tau)e^{2i\int^\tau W_{\mathbf{k}}d\tau'}],$$

$$\beta_{\mathbf{k}}(t) = i\int_{t_0}^t d\tau S_{\mathbf{k}}(\tau)[\beta_{\mathbf{k}}(\tau) + \alpha_{\mathbf{k}}(\tau)e^{-2i\int^\tau W_{\mathbf{k}}d\tau'}].$$

Up to a trivial change in sign, caused by a slightly different convention, the above equations coincide with those, obtained by Parker in [2]. The choice of initial conditions will be discussed later.

We have, so far, established a nice geometric formalism for the quantization in curved spaces, that is to say, we already have what is called a "kinematical arena". The problem of dynamics is, however, still open, since we don't know any details about the "shifted" frequency ω' we have arbitrarily involved into the description. This free parameter requires some physical input, which must be given. There are many possible ways of fixing ω'. One of them is the so-called Hamiltonian diagonalization which, however, for several reasons fails to properly describe the reality (see e.g. [12], [13]). An alternative concept is the adiabatic hypothesis, first

[4] The operator W is diagonal in U'.

proposed by Parker [2], and in its final version formulated by Parker and Fulling [3]. According to this hypothesis, the number of quanta should be an adiabatic invariant and the quanta production rate should vanish in the limit of infinitely slowly varying metric. The same, of course, applies to the commutator $[J, \xi_t]$, for it governs via equation (10) the change in J, which, in turn, is just called quanta production [5]. If we choose as ξ_t the Parker connection, we must demand it to be adiabatic invariant and fall of sufficiently fast in the limit of infinitely slowly varying metric. Following [3] (but see also [14]), we introduce the *adiabatic parameter* T, which allows to rescale time $t \to t/T$ and to measure the slowness. Requiring ξ_t to vanish up to the adiabatic order $2n$ as $T \to \infty$, i.e. to behave like $O(T^{-2n-2})$, we obtain from (12) and (13)

$$\Omega\omega^2\Omega^{-1} - W^2_{(2n)} + W^{1/2}_{(2n-2)} \frac{d^2}{dt^2} W^{-1/2}_{(2n-2)} = 0 .$$

This allows to establish the following iterative definition of W.

Definition 4 *The 2n-th adiabatic order operator* $W_{(2n)}$ *is defined by means of the following recurrence relation*

(i) $W_{(0)} := Ad_\Omega \omega$,

(ii) $W^2_{(2n)} := Ad_\Omega \omega^2 - W^{1/2}_{(2n-2)} \frac{d^2}{dt^2} W^{-1/2}_{(2n-2)}$ for $n > 0$.

Finally, we complete the formalism with

Definition 5 *For a given adiabatic order 2n*

$$\omega'^2 := W^2_{(2n)} - W^{1/2}_{(2n)} \frac{d^2}{dt^2} W^{-1/2}_{(2n)} .$$

A few remarks should be made at this point. First, from the logical point of view, the above definition belongs to the section 3, since it specifies the notion of the Parker bundle for each n. For obvious pedagogical reasons we decided, however, to keep ω' free up to this section.

Second, the presented mathematical structure is well defined for each n and there is nothing like "approximated Parker bundle". It is physical reasons that tell us to consider higher adiabatic orders to be, in a sense, better than the lower ones [3].

Third, the quantization of the theory is based on *one* vacuum state (called adiabatic vacuum), although there is a one-parameter family of such states, each of them assigned to a different choice of initial data for equation (5), i.e. to a choice of t_0, such that $\Omega(t_0) = I$. The complex structure remains here constant. One chooses it to be diagonal in the modes $U = U'\Omega(t)$, interpreting (11) as the positive-frequency solution for t_0. However, we must stress once again the fact that there is generally no criterion telling us, how to choose t_0 (unless the space-time possesses a static region, where it is reasonable to require the adiabatic vacuum to coincide with the natural Minkowski vacuum, or other special cases occur), but all solutions of (5) give by construction adiabatically equivalent vacua.

[5] We avoid identifying those "quanta" with physical particles which should, in our opinion, be attached rather to some precisely defined physical detector.

The time-dependent complex structure, given by equation (10), has been important for establishing the notion of adiabatic vacuum. It is also useful for investigating production of quanta (whatever this means physically) and, as will be shown in [4], for the geometric definition of the adiabatic regularization procedure.

7 Conclusions

This investigation is based on a certain analogy between local gauge transformations and the local (i.e. time-dependent) Bogolubov transformations, and the proposed geometrical framework is in a natural way a copy of the one used for Yang-Mills theories. There is, however, a significant difference: our base-space is one-dimensional, and the curvature of the bundle (as a two-form) must vanish identically. The available structure is therefore, in a sense, too poor to write anything like the Yang-Mills equations. On the other hand, the one-dimensionality of our base-space is a direct consequence of the fact that one always needs a notion of time in the Hamiltonian formalism. *Notice that the alternative method* of quantization - the path-integral approach - suffers from troubles with the definition of measure for the functional integral, which are often considered to be more serious.

The temptation to have a quantization scheme, similar to the proposed here, but with a higher dimensional base-space for the corresponding Bogolubov bundle, is strong not only for mathematical reasons. If it didn't require any pre-choice of foliation of space-time, one could be happy to find a proper tool for handling the back-reaction problem and, maybe, also for quantizing gravity, where nothing connected with the space-time geometry should remain a priori fixed! Such a formulation could e.g. be based on the Bogolubov bundle over the space of *all* Cauchy surfaces of a given space-time. The horizontal lifts of curves (corresponding to foliations) would then play a similar role to that of the horizontal sections in the one-dimensional case. This is in fact, as pointed out in [1], the long-range goal, and the present investigation is only the starting point or a sort of toy-formalism, rather than some final result.

The concept of local Bogolubov transformations provides us with a nice geometric tool for treating the adiabatic approach which, in turn, is very fruitful in defining the (regularized) stress-energy tensor (see [3], [14] and many references quoted there). Thus, we can hope to be able to understand geometrically the adiabatic regularization procedure and to express it in terms of a connection on the Bogolubov bundle. One can further use this connection for solving some particular back-reaction problems and, if one succeeds, i.e. if the resulting solution has some good geometric properties, one can even believe the connection to properly describe the gravitational interactions on the quantum level.

Here we arrive again at the quantization of gravity, although, as emphasized in the introduction, we don't intend to arbitrarily propose any particular scheme, but rather to learn about the problem as much as possible, while proceeding in the way sketched above. The next step to be done is the geometric formulation of the adiabatic regularization procedure.

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References

[1] Bombelli, L. and Wyrozumski, T., (1989), "The Bogolubov group and quantization of scalar fields", Class. Quantum Grav. **6**, 919-32

[2] Parker, L. (1969), "Quantized Fields and Particle Creation in Expanding Universes. I", Phys.Rev. **183**, 1057-68.

[3] Parker, L. and Fulling, S.A. (1974), "Adiabatic regularization of the energy-momentum tensor of a quantized field in homogeneous spaces", Phys.Rev. **D9**, 341-54.

[4] Wyrozumski, T., "The geometrical meaning of the adiabatic regularization technique", in preparation.

[5] Ashtekar, A. and Magnon, A. (1975), "Quantum fields in curved space-times", Proc.Roy.Soc.(London) **A346**, 375-94.

[6] Bongaarts, P.J.M., "Linear fields according to I.E.Segal", in Streater, R.F. (ed.) (1972), *Mathematics of contemporary physics*, Academic Press, London and New York.

[7] Shale, D. (1962), "Linear symmetries of free boson fields", Trans. Am. Math. Soc. **103**, 149-67.

[8] Daniel, M. and Viallet, C.M., (1980), "The geometrical setting of gauge theories of the Yang-Mills type", Rev. Mod. Phys. **52**, 175-97.

[9] Nash, C. and Sen, S. (1983), *Topology and geometry for physicists*, Academic Press, London.

[10] Choquet-Bruhat Y., De Witt-Morette C. and Dillard-Bleick M. (1977) *Analysis, manifolds and physics*, North Holland Publishing Company, Amsterdam, New York, Oxford.

[11] Kobayashi S. and Nomizu K. (1969), *Foundations of differential geometry*, Vol. I and II, Wiley, New York.

[12] Fulling, S.A. (1979), "Remarks on positive frequency and Hamiltonians in expanding universes", Gen. Rel. Grav. *10*, 807-24.

[13] Raine, D.J. and Winlove, C.P. (1975), "Pair creation in expanding universes", Phys.Rev. **D12**, 946-51.

[14] Birrell, N.D. and Davies, P.C.W. (1982), *Quantum fields in curved space*, Cambridge Univ. Press.