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EXACT CRITICAL PROPERTIES OF TWO-DIMENSIONAL POLYMER NETWORKS  
FROM CONFORMAL INVARIANCE

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EXACT CRITICAL PROPERTIES OF TWO-DIMENSIONAL POLYMER NETWORKS  
FROM CONFORMAL INVARIANCE

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Abstract

An infinity of exact critical exponents for two-dimensional self-avoiding walks can be derived from conformal invariance and Coulomb gas techniques applied to the  $O(n)$  model and to the Potts model. They apply to polymer networks of any topology, for which a general scaling theory is given, valid in any dimension  $d$ . The infinite set of exponents has also been calculated to  $O(\epsilon^2)$ , for  $d=4-\epsilon$ . The 2D study also includes other universality classes like the dense polymers, the Hamiltonian walks, the polymers at their  $\theta$ -point. Exact correlation functions can be further given for Hamiltonian walks, and exact winding angle probability distributions for the self-avoiding walks.

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I. INTRODUCTION

Polymers are random walks, whose configurations are drastically restricted by the self-avoidance constraint : the configurations have no double points<sup>[1]</sup> and are called self-avoiding walks (SAW). Physically, the origin of this excluded volume effect is, in a good solvent, the preferential solvation of monomers by the solvent molecules rather than by other monomers. This results in an effective short range repulsion between monomers<sup>[1]</sup>. Mathematically, the problem is then fairly well abstracted by the model of SAW. It is now well known that long polymers form a geometrical *critical* system, whose universality class is that of a Heisenberg ferromagnet (or  $O(n)$  model) with a number of components  $n \rightarrow 0$ <sup>[1]</sup>. The simplest consequences are the existence of well-known critical exponents  $\nu$  and  $\gamma$  for a single SAW. The number of configurations of a single polymer (e.g. drawn on a lattice) grows like

$$\mathcal{Z} \sim \mu^N N^{\gamma-1} \quad (N \rightarrow \infty) \quad (1)$$

where  $N$  is the length of the polymer (or its number of monomers),  $\mu$  is the growing factor, or "effective connectivity constant" (it is non universal and lattice dependent) and  $\gamma$  is a *universal* critical exponent (the magnetic susceptibility exponent of the  $O(n)$  model, for  $n \rightarrow 0$ ). The end-to-end distance of the chain scales like

$$R^2 \sim N^{2\nu} \quad (N \rightarrow \infty)$$

where  $\nu$  is the *universal* correlation length exponent of the  $O(n=0)$  model. This identification led to the calculation of  $\epsilon$ -expansions of these exponents for a space dimension  $d=4-\epsilon$  [1,2], and of other universal functions. In 1982 new progresses were made in two-dimensional (2D) critical theories [3], and the  $O(n)$  and Potts critical models were shown to be equivalent to some interacting Coulomb gas. From this, the exact values  $\gamma = 43/32$ ,  $\nu = 3/4$  were derived [3]. It has been realized more recently [4-7] that many more critical exponents (actually infinities) exist in the polymer theory (and also in the  $O(n)$  model), and furthermore they can all be calculated exactly in 2D, using conformal invariance. In particular, in ref. [4] a general scaling theory of polymer networks of any topology, valid in any dimension, was established which gave the universal configuration exponent  $\gamma_{\mathcal{G}}$  of any networks  $\mathcal{G}$  as an explicit function of the topology of  $\mathcal{G}$ . The exact formulas for  $\gamma_{\mathcal{G}}$  in two dimensions (based on conformal identifications [5]) and to  $O(\epsilon)$  in  $d=4-\epsilon$  were given [4]. More recently the expansion of  $\gamma_{\mathcal{G}}$  to second order in  $\epsilon$  was performed [8]. The purpose of this short contribution is to describe how infinities of critical exponents naturally appear in polymer theory, how they can be classified by an exact scaling theory, and what are the principal results known both in  $d=2$  and  $d=4-\epsilon$ .

## II. GEOMETRICAL EXPONENTS

In polymer theory, one can ask many questions, when one considers polymer objects which are more complicated than a single polymer chain. For instance, consider star polymers, with  $L$  arms of approximately equal lengths  $N$  (Fig.1). Then one expects the configuration number to behave asymptotically as

$$\mathcal{Z}_L \sim \mu^{LN} N^{\gamma_L} \quad (N \rightarrow \infty) \quad (2)$$

generalizing (1). Note that  $LN$  is the total mass of the object, with which the number of configurations grows exponentially. Now,  $\gamma_L$  is a new universal critical exponent, characteristic of the star. One can ask also: what is the relative probability that the cores of two  $L$  and  $L'$ -arm stars approach each other at a distance  $r$  (Fig.1)? Due to exclude volume it vanishes at short distance like

$$P_{L..L'}(r) \sim r^{\theta_{L..L'}} \quad (r \rightarrow 0) \quad (3)$$

where  $\theta_{L..L'}$  is a new *universal contact exponent*, function of  $L$ ,  $L'$  and  $d$  only.

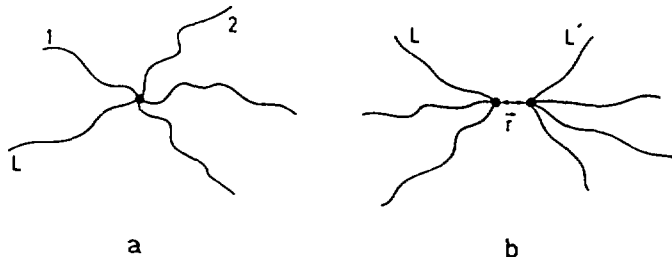


Fig.1 - a) L-arm star b) Contact of the cores of  $L$  and  $L'$  stars.

The scaling theory we gave [4,6,8] allows to calculate all of these exponents, exactly in  $2D$  [4] with help of conformal invariance [5] and in  $d=4-\epsilon$  with renormalization theory. They are respectively [4]

$$\gamma_L - 1 = [4 + 9L(3-L)]/64 \quad (2D) \quad (4)$$

and in  $d=4-\epsilon$  [8]

$$\gamma_L - 1 = \frac{1}{2} L(3-L) \frac{\epsilon}{8} + \frac{L}{8} (8L^2 - 37L + 55) \left(\frac{\epsilon}{8}\right)^2 + O(\epsilon^3) \quad (5)$$

Of course for  $L=1$ , one recovers the well known susceptibility exponent  $\gamma = 43/32$  in  $2D$  [3] and [1,2]  $\gamma = 1 + \frac{\epsilon}{8} + \frac{13}{4} \left(\frac{\epsilon}{8}\right)^2 + \dots$  in  $d=4-\epsilon$ . The exact values of the contact exponents  $\theta_{L,L'}$  are [6,8]

$$\theta_{L,L'} = (9LL' + 2)/24 \quad (2D) \quad (6)$$

$$\theta_{L,L'} = 2LL' \left\{ \frac{\epsilon}{8} + \left(\frac{\epsilon}{8}\right)^2 \left[ \frac{33}{4} - 3(L+L') \right] + O(\epsilon^3) \right\} \quad d=4-\epsilon \quad (7)$$

From this, one has in particular the usual single chain contact exponents [9]  $\theta_0, \theta_1, \theta_2$  of the two extremities of a single chain, of one extremity and the interior, and of two interior points (Fig.2). They read in our formalism (see Fig.1)  $\theta_0 \equiv \theta_{1,1}, \theta_1 \equiv \theta_{1,2}, \theta_2 \equiv \theta_{2,2}$ , and we find [6] the exact values in  $2D$  from (6)

$$\theta_0 = \frac{11}{24} (\equiv \gamma - 1), \quad \theta_1 = \frac{5}{6}, \quad \theta_2 = \frac{19}{12}, \quad (8)$$

while from (7) we recover the des Cloizeaux' values [9] to  $O(\epsilon^2)$ . Note that the connectivity of the chain does not play any role in the contact exponents  $\theta_0, \theta_1, \theta_2$  which depend only on the star vertices upon contact (compare Fig.1 and Fig.2).

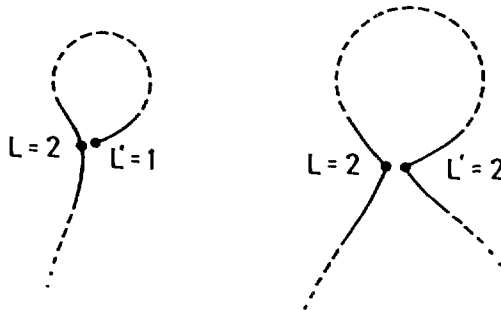


Fig. 2 - Contacts inside a single linear chain and the relation to equivalent contacts of stars.

This remark is crucial for understanding the scaling theory which underlies results (4)-(7). How can one derive these results ? They are all

consequences of the basic idea that the polymer theory can be entirely renormalized in terms of the physical vertices which build a polymer network<sup>[4]</sup>. Each polymer L-vertex or branching point has its own anomalous scaling dimension  $\sigma_L$  (or  $x_L$ ) (see below), and any exponent can be expressed in terms of this basic series  $\{\sigma_L, L \geq 1\}$ . Thus, all the polymer theory is unified in terms of a *single infinite series* of independent and universal critical exponents  $\sigma_L$ , indexed by an integer  $L \geq 1$ . They correspond to the new anomalous critical dimensions associated with the physical branching points of L polymers tied together. In other words, only vertices matter in polymer theory, and one can (almost) forget about the polymer lines connecting them ! L=1 corresponds to the single *extremity* of a linear polymer and is associated with the usual magnetic exponent  $\eta$  (or  $\gamma$ ), L=2 corresponds to a *two-leg* vertex, hence to a single monomer inside a polymer chain, and its anomalous dimension is simply associated with the fractal dimension exponent  $\nu$ . For  $L \geq 3$ , the  $\sigma_L$  are new critical exponents.

All geometrical critical exponents like  $\gamma_L$  in (2) or  $\theta_{L,L'}$  in (3) can be expressed in terms of the  $\sigma_L$ 's. For example the "magnetic" enhancement exponent  $\gamma_L$  for a star reads simply<sup>[4]</sup>

$$\gamma_L - 1 = \sigma_L + L\sigma_1 \quad (9)$$

This is very easy to understand: the core contributes an exponent  $\sigma_L$  while the L=1 extremities in number L contribute a  $\sigma_1$  each (Fig.3). In a similar way, the two-star contact exponent  $\theta_{L,L'}$  in (3) reads<sup>[8]</sup>

$$\theta_{L,L'} = \frac{1}{\nu} (\sigma_L + \sigma_{L'} - \sigma_{L+L'}) \quad (10)$$

which is entirely reminiscent of an operator product expansion (OPE) in field theory. Indeed,  $\sigma_L$  and  $\sigma_{L'}$  are the exponents of the two cores, while  $\sigma_{L+L'}$  is that of the new core made by the fusion of the two stars upon contact (Fig.3).

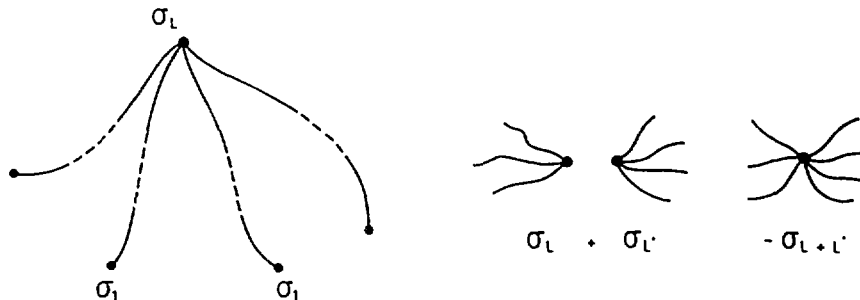


Fig. 3 - Additivity rules for the contributions of the polymer vertices to critical exponents.

From the "vertex structure" of the theory, one can extract new scaling relations between exponents. For instance, from (9)(10) one can eliminate the  $\sigma_L$ 's and get

$$\theta_{L,L'} = \frac{1}{\nu} (\gamma_L + \gamma_{L'} - \gamma_{L+L'} - 1) \quad (11)$$

How does all this fit in the usual field theoretic description of critical phenomena ?

The scaling dimensions  $\sigma_L$  correspond <sup>[4]</sup> actually in  $(\varphi^2)^2$   $O(n)$ -field theory to the anomalous dimensions of successive powers of the field  $\varphi^L$ . Usually, one is interested in the usual statistical mechanics of critical points only in the scaling dimensions of the first powers  $\varphi$  and  $\varphi^2$ , which lead to the usual exponents  $\eta$  and  $\nu$ , i.e. also to  $\gamma = (2-\eta)\nu$ .

But there exists also higher independent scaling dimensions of  $\varphi^L$ , leading to new higher critical exponents, but these operators are not so much considered, perhaps both because they have no immediate physical interpretation, and because their field-theoretic renormalization for high  $L$ 's is not simple to perform. In polymer physics [which corresponds to the particular case  $n \rightarrow 0$ ] the situation is quite different. These exponents have an immediate geometrical meaning, and furthermore they are relatively easily accessible <sup>[8]</sup> by the technique of *direct* renormalization, specially tailored for polymers <sup>[2]</sup>.

Let us now return to polymers. From the basic exponents  $\sigma_L$ , one can now construct the scaling theory of any polymer network <sup>[4]</sup>.

Consider indeed a branched polymer  $\mathcal{G}$  (Fig.4) of arbitrary but fixed topology, made of  $N$  chains, of equal lengths  $N$ , tied together at vertices of functionalities  $L$  ( $L \geq 1$ ) and in number  $n_L$ , and ask about the asymptotic number of self-avoiding configurations  $\mathcal{Z}_{\mathcal{G}}$  of  $\mathcal{G}$ . The answer is <sup>[4]</sup>

$$\mathcal{Z}_{\mathcal{G}} \sim \mu^{N_{\mathcal{G}}} N^{\gamma_{\mathcal{G}}} \quad (N \geq 1) \quad (12)$$

where  $\gamma_{\mathcal{G}}$  is the topology dependent critical exponent :

$$\gamma_{\mathcal{G}} - 1 = -\nu d \ell + \sum_{L \geq 1} n_L \sigma_L \quad (13)$$

where  $\ell$  is the number of physical loops in the polymer network  $\mathcal{G}$ . In (13) one observes the fundamental result that each  $L$ -vertex contributes by its scaling dimension  $\sigma_L$  as many times  $n_L$  it appears in  $\mathcal{G}$ .  $L=1$  corresponds to the free extremities of polymer chains. Note that the loop number is given by Euler relation as <sup>[4]</sup>  $\ell = 1 + \frac{1}{2} \sum_{L \geq 1} n_L (L-2)$ . Hence (13) can also be trivially written as

$$\gamma_{\mathcal{G}} - 1 = -\nu d + \sum_{L \geq 1} n_L \Delta_L \quad (13\text{bis})$$

where the total contribution of an  $L$ -vertex reads  $\Delta_L = \sigma_L - \nu d(L-2)/2$ . A necessary requirement of the theory is that  $\sigma_2 \equiv 0$  since  $L=2$  corresponds to the two-leg vertices, i.e. to any "monomer" or any point in the network, which forms a continuum set, hence  $n_2 = \infty$ . We shall see that this is indeed always the case.

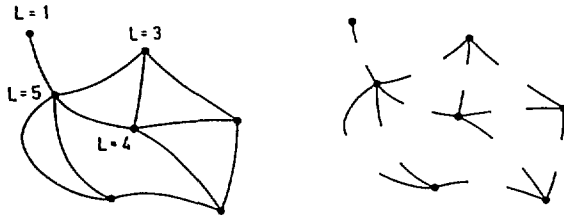


Fig. 4 - A general network  $\mathcal{G}$  and its decomposition into vertices.

All the fundamental polymer scaling theory described here applies in any dimension  $d$ . Above four dimensions ( $d > 4$ ) the self-avoidance effects are irrelevant. Hence all critical exponents are trivial, and we have simply  $\sigma_L = 0, \forall L \geq 1, d > 4$ . When  $d \leq 4, d = 4 - \epsilon$ , the  $\sigma_L$  can be calculated analytically via the  $\epsilon$ -expansion<sup>[8]</sup>

$$\sigma_L = \frac{\epsilon}{8} (2-L) \frac{L}{2} + \left(\frac{\epsilon}{8}\right)^2 \frac{L}{8} (L-2) (8L-21) + O(\epsilon^3) \quad (14)$$

The theory applies equally well for self-avoiding walks in *two dimensions*. We know there the infinite set of exact values<sup>[4]</sup>

$$\sigma_L = (2-L)(9L+2)/64, \quad (L \geq 1) \quad (15)$$

based on conformal invariance<sup>[5]</sup> or Coulomb gas<sup>[3,10]</sup> results.

From these results (14)(15) *all geometrical exponents* for polymers are known, to  $O(\epsilon^2)$  in  $d=4-\epsilon$ <sup>[8]</sup> and *exactly* in 2D<sup>[4,5,6]</sup>. For instance the fore mentioned results (4)(5) for  $\gamma_L$  and (6)(7) for  $\theta_{L,L}$ , are derived from the scalar relations (9)(10) augmented by exact results (14)(15). The exact configuration exponent  $\gamma_{\mathcal{G}}$  of any network  $\mathcal{G}$  is also obtained in 2D as<sup>[4]</sup>

$$\gamma_{\mathcal{G}} = -\frac{1}{2} + \frac{1}{64} \sum_{L \geq 1} n_L (2-L)(9L+50) \quad (16)$$

while the  $O(\epsilon^2)$  value is easily calculable from (13)(14). Thus we see that the polymer theory embodies infinities of critical exponents, which all have a direct geometrical meaning, and are now exactly known in two dimensions, as well as in  $d=4-\epsilon$ .

### III. $O(n)$ MODEL, CONFORMAL INVARIANCE AND COULOMB GAS METHODS

Until now, we have not made clear how the basic exponents  $\sigma_L$  (14) in  $d=4-\epsilon$  and (15) in 2D are obtained. In  $d=4-\epsilon$ , the best way is to work directly with polymers, and use direct renormalization<sup>[2]</sup>. We have so calculated directly<sup>[8]</sup> with the Edwards model the configuration exponents  $\gamma_L$  (2)(5) of star polymers to  $O(\epsilon^2)$ . From these exponents one derives immediately the  $\sigma_L$ 's (Eq.(9)) and reconstructs all other exponents. In two dimensions, one takes a different step and uses the  $O(n)$  model, i.e. the "magnetic" or "field theoretic"

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version of polymers (Of course, one can also use the  $\phi^4$ - $O(n)$  theory in  $d=4-\epsilon$  to get the  $\sigma_L$ 's, but this is not necessary<sup>[8]</sup>). In the 2D  $O(n)$ -model the  $\sigma_L$ 's are then obtained from an equivalent set of exponents  $x_L$ , characteristic of the critical decay of the correlation function of  $L$  polymer lines tied together at their extremities (watermelon topology<sup>[4,5]</sup>) (Fig.5) and with fluctuating lengths. More precisely, let us denote by  $\mathcal{Z}(N_1, \dots, N_L; \vec{X}, \vec{Y})$  the partition function of such an  $L$ -watermelon with extremities fixed at  $\vec{X}$  and  $\vec{Y}$ ,  $N_1, \dots, N_L$  being the lengths of the  $L$  chains. Then we introduce the Laplace transform (generalizing the original de Gennes  $n=0$  mapping for  $L=1$ )

$$G_L(\vec{X}-\vec{Y}, T) = \int_0^\infty \prod_{i=1}^L dN_i e^{-T \sum_{i=1}^L N_i} \mathcal{Z}(\{N_i\}; \vec{X}, \vec{Y}) \quad (17)$$

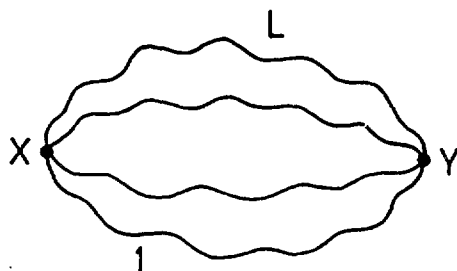


Fig. 5 - The watermelon topology.

This correlation function is, in the  $O(n)$  model, the correlator of  $L$  spins at  $\vec{X}$  and  $L$  at  $\vec{Y}$ , of the type  $\langle S_{\alpha_1} \dots S_{\alpha_L}(\vec{X}) S_{\alpha_1} \dots S_{\alpha_L}(\vec{Y}) \rangle$ , where all the indices  $\{\alpha_i\}$  are different, in such a way that  $L$  lines pair  $\vec{X}$  to  $\vec{Y}$ <sup>[5,8,11,12]</sup>. It can be formally written as<sup>[5,7,11,12]</sup>  $G_L(\vec{X}-\vec{Y}, T) = \langle \phi_L(\vec{X}) \phi_L(\vec{Y}) \rangle_T$  where  $\phi_L$  is the abstract vertex operator of the  $O(n)|_{n=0}$  model, associated with the  $L$  polymer lines (This actually holds for any  $O(n)$  theory<sup>[3,11-13]</sup>). At criticality ( $T=T_c$ ,  $e^{T_c} \equiv \mu$ )  $G_L$  decays algebraically as

$$G_L(\vec{X}-\vec{Y}, T_c) = |\vec{X}-\vec{Y}|^{-2x_L} \quad (18)$$

where  $x_L$  is the scaling dimension of  $\phi_L$ . In particular for  $L=1$  we recover the usual correlation function, and thus  $2x_1 \equiv \eta$ . Now, note that a  $L$ -watermelon network is made of two  $L$ -leg vertices only, thus its configuration exponent  $\gamma$  (13) includes two times the vertex exponent  $\sigma_L$ . Then, using definition (17), and the critical scaling relation (18), one can easily relate  $\sigma_L$  to  $x_L$ . The result reads<sup>[4,8]</sup>

$$\sigma_L = -\nu x_L + L(\nu-1/2) \quad (19)$$

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Hence the families  $\{\sigma_L\}$  or  $\{x_L\}$  are essentially the same. One  $(\sigma_L)$  acts in the "space" of polymer lengths  $N$  (13) while the other  $(x_L)$  acts in physical  $d$ -space. Note also that the contact exponent  $\theta_{L,L}$ , (10) reads in terms of the  $\{x_L\}$ , owing to (19)

$$\theta_{L,L} = x_{L+L} - x_L - x_L \quad (20)$$

which is nothing but a fusion rule in a standard operator product expansion in field theory. Now the scaling dimensions  $x_L$  can be calculated in two dimensions by the so-called coulomb gas technique [3,10,12]. In this method [3], the  $O(n)$  model is transformed into a Solid-on-Solid (SOS) model. Then the  $L$  polymer lines joining  $\vec{X}$  to  $\vec{Y}$  in the correlator  $G_L$  (17) correspond to a dislocation of  $L$  steps between  $\vec{X}$  and  $\vec{Y}$  in the SOS model. As a result, the vertices  $\vec{X}$  and  $\vec{Y}$  interact like vortices in the XY model as  $+$  and  $-$  magnetic monopoles of strengths  $m_x = -m_y = \frac{L}{2}$ . There exist also some electric charges at  $\vec{X}$  and  $\vec{Y}$  which are necessary to reconstruct in the SOS model the correct  $O(n)$  statistical weight. Now, the critical correlation function (18) can be written as

$$G_L(\vec{X}-\vec{Y}, T_c) = e^{-2x_L \ln|\vec{X}-\vec{Y}|} \quad (21)$$

where

$$\begin{aligned} x_L &= \frac{1}{2} g m_x m_y + \frac{1}{2g} e_x e_y \\ &= g \frac{L^2}{8} - \frac{1}{2g} (1-g)^2 \end{aligned} \quad (22)$$

$g$  being the renormalized Coulomb gas coupling constant parametrizing the  $O(n)$  model [3]:  $n = -2 \cos \pi g$ ,  $g \in [1, 2]$ , for  $n \in [-2, 2]$ , where the  $O(n)$  model possesses a critical point. The name "Coulomb gas" comes from this writing of  $G_L$  in (21) as the Gibbs statistical weight of magnetic and electric charges interacting via the logarithmic 2D coulomb potential [3]. For polymers,  $n=0$ , thus the coupling constant is  $g=3/2$ , giving [10,3] in (22)

$$x_L = (9L^2 - 4)/48 \quad (23)$$

From  $\nu=3/4$  [3], one finds the exact value (15) of  $\sigma_L$  in 2D, and all other exponents, QED.

#### DENSE POLYMERS

It is now very interesting to remark, following Nienhuis [3], that other analytic determinations of  $g$  in  $n=-2 \cos \pi g$  exist. The other branch  $g \in [0, 1]$  describes the *low-temperature phase* of the  $O(n)$  model. This low-T phase is also critical, but in a different universality class from that of the critical point! By this, we mean that the geometrical properties (not necessarily the magnetic ones) of the  $O(n)$  loop model [3] are critical. This low-T phase represents a system of *dense* loops (with a loop fugacity  $n$ ) which *fill* a finite fraction of the (infinite) lattice [11,3]. In the polymer case ( $n=0$ ), it represents a dense polymer chain or a dense network (or a finite number of these) which fill the infinite lattice, with a finite density  $f$ . It is important

to note that the universality class does not change with  $f$ , for  $0 < f \leq 1$ , as soon as  $f \neq 0$ ,  $f = 0$  corresponding to usual SAW. Setting for  $n=0$ ,  $g=1/2$  in (22) we find the exact basic scaling dimensions of dense polymers in 2D<sup>[11,12]</sup>

$$x_L^D = (L^2 - 4)/16 \quad (24)$$

These dense polymers are compact, this<sup>[11]</sup>  $\nu^D = 1/d = 1/2$ . Hence, one has from (19)

$$\sigma_L^D = -(L^2 - 4)/32 \quad (25)$$

All the scaling theory for networks given above applies to dense polymers. In particular the contact exponents read from (20)

$$0_{L,L'}^D = (LL' + 2)/8 \quad (26)$$

while the  $\gamma_L$  exponents for dense stars are from (9)

$$\gamma_L^D - 1 = [4 + L(3-L)]/32 \quad (27)$$

and for a network  $\mathcal{G}$ <sup>[11,14]</sup>

$$\gamma_{\mathcal{G}}^D = \sum_{L \geq 1} \frac{1}{32} n_L (2-L)(L+18) . \quad (28)$$

It is interesting to note that this dense system has been much less investigated numerically than usual (dilute) SAW. One can even say that the existence of this new whole critical phase was not really suspected before<sup>[3]</sup>. Let us mention however that the configuration exponents depend on the boundary conditions for a dense system<sup>[12]</sup>, and the interpretation of the universal configuration exponents  $\gamma_L^D$  (27) and  $\gamma_{\mathcal{G}}^D$  (28) is subtle. We refer the reader to ref.[12] for a detailed discussion.

It is also interesting to note that the dense phase for polymers in  $d > 2$  or  $d = 4 - \epsilon$ , has never been studied by renormalization techniques, in contrast to the usual dilute case of SAW, equivalent to a  $\phi^4$  field theory near the critical point, and treated by the  $\epsilon$ -expansion.

#### CONFORMAL INVARIANCE

Two-dimensional critical systems fit into the classification of conformal invariance<sup>[15]</sup>. This also applies to polymers<sup>[5,7,11,12,13]</sup>. Conformal field theories can be parametrized by their (universal) central charge<sup>[15]</sup>  $C$ . Unitary theories with  $C < 1$ , are parametrized by<sup>[15,16]</sup>

$$C = 1 - \frac{6}{m(m+1)} \quad m \in \mathbb{N}^* - \{1,2\} \quad (29)$$

and the scaling dimensions of their (primary) operators are given by the Kac formula<sup>[15,16]</sup>

$$h_{p,q} = \frac{(p(m+1) - mq)^2 - 1}{4m(m+1)} , \quad p, q \in \mathbb{N}. \quad (30)$$

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 $\gamma_1$

We

$$1 \leq p \leq m-1, 1 \leq q \leq p \quad (30bis)$$

Interestingly enough, polymers correspond to the two missing values of integer  $m$  in the set (2.9), namely  $m=2$  (dilute-polymers) or  $m=1$  (dense polymers). The reason is that polymers do not correspond to a unitary theory (the number of components is  $n=0$  and comes from an analytical continuation). The polymer central charges are  $C=0$  (dilute SAW) and  $C=-2$  (dense polymers). The scaling dimensions  $x_L$  (23)(24) of the L-vertex operators  $\phi_L$  are then<sup>[5,11,12]</sup>

$$\begin{aligned} x_L &= 2h_{L/2,0} & \text{for } m=2, C=0 & \text{ (dilute SAW)} \\ x_L &= 2h_{0,L/2} & \text{for } m=1, C=-2 & \text{ (dense SAW)} \end{aligned} \quad (31)$$

Note that this implies some half-integer values<sup>[5]</sup> of indices  $p,q$ , and that the exponents (31) lie *outside* of the minimal block (30bis), which for  $m=2$  is reduced to  $(p,q)=(1,1)$  (identity operator) and is empty for  $m=1$ . This is characteristic of non unitary geometrical models. Note also that exact partition functions can be calculated on the torus, at the critical point or for dense polymers<sup>[11,17]</sup>.

#### IV. SURFACE EXPONENTS

Interesting new phenomena appear when polymer chains approach a surface i.e. a line in two dimensions<sup>[18,19]</sup>. We consider here the case of free boundary conditions, which correspond in the continuum limit to Dirichlet ones<sup>[15]</sup>, or in terms of surface critical phenomena to the ordinary surface transition<sup>[20]</sup>. Consider first a L-watermelon network grafted along the surface by its two extremities (Fig. 4). Its critical correlation function  $G_L$  along the surface will decay like<sup>[7]</sup>

$$G_L^S(\vec{X} - \vec{Y}) = |\vec{X} - \vec{Y}|^{-2x_L^S}$$

where the  $x_L^S$  are new surface scaling dimensions, characteristic of L-line polymer vertices. We have found their exact expressions from conformal invariant<sup>[7]</sup>

$$x_L^S = (3L + 2)L/8 \quad (L \geq 1) \quad (32)$$

and for dense polymers<sup>[11]</sup>

$$x_L^{D,S} = (L - 2)L/8 \quad (L \geq 1)$$

with respective conformal expressions

$$x_L^S = h_{L,1,1} \quad , \quad m = 2, \quad C = 0 \quad \text{(dilute SAW)}$$

$$x_L^{D,S} = h_{1,L,1} \quad , \quad m = 1, \quad C = -2 \quad \text{(dense SAW)}$$

For  $L = 1$  (single polymer chain),  $x_1^S = 5/8$  gives the known values<sup>[18,19]</sup> for  $\gamma_1 = (2 - x_1^S - x_1)\nu$  and  $\gamma_{11} = (2 - 2x_1^S)\nu$ .

The exponents  $x_L^S$  can be also calculated within the  $\epsilon$ -expansion in  $d=4-\epsilon$ . We have found<sup>[8]</sup>

$$x_L^S = L \frac{d}{2} + \frac{\epsilon}{8} L(L-2) + O(\epsilon^2) \quad (33)$$

From the knowledge of these exponents  $x_L^S$  one deduces any topological exponent for polymer networks grafted onto a surface. Consider indeed a polymer network  $\mathcal{G}$  grafted onto a surface by some of its vertices (Fig. 5). (One is fixed, the other move freely along the surface, or in the bulk if they are bulk vertices). Then the total number of configurations of  $\mathcal{G}$  will scale like

$$\mathcal{Z}_{\mathcal{G}} \sim \mu^N N^{\gamma_{\mathcal{G}}^S - 1} \quad (N \rightarrow \infty)$$

where, as before  $N$  is the total number of chains of lengths  $N$ . The exact expression of the new surface configuration exponent  $\gamma_{\mathcal{G}}^S$  is then<sup>[7,8]</sup>

$$\gamma_{\mathcal{G}}^S = \nu \left[ d\nu + (d-1)(\nu_s - 1) - \sum_{L \geq 1} (n_L x_L + n_L^S x_L^S) \right] - N \quad (34)$$

where  $\nu = \sum_{L \geq 1} n_L$  is the total number of bulk vertices,  $\nu_s = \sum_{L \geq 1} n_L^S$  that of surface vertices,  $\{n_L\}$  and  $\{n_L^S\}$  being respectively the number of  $L$ -leg vertice in the bulk or onto the surface<sup>[7,8]</sup>.

Note that the formula (34) is valid in any dimension  $d$ . We have chosen here to write it in terms of scaling dimensions  $x_L, x_L^S$ . A formulation similar to eq.(13) of course exists in terms of the  $\{\sigma_L\}$  exponents<sup>[8]</sup>. Applied in two dimensions, with values (32), this formula gives<sup>[7]</sup> the exact result

$$\gamma_{\mathcal{G}}^S = \frac{1}{4} + \frac{1}{64} \sum_{L \geq 1} n_L (2-L)(9L+50) - \frac{1}{32} \sum_{L \geq 1} n_L^S (9L^2 + 22L - 24) \quad (35)$$

valid for any network grafted onto a surface line in two-dimensions. Note that a surface line can be folded into a wedge of angle  $\alpha$  by the conformal mapping<sup>[18]</sup>  $w(z) = z^{\alpha/\pi}$ . From this one can derive similar exact wedge exponents for networks<sup>[7]</sup>. The scaling dimensions give also access to surface contact exponents : for instance the probability that the core of a star approaches the surface scales like  $P_L(r) \sim r^{\zeta_L}$  where  $\zeta_L = x_L^S - x_L$ . One finds in 2D<sup>[7]</sup>  $\zeta_L = (3L+2)^2/48$ .

## V. HAMILTONIAN WALKS AND DENSE POLYMERS

A Hamiltonian walk (HAW) is a walk which visits all sites of a given lattice once and only once. Thus it is self-avoiding and fills completely the lattice, with a fraction of occupied sites  $f = 1$ . In the limit of an infinite lattice, the system becomes *critical*. One may wonder if the Hamiltonian constraint modifies the universality class. The answer is that the universality class of Hamiltonian walks is exactly that of dense polymers<sup>[12]</sup>. A proof of this is furnished by Hamiltonian walks on the Manhattan oriented lattice, introduced by P.W. Kasteleyn in 1963<sup>[21]</sup>. On this lattice, the horizontal or

vertical lines are oriented in alternating directions. Kasteleyn solved the enumeration problem of single HAW on any finite Manhattan lattice, by a determinant technique. When the  $M \times N$  lattice size goes to infinity, the number of distinct closed HAW configurations on the *periodic* lattice is asymptotically

$$N_H = 2 e^{\frac{G}{\pi} MN} \eta^4(q) \frac{M}{N} \quad (36)$$

where  $\eta(q)$  is Dedekind's function

$$\eta(q) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n), \quad q = e^{-2\pi M/N}$$

and  $G$  is Catalan's constant  $G = 1 - 3^2 + 5^{-2} - \dots$ . The result (36) is actually the simplest example of modular invariance for a critical system on the torus. It can be shown that (36) is related to the free field partition function

$$N_H = 4 Z_1^{-2} \quad (37)$$

where  $Z_1$  is the partition function of a massless free field on the torus<sup>[17]</sup>

$$Z_1 = [\det'(-\Delta)]^{-1/2}$$

( $\Delta$  is the Laplacian). The central charge of a critical free field is  $C = 1$ <sup>[15]</sup> and from (37) one gets that of Manhattan Hamiltonian walks:  $C = -2$ , i.e. exactly that of dense polymers. The equivalence goes much deeper. It is possible to establish an exact equivalence<sup>[12]</sup> of Manhattan HAW's to the critical  $Q$ -state Potts model, in the limit  $Q \rightarrow 0$ , using the standard Baxter-Kelland-Wu polygon decomposition of the Potts model. From this, one can obtain by Coulomb gas techniques all the scaling dimensions  $x_L^D$  (24) and  $x_L^{D,S}$  (32) of dense polymers or of the low- $T$  phase of the  $O(n=0)$  model, as exact exponents of the  $Q=0$  Potts model at its critical point, i.e. of Manhattan HAWs. This shows the *universality* of Manhattan HAWs, which is that of dense SAWs. Many other results can be obtained, like exact partition functions and correlation function of multiple Hamiltonian walks on the Manhattan lattice, for various topological situations. We refer the reader to ref.[17] for an extensive study of this system in the light of conformal invariance and exact techniques.

## VI. THETA POINT

The theta point of polymers is a special tricritical transition point<sup>[1]</sup> where attractive interactions lead to a sudden contraction of the polymer chains. When more attraction is induced by (e.g.) still lowering the solvent temperature, the compact or dense polymer phase is reached.

All the above scaling theory of polymer networks applies to the  $\theta$ -point<sup>[8]</sup>. Since the upper tricritical dimension is  $d=3$ , one can evaluate exactly the network partition function in  $d=3$  by renormalization theory. This has been done recently<sup>[8]</sup>

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$$\chi_g^{\theta} \sim N^{-\frac{3}{2} \mathcal{L}} (\ell_n N)^{-\frac{1}{132} \sum_{L \geq 1} n_L L(L-1)(L-2)} \quad (d=3)$$

where  $\mathcal{L} = 1 + \sum_{L \geq 1} n_L (L-2)/2$  is the number of loops of the network.

In two-dimensions, it has been a long standing problem to finding the exact tricritical  $\theta$  exponents (see ref. [22] and references therein). An annealed disorder model has been proposed in [22] for simulating the attractions inside the SAW by random forbidden hexagons (on the hexagonal 2D lattice). From an exact mapping to the  $O(n)$  loop model, one can show an equivalence to the  $O(n=1)$  model in its critical low-temperature phase, and find the exact exponents

$$\nu = \frac{4}{7}, \quad \gamma = \frac{8}{7}, \quad \phi = \frac{3}{7} \quad (\text{crossover})$$


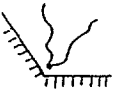


and

$$\gamma_1 = \frac{8}{7}, \quad \gamma_{11} = \frac{4}{7} \quad (\text{surface exponents})$$

The network configuration exponent  $\gamma_g^{\theta}$  (and  $\gamma_g^{\theta, s}$ ) is also known exactly in this case for any network, like in (13)<sup>[22]</sup>. However, the annealed forbidden hexagon model happens to be equivalent to attractive interactions between nearest and a special *subclass* second nearest neighbors on the SAW. This has led to suggesting that several universality classes could exist with different  $\theta$  points, depending on local details of the model<sup>[23]</sup>. Theoretical work is presently devoted to clarifying this question<sup>[23-25]</sup>.

#### VII. COMPARISON TO NUMERICAL SIMULATIONS

It is very interesting to compare the exact predictions in two dimensions with numerical simulations. This will be a good test of two important steps: of the scaling theory decomposing polymer networks into their vertices, and of the exact values of the vertex scaling dimensions given by conformal invariance and Coulomb gas techniques. Over the years, before the advent of conformal theory, numerical information has been accumulated<sup>[18,19,26-34]</sup> (by series or Monte Carlo methods) on exponents for polymers of topologies more complicated than single chains (like star polymers). We give the present numbers and the predictions of the above 2D theory in the table.

L-Stars	Exact	Numerical	
$\gamma(3)$	$\frac{17}{16} = 1.0625$	$1.07 \pm 0.02^a$	
$\gamma(4)$	$\frac{1}{2}$	$0.52 \pm 0.04^a$	
$\gamma(5)$	$-\frac{11}{32} = -0.34375$	$-0.29 \pm 0.04^a$	
$\gamma(6)$	$-\frac{47}{32} = -1.46875$	$-1.33 \pm 0.05^a$	
H-Comb $\gamma(H)$	$\frac{25}{32} = 0.78125$	$0.79 \pm 0.02^b$	
ring closure exponent <sup>[6]</sup>			
$\gamma = (2 + \theta_1)\nu$	$\frac{17}{8} = 2.125$	$2.13 \pm 0.01^c$ $2.10 \pm 0.10^d$ $2.15 \pm 0.30$ - 0.15	
Contact exponent			
$\theta_1$	$\frac{5}{6} = 0.833\dots$	$0.84 \pm 0.01^c$ $0.84 \pm 0.13^d$	
$\theta_2$	$\frac{19}{12} = 1.5833\dots$	$1.93 \pm 0.27^e$	
Surface and wedge exponents			
 $\gamma_1$	$\frac{61}{64} [18,19] = 0.9531\dots$	$0.945 \pm 0.005^f$ $0.956 \pm 0.014^g$ - 0.006	
	$\gamma_2(\pi/2)$	$-\frac{37}{32} = -1.15625$	$-1.15 \pm 0.05^h$
	$\gamma_2(2\pi/3)$	$-\frac{13}{32} = -0.40625$	$-0.4 \pm 0.05^h$
	$\gamma_2(\pi)$	$\frac{11}{32} = 0.34375$	$0.35 \pm 0.05^h$
	$\gamma_3(\pi)$	$-\frac{53}{64} = -0.828\dots$	$-0.82 \pm 0.05^h$
	$\gamma_{1,3}(\pi)$	$\frac{43}{64} = 0.671\dots$	$0.68 \pm 0.05^h$

- a Wilkinson et al.<sup>[26]</sup>  
 b Gaunt et al.<sup>[27]</sup>  
 c Trueman, Whittington<sup>[28]</sup>  
 d Guttman, Sykes<sup>[29]</sup>  
 e Redner<sup>[30]</sup>  
 f Barber et al.<sup>[31]</sup>  
 g De Bell and Essam<sup>[32]</sup>  
 h Colby et al.<sup>[33]</sup>

Note also that a careful series enumeration allowed Guttman and Torrie<sup>[19]</sup> to guess the correct exact formulas like  $\gamma_1(\alpha)$  of a single chain attached in a wedge, conjectured by Cardy from conformal invariance<sup>[18]</sup>, and which is a particular case of our general formula (35) for networks grafted onto a surface on in a wedge in 2D.

$$\mathcal{F} \sim \mu^N N^{\gamma-1} \quad (N \rightarrow \infty) \quad (1)$$

where  $N$  is the length of the polymer (or its number of monomers),  $\mu$  is the growing factor, or "effective connectivity constant" (it is non universal and lattice dependent) and  $\gamma$  is a *universal* critical exponent (the magnetic susceptibility exponent of the  $O(n)$  model, for  $n \rightarrow 0$ ). The end-to-end distance of the chain scales like

$$R^2 \sim N^{2\nu} \quad (N \rightarrow \infty)$$

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In conclusion, a general scaling theory has been derived for polymer networks where it is shown that vertices only matter, each generating its own anomalous dimension. Next, these scaling dimensions can be found by renormalization techniques in  $d = 4 - \epsilon$  dimensions, or exactly in  $d=2$  by conformal invariance. This provides infinities of exact critical exponents for polymers, which are universal and cover all possible topological situations, and various universality classes: dilute SAW, dense polymers, Hamiltonian walks, polymers at the disorder  $\Theta$ -point...

Similar scaling and conformal methods can be applied to other geometrical critical problems, like the windings of SAW in 2D<sup>[35]</sup>, or the geometrical properties of percolation clusters<sup>[36]</sup>.

It is a pleasure to thank F. David and H. Saleur with whom I collaborated on these subjects.

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Fig.1 - a) L-arm star b) Contact of the cores of L and L' stars.

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