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S. Srinivas Rau

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ON AN ISOSPECTRALITY QUESTION OVER COMPACT RIEMANN SURFACES *

S. Srinivas Rau **
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

It is proved that for a generic compact Riemann surface X of genus $g > 1$, (i) there are at most 2^{2g} unitary characters of $\pi_1(X)$ whose associated line bundles have laplacians of identical spectrum, (ii) generating cycles for $\pi_1(X)$ can be chosen to be closed geodesics whose length multiplicity is 1.

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** Permanent address: School of Mathematics and Computer/Information Sciences, University of Hyderabad, Hyderabad – 500 134, India.

Let X be a compact Riemann surface of genus > 1. Let $\chi : \pi_1(X) \to U(1)$ be a unitary character and $E_\chi \to X$ the associated line bundle of χ. The standard laplacian Δ_χ on the smooth sections of E_χ has a purely discrete spectrum $\{0 \le \lambda_0 \le \lambda_1 \le \ldots \to \infty\}$ (cf. Selberg [3]). In [5], the question was posed whether distinct characters χ could have the same spectrum for Δ_χ. We prove here the following

Proposition : For a generic compact Riemann surface X of genus $g > 1$, there exist at most 2^{2g} unitary characters χ of $\pi_1(X)$ with a given spectrum $\{0 \le \lambda_0 \le \lambda_1 \le \ldots \to \infty\}$ for Δ_χ.

Here 'generic' refers to an open dense subset of the standard space of marked Riemann surfaces T_g (cf. Wolpert [4]). The Proposition follows readily from Lemmas 1 and 2 below.

We first recall some standard facts and notation (cf. Selberg [3]). Let X be a compact Riemann surface of genus $g > 1$. $\pi_1(X)$ is generated by $2g$ elements $a_1, a_2, \ldots, a_{2g}$ with the single relation

$$\prod_{i=1}^{g} (a_{2i-1} a_{2i} a_{2i-1}^{-1} a_{2i}^{-1}) = identity\ Id \tag{1}$$

$\pi_1(X)$ may be regarded as imbedded in $PSL_2(\Re)$ as a discrete subgroup Γ, by the Uniformisation Theorem. Γ acts on the upper half plane $\{z = x+iy \mid y > 0\}$ by Moebius maps. Each element γ of Γ is conjugate in $PSL_2(\Re)$ to a unique magnification map $z \mapsto N(\gamma)z$ with $N(\gamma) > 0$. $N(\gamma)$ is called the norm of γ. $\gamma \mapsto N(\gamma)$ is a class function on Γ. Consider the pullback group $\hat{\Gamma}$ of Γ under the natural covering map $SL_2(\Re) \to PSL_2(\Re)$. Let $\hat{\gamma}$ denote the matrix of positive trace in $\hat{\Gamma}$ corresponding to $\gamma \in \Gamma$ ($-\hat{\gamma}$ also yields γ.) $\hat{\gamma}$ is conjugate in $SL_2(\Re)$ to the positive diagonal matrix (a_{ij}) with $a_{11} = N(\gamma)^{\frac{1}{2}}$, $a_{22} = N(\gamma)^{-\frac{1}{2}}$ and zeroes elsewhere. Defining $Tr\ \gamma = Tr\ \hat{\gamma}$ one has the relations among traces

$$Tr\ \gamma = N(\gamma)^{\frac{1}{2}} + N(\gamma)^{\frac{-1}{2}} = Tr\ \gamma^{-1}$$

The multiplicity of γ is defined to be the number of conjugacy classes $\{\gamma'\}$ in Γ such that $Tr\ \gamma = Tr\gamma'$. If $\gamma \ne identity$, then γ and γ^{-1} are never conjugate in Γ and the multiplicity of such a γ is at least 2.

The centralizer Z_γ of γ in Γ , $\gamma \ne identity$ is an infinite cyclic group generated by a unique element γ_0 such that γ is a positive power of γ_0. Such a generator γ_0 is called a primitive element of Γ.

Lemma 1: If $\gamma' \in \Gamma$ is primitive and of multiplicity 2, then the spectrum of Δ_χ determines $\chi(\gamma')$ upto complex conjugation .

Proof : We imitate Randol in using the Selberg Trace Formula for suitable pairs of functions (h_1, h_2) (cf. Randol [2]). For a wide class of functions h_1 the Formula states (cf. [3] p74)

$$\sum_n h_1(r_n) = \frac{area\ (X)}{4\pi}\int_0^\infty r\ tanh\ \pi r\ h_1(r)dr + 2\sum_{\{\gamma\}} \frac{\chi(\gamma) log\ N(\gamma_0)}{N(\gamma)^{\frac{1}{2}} - N(\gamma)^{\frac{-1}{2}}} h_2(log\ N(\gamma))$$

where $r_n^2 + \frac{1}{4} = \lambda_n$, h_2 is the Fourier transform of h_1 and the sum on the right is over all the non-identity conjugacy classes in Γ.

Let $\alpha = log\ N(\gamma\prime)(= log\ N(\gamma_0\prime))$. Let h_2 be an even smooth function $\Re \to \Re$, of compact support centred at $\pm\alpha$. Suppose no number $log\ N(\gamma), (\{\gamma\} \neq \{\gamma\prime^{\pm 1}\})$ lies in the support of h_2. (This is possible since these numbers form a discrete subset of $\Re\backslash\{0\}$.) Normalise h_2 as follows:

$$h_2(\pm\alpha) = \frac{N(\gamma\prime^{\frac{1}{2}}) - N(\gamma\prime^{\frac{-1}{2}})}{log\ N(\gamma\prime)}$$

Let h_1 be the inverse Fourier transform of h_2. This choice of (h_1, h_2) in the Selberg Trace Formula yields

$$\chi(\gamma\prime) + \chi(\gamma\prime^{-1}) = \sum_n h_1(r_n) - \frac{area\ (X)}{4\pi}\int_0^\infty r\ tanh\ \pi r\ h_1(r)dr = a \ \ (say)$$

Since χ is a unitary character,

$$\chi(\gamma\prime^{-1}) = \chi(\gamma\prime)^{-1} = \overline{\chi(\gamma\prime)} = a - \chi(\gamma\prime)$$

The decomposition of a above as a sum of two complex numbers of unit modulus is unique. This proves Lemma 1.

Lemma 2: For a generic compact Riemann surface X of genus $g > 1$, it is possible to choose generators $a_1, a_2, \ldots, a_{2g}$ of $\Gamma = \pi_1(X)$ such that each a_i is of multiplicity 2.

Proof : We use results of Fricke-Klein, Ahlfors and Wolpert on the manifold structure of T_g (cf. [5] § 3) and of Horowitz (cf. [1] § 3,4) on formulas for traces of words in the generators.

Step 1. For each word W in the generators $\alpha_1, \alpha_2, \ldots, \alpha_{2g}$ of a subgroup G of $SL_2(\Re)$, there exists a polynomial with integer coefficients

$$P_W(Tr\ \alpha_i, Tr\ \alpha_i\alpha_j, \ldots, Tr\ \alpha_1\alpha_2\ldots\alpha_{2g}) = Tr\ W$$

P_W is universal in the sense that it is the same for all representations $A_i \mapsto \alpha_i$ of the free group on $2g$ generators $F_{2g} = \langle A_1, \ldots A_{2g}\rangle$. ([1] 3.1, 3.4, § 4 p641)

Step 2 : A group of the type $\hat{\Gamma} \subset SL_2(\Re)$ is the image of a representation of F_{2g} :

$$A_i \mapsto \hat{a_i}, \text{ with } \prod_{i=1}^{g}[\hat{a_{2i-1}}, \hat{a_{2i}}] = -Id \ \text{---------}(2)$$

Horowitz' polynomial P_W simplifies over this class of representations by use of (2) .

Step 3 : A subcollection of $(6g-6)$ traces from $\{Tr\ a_i, Tr\ a_ia_j, Tr\ a_ia_ja_k\}_{i<j<k}$ gives local coordinates on an open subset of T_g making T_g a manifold homeomorphic to $\Re^{6g-6}$ (Theorem of Fricke-Klein [4] Th 3.2).

Step 4 : Let W be a non-trivial word, i.e., W is not conjugate to $a_i^{\pm 1}$ for any $i = 1, 2, \ldots, 2g$, after applying the condition (1). Consider (over an open subset of T_g) for a fixed $i \in \{1, 2, \ldots, 2g\}$ the eqation

$$P_W - Tr\ a_i = 0$$

Recall Ahlfors' normalisation $a_1(0) = 0$, $a_1(\infty) = \infty$, $a_2(1) = 1$. ([5] § 3)

Case 1 : If $i \neq 1, 2$, $Tr\ a_i$ is a coordinate function on an open subset U.

Case 2 : If $i = 1$ or 2, $Tr\ a_i$ is an algebraic function (of the local coordinates) determined by (2).

Claim : Now the above equation can hold identically on an open subset of T_g if and only if W is conjugate to $a_i^{\pm 1}$.

For clearly such an identity can hold only in Case 2. But then by (2) $Tr\ a_1$, $Tr\ a_2$ are $\pm$ traces of words which are inverses of a_1, a_2 respectively.

By hypothesis W is a non-trivial word and so the zero set

$$H_i(W) = \{P_W - Tr\ a_i = 0\}$$

is a proper hypersurface in an open subset of T_g. But then there are only countably many non-trivial words W and $i \in \{1, 2, \ldots, 2g\}$. Thus there are only countably many hypersurfaces $H_i(W)$. The required genericity holds outside the union of these hypersurfaces. This proves Lemma 2 and the Proposition once we observe that the open subsets U of Case 1 cover a dense open subset of T_g.

Recall ([2]) that the length spectrum of X is the list of numbers $log\ N(\gamma)$ as γ varies in $\Gamma = \pi_1(X)$ over conjugacy classes of norm > 1. Lemma 2 is equivalent to

Corollary : For a generic X in T_g there exist generators $a_1, a_2, \ldots, a_{2g}$ for $\pi_1(X)$ such that no number $log\ N(a_i)$ is repeated in the length spectrum.

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