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THE COUSIN PROBLEMS IN THE VIEWPOINT OF PARTIAL DIFFERENTIAL EQUATIONS *

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ABSTRACT

In this paper we consider the Cousin problems for overdetermined systems of partial differential equations, which are generalizations of the Cauchy–Riemann system. The general methods for solving these problems are given. Applying the given methods we can solve the Cousin problems for many important systems in theoretical physics.

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1. INTRODUCTION

It is well known, that the Mittag-Leffler, about the possibility of constructing a meromorphic function having prescribed poles, and the Weierstrass Theorem of constructing a holomorphic function having prescribed zeros in the Complex Plane C, both play a very important role in the Complex Analysis of one variable.

For several complex variables these theorems are generalized as the Additive and Multiplicative Cousin Problems respectively (we call them the Cousin Problems 1 and 2 for short), which allow to construct a meromorphic function in a domain Ω of C^n ($n \succ 2$) by its local prescribed singularities and to construct a holomorphic function in Ω by its local prescribed zeros.

The solvability of these Problems is in close connection with the geometrical properties of $\partial\Omega$ and the solvability of many other important problems in the Mathematical Analysis, as the $\overline{\partial}$-Neumann Problem,... (see[5,6]).

From the viewpoint of partial differential equations, holomorphic functions are considered as solutions of a special system of partial differential equations, namely the Cauchy-Riemann system (in one or several complex variables):

(1.1) $\frac{\partial W}{\partial \overline{z}_j} = 0$

j = 1,...,n.

Therefore many mathematicians have tried to generalize the beautiful properties of holomorphic functions for solutions of a more general system of partial differential equations.

One of these directions is the theory of I.N.Vekua and L.Bers about the generalized analytic functions, namely the solutions of system:

(1.2) $\frac{\partial W}{\partial \overline{z}} = A(z)W + B(z)\overline{W}$

in the complex plane C.

Many important properties of holomorphic functions in one complex variable as the Identity Theorem, Liouville's Theorem, the Cauchy Integral formula... are proved for solutions of (1.2),(see[4,19]).

For several complex variables there are various generalizations of the theory of holomorphic functions as follows.

For solutions of the Vekua-system:

(1.3) $\frac{\partial W}{\partial \overline{z}_j} = A_j(z)W + B_j(z)\overline{W}$

$j = 1,...,n.$

many properties of holomorphic functions on several complex variables as the Identity theorem, the Hartogs extension theorem...are proved. (see [8, 14, 15, 17, 18]).

Applying the theorem of Newlander-Nirenberg (see[16]), the solutions of Beltrami-system:

(1.4) $\frac{\partial W}{\partial \bar{z}_j} = \sum_{k=1}^{n} B_{jk}(z) \frac{\partial W}{\partial z_k}$

$j = 1,...,n;$

in a domain Ω of C^n can be considered as holomorphic functions on a complex manifold. Therefore, many properties of holomorphic functions on several complex variables are true for solutions of (1.4).(see[7,10]).

Another important generalization of the theory of holomorphic functions is the theory of functions with values in a Clifford algebra developed by R.Delanghe, F.Sommen,P.Gilbert and many other authors,(see:[1,2,3, 10,11]). In this theory the generalized Cauchy-Riemann operator

$$D = \frac{\partial}{\partial x_1} e_1 + ... + \frac{\partial}{\partial x_n} e_n$$

is introduced and the solutions of systems:

(1.5) $Df = 0$

and

(1.6) $D_{x^{(k)}} = 0$

$k = 1,...,m;\ x^{(k)} = (x_1^{(k)}, ..., x_{n_k}^{(k)}), 1 \leq n_k \leq n.$

are considered. The solutions of (1.5) and (1.6) are called the monogenic and multi-monogenic (or biregular) functions, respectively.

Many properties of holomorphic functions in one complex variable (resp. in several complex variables) are proved for monogenic (resp. multi-monogenic) functions.

In the present paper we shall discuss the generalization of the Cousin problems for a general system of partial differential equations, which contains as particular cases the above quoted systems (1.1)-(1.6), namely the following system:

$$(1.7) \quad T^{(l)}u = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk}^{(l)}(x) \frac{\partial u_j}{\partial x_k} + \sum_{j=1}^{m} b_j^{(l)}(x) u_j = 0$$

$l = 1,...,L.$

In the second section, after introducing some definitions and notations the generalized Cousin problems for system (1.7) are stated. The general methods for solving these problems are discussed in section 3 and the applications of these methods to various special systems are given in section 4.

2. PRELIMINARIES

From now on we consider the system (1.7) in a domain G of the Euclidean space R^n.

Suppose that

$$a_{jk}^{(l)}(x) \in c^{\infty}(G), \quad b_j^{(l)}(x) \in c^{\infty}(G)$$

for all j,k,l.

Let Ω be a subset of G,

Definition 2.1:

The vector function

$$u(x) = \{u_1(x), ..., u_m(x)\}$$

is called a regular solution of (1.7) in Ω iff $u_j(x) \in c^1(\Omega)$ for j = 1,...,m and satisfy (1.7).

We denote by $R(\Omega)$ the set of all regular solutions in Ω.

Definition 2.2:

The point $x = (x_1, ..., x_n) \in R^n$ is called a regular point for the solution u(x) iff there exists an open neighborhood U of x such that $u \in R(U)$. Otherwise x is called the singular point of u(x).

The set of solutions u(x) having singularities in Ω is denoted by $\tilde{R}(\Omega)$.

Let $\{V_\alpha, \alpha \in I\}$ be an open covering of $\Omega(\subset G)$. We shall consider the following problems:

Problem 1:

In every V_α is given $u^{(\alpha)} \in \tilde{R}(V_\alpha)$ such that

(2.1) $u^{(\alpha)} - u^{(\beta)} \in R(V_\alpha \cap V_\beta)$

for all α, β with $V_\alpha \cap V_\beta \neq 0$.

Find a vector function $u \in \tilde{R}(\Omega)$, so that

(2.2) $u - u^{(\alpha)} \in R(V_\alpha)$ for $\alpha \in I$.

Problem 2:

In every V_α is given $u^{(\alpha)} \in R(V_\alpha)$ such that

(2.3) $\frac{u^{(\alpha)}}{u^{(\beta)}} \in R(V_\alpha \cap V_\beta)$.

Find $u \in R(\Omega)$ so that

(2.4) $\frac{u}{u^{(\alpha)}} \in R(V_\alpha), \alpha \in I$.

Remark:

Since systems (1.1)-(1.6) are the special cases of (1.7), Problems 1 and 2 are the generalizations of Mittag-Leffler Theorem (the Cousin problem 1) and of Weierstrass theorem (the Cousin problem 2) respectively. In case of Problem 2 it is assumed, that the quotient of two solutions is a solution, too.

3.GENERAL METHODS TO SOLVE PROBLEMS 1 AND 2

As it is mentioned in the introduction, this section is devoted to several methods for solving Problems 1 and 2.

3.1. Analytical Method

Let $\{u^{(\alpha)}\}$ be the given data of Problem 1. We set

(3.1) $g^{(\alpha\beta)} = u^{(\alpha)} - u^{(\beta)}$

then

(3.2) $g^{(\alpha\beta)} \in R(V_\alpha \cap V_\beta)$

(3.3) $g^{(\alpha\beta)} + g^{(\beta\alpha)} = 0$ in $V_\alpha \cap V_\beta$.

(3.4) $g^{(\alpha\beta)} + g^{(\beta\gamma)} + g^{(\gamma\alpha)} = 0$ in $V_\alpha \cap V_\beta \cap V_\gamma$

for $\alpha, \beta, \gamma \in I$.

If there exists $g^{(\alpha)} \in R(V_\alpha)$ such that

(3.5) $g^{(\alpha\beta)} = g^{(\beta)} - g^{(\alpha)}$ in $V_\alpha \cap V_\beta$ for all $\alpha, \beta \in I$; then $\{g^{(\alpha)} + u^{(\alpha)}\}$ define a vector function $u \in \tilde{R}(\Omega)$, which solves the Problem 1. Hence, the Problem 1 reduces to the following problem:

Problem 1':

Assume, that in every $V_\alpha \cap V_\beta$ is given

$$g^{(\alpha\beta)} \in R(V_\alpha \cap V_\beta),$$

which satisfy conditions (3.3) and (3.4). Find

$$g^{(\alpha)} \in R(V_\alpha)$$

such that (3.5) holds.

Remark 3.1:

If Problem 1' is solvable on Ω, then Problem 1 can be solved on it.

In the sequel instead solving Problem 1 we shall solve Problem 1'.

Let $\{\varphi_j\}$ be a subordinate partition of Unity for the covering $\{V_\alpha\}$ with $\varphi_j \in c_0^\infty(V_{\alpha_j})$.

Further, set:

(3.6) $h^{(\beta)} = \sum_j \varphi_j g^{(\alpha_j\beta)}$ for fixed $\beta \in I$;

then

$$h^{(\beta)} \in R(V_\beta)$$

and

$$h^{(\beta)} - h^{(\alpha)} = \sum_j \varphi_j (g^{(\alpha_j\beta)} - g^{(\alpha_j\alpha)}) = \sum_j \varphi_j (g^{(\alpha\alpha_j)} + g^{(\alpha_j\beta)}) = \sum_j \varphi_j g^{(\alpha\beta)} = g^{(\alpha\beta)}.$$

Hence it follows

$$(3.6)' \quad T^{(l)} h^{(\beta)} - T^{(l)} h^{(\alpha)} = T^{(l)} g^{(\alpha\beta)} = 0$$

in $V_\alpha \cap V_\beta$ for l = 1,...,L.

This means that $\{T^{(l)} h^{(\beta)}\}$ define a function $f^{(l)} \in c^\infty(\Omega)$ for all l = 1,...,L.

Now let us consider the system:

(3.7) $T^{(l)} W = -f^{(l)}; l = 1, ..., L$

Theorem 3.1:

If the system (3.7) is solvable in c^∞, then Problem 1' is solvable.

Proof:

By assumption, there exists a vector function

$$W = \{W_1(x), ..., W_m(x)\}$$

vith $W_j(x) \in c^\infty(\Omega)$, which is a solution of the system (3.7). Now set:

(3.8) $g^{(\alpha)} = h^{(\alpha)} + W$ in V_α.

We have:

$$T^{(l)} g^{(\alpha)} = T^{(l)} h^{(\alpha)} + T^{(l)} W = T^{(l)} h^{(\alpha)} - f^{(l)} = 0; l = 1, ..., L$$

Ience

$$g^{(\alpha)} \in R(V_\alpha), \forall \alpha \in I$$

ecause of (3.6)' and (3.8) we obtain

$$g^{(\beta)} - g^{(\alpha)} = (h^{(\beta)} + W) - (h^{(\alpha)} + W) = h^{(\beta)} - h^{(\alpha)} = g^{(\alpha\beta)},$$

ıerefore $\{g^{(\alpha)}\}$ solve the Problem 1'. Q.e.d.

3.2. The Algebra-topological Method:

Let R be the sheaf of germs of regular solutions of system (1.7) and $\tilde{R}$ be the sheaf of germs of solutions having singularities over Ω. Consider the exact sequence:

$$0 \longrightarrow R \xrightarrow{i} \tilde{R} \xrightarrow{\varphi} \tilde{R}/R \longrightarrow 0, \tag{3.9}$$

where i is the identity map and φ is the natural map in the quotient sheaf.

From (3.9) we obtain the following exact sequence (see [4],p.180)

$$0 \longrightarrow \Gamma(\Omega, R) \longrightarrow \Gamma(\Omega, \tilde{R}) \longrightarrow \Gamma(\Omega, \tilde{R}/R) \longrightarrow H^1(\Omega, R) \longrightarrow \tag{3.10}$$

$H^1(\Omega, \tilde{R}) \longrightarrow \dots,$

where $\Gamma(\Omega, R)$ is the set of sections of sheaf R over Ω, $H^q(\Omega, R)$ is the q-th cohomology groups of Ω with coefficients in the sheaf R. The others are similarly denoted.

Theorem 3.2:

Assume that

(3.11) $H^1(\Omega, R) = 0$

then the Problem 1' can be solved on Ω.

Proof:

From (3.10) and (3.11) it follows that the sequence:

$$0 \longrightarrow \Gamma(\Omega, R) \longrightarrow \Gamma(\Omega, \tilde{R}) \xrightarrow{\varphi} \Gamma(\Omega, \tilde{R}/R) \longrightarrow 0 \tag{3.12}$$

is exact. Hence the map

$$\varphi : \Gamma(\Omega, \tilde{R}) \longrightarrow \Gamma(\Omega, \tilde{R}/R)$$

is surjective. Let $\{u^{(\alpha)}\}$ be the given data of the Problem 1, then

$$u^{(\alpha)} \in \Gamma(V_\alpha, \tilde{R})$$

and

$$\varphi u^{(\alpha)} \in H^0(V_\alpha, \tilde{R}/R); \varphi u^{(\alpha)} = \varphi u^{(\beta)}$$

in $V_\alpha \cap V_\beta$.

Thus we can define a section $\sigma \in H^0(\Omega, \tilde{R}) = \Gamma(\Omega, \tilde{R})$ by $\sigma = \varphi u^{(\alpha)}$. Then there exists $u \in H^0(\Omega, \tilde{R}) = \Gamma(\Omega, \tilde{R})$ such that $\varphi u = \sigma$. Thus $\varphi u =$

$\varphi u^{(\alpha)}$ in V_α, so $u - u^{(\alpha)}$ is a regular solution in V_α. Hence u is the required solution of Problem 1. Q.e.d.

Denoting by R^* the sheaf of the germs of invertible regular solutions of (1.7) and consider the sequence

$$0 \longrightarrow C \longrightarrow R \longrightarrow R^* \longrightarrow 0. \tag{3.13}$$

Suppose that the sequence (3.13) is exact, then the sequence

$$H^1(\Omega, R) \longrightarrow H^1(\Omega, R^*) \longrightarrow H^2(\Omega, C) \longrightarrow H^2(\Omega, R) \longrightarrow \dots \tag{3.14}$$

is exact.

Now if

(3.15) $H^1(\Omega, R) = H^2(\Omega, R) = 0$

then

(3.16) $H^1(\Omega, R^*) \approx H^2(\Omega, C)$.

Further, suppose that

(3.17) $H^2(\Omega, C) = 0$,

then from (3.16) it follows

(3.18) $H^1(\Omega, R^*) = 0$

On the other hand from the exact sequence

$$0 \longrightarrow R^* \longrightarrow R \longrightarrow R/R^* \longrightarrow 0$$

we obtain the following exact sequence

$$\Gamma(\Omega, R) \longrightarrow \Gamma(\Omega, R/R^* \longrightarrow H^1(\Omega, R^*) \longrightarrow \dots$$

Because of (3.18) the sequence

$$\Gamma(\Omega, R) \longrightarrow \Gamma(\Omega, R/R^*) \longrightarrow 0$$

is exact. Hence the map

$$\varphi : \Gamma(\Omega, R) \longrightarrow \Gamma(\Omega, R/R^*)$$

is surjective. The given data $\{g^{(\alpha)}\}$ of Problem 2 define an element $\sigma^* \in \Gamma(\Omega, R/R^*)$. Then there exists an element $u \in \Gamma(\Omega, R)$ such that $\sigma^* = \varphi u$. It is easily to verify that u is the required solution of the Problem 2. Thus we have proved the following

Theorem 3.3:

If the conditions (3.15) and (3.17) hold, then the Problem 2 can be solved on Ω, for every given data $\{g^{(\alpha)}\}$, which satisfy the condition (2.3)

3.3. The reduction to The Cousin Problems in Complex Analysis:

Suppose that there exist the maps

$$T : R(\Omega) \longrightarrow H(\Omega)$$

$$\tilde{T} : \tilde{R}(\Omega) \longrightarrow M(\Omega)$$

which satisfy the following properties:

(i) T and $\tilde{T}$ are the "one-one" maps.

(ii) There exist T^{-1} and $\tilde{T}^{-1}$

(iii) $T(u^{(1)} \pm u^{(2)}) = T(u^{(1)}) \pm T(u^{(2)})$

$$\tilde{T}(u^{(1)} \pm u^{(2)}) = \tilde{T}(u^{(1)}) \pm \tilde{T}(u^{(2)})$$

(iv) $T(u^{(1)}.u^{(2)}) = T(u^{(1)}).T(u^{(2)})$

$$\tilde{T}(u^{(1)}.u^{(2)}) = \tilde{T}(u^{(1)}).\tilde{T}(u^{(2)})$$

where $u^{(1)}, u^{(2)} \in R(\Omega)$ ore $\in \tilde{R}(\Omega)$

(v) $\tilde{T} = T$ in $R(\Omega)$

$\tilde{T}^{-1} = T^{-1}$ in $H(\Omega)$.

Hereby $H(\Omega)$ is the space of holomorphic functions and $M(\Omega)$ is the space of meromorphic functions in Ω.

Theorem 3.4:

If there exist the operators T and $\tilde{T}$, which satisfy properties (i),(ii),(iii) and (v), and if Ω is a domain of holomorphy; then the Problem 1 can be solved on it.

Proof:

Let $\{g^{(\alpha)}\}$ be the given data of the Problem 1, we set

(3.19) $\Phi^{(\alpha)} = \tilde{T}(g^{(\alpha)}), \alpha \in I.$

Then

$$\Phi^{(\alpha)} \in M(V_\alpha)$$

$$\Phi^{(\alpha)} - \Phi^{(\beta)} \in H(V_\alpha \cap V_\beta)$$

for all $\alpha, \beta \in I$.

This means that $\{\Phi^{(\alpha)}\}$ are the given data of the Cousin problem 1, which can be solved on the domain of holomorphy Ω. Hence there exists a function

$$\Phi \in M(\Omega)$$

such that

$$\Phi - \Phi^{(\alpha)} \in H(V_\alpha).$$

Now we set

(3.20) $u = \tilde{T}^{-1}(\Phi)$

It is $u \in \tilde{R}(\Omega)$

and

$$u - u^{(\alpha)} = \tilde{T}^{-1}(u) - \tilde{T}^{-1}(u^{(\alpha)}) = \tilde{T}^{-1}(u - u^{(\alpha)}) = T^{-1}(u - u^{(\alpha)}) \in R(V_\alpha).$$

Thus u is the required solution of Problem 1. Q.e.d.

Theorem 3.5:

Suppose that there exists operators T and $\tilde{T}$, which satisfy properties (i),(ii),(iv) and (v), Ω is a domain of holomorphy and

(3.21) $H^2(\Omega, Z) = 0$

Then Problem 2 can be solved on it.

Proof:

Let $\{g^{(\alpha)}\}$ be the given data of Problem 2, set

$$\Phi^{(\alpha)} = T(g^{(\alpha)})$$

then

$$\frac{\Phi^{(\alpha)}}{\Phi^{(\beta)}} = \frac{T(g^{(\alpha)})}{T(g^{(\beta)})} \in H(V_\alpha \cap V_\beta)$$

Hence $\{\Phi^{(\alpha)}\}$ are the given data of the Cousin Problem 2, which, by assumptions of the theorem, can be solved on Ω. Let Φ be a solution of the Cousin problem 2, we set

$$g = T^{-1}(\Phi)$$

Then

$$\frac{g}{g^{(\alpha)}} = \frac{T^{-1}(\Phi)}{T^{-1}(\Phi^{(\alpha)})} = T^{-1}(\frac{\Phi}{\Phi^{(\alpha)}}) \in R(V_\alpha).$$

Hence g is the required solution of the Problem 2. Q.e.d.

4. APPLICATIONS

In this section we shall give some examples in which the methods given in section 3 can be applied for solving the Problems 1 and 2.

4.1 As the first example we consider the Problem 1 in the theory of biregular functions with values in a Clifford Algebra.

Let $\mathcal{A}$ be the real Clifford algebra over R^n, then a general element $a \in \mathcal{A}$ may be written in the form

(4.1) $\quad a = \sum_{A \subset N} a_A e_A \quad N = \{1..., n\}; \quad a_A \in R$

where

$$e_A = e_{\alpha_1} ... e_{\alpha_h}; A = \{\alpha_1, ..., \alpha_h\}; 1 \leq \alpha_1 \prec ... \prec \alpha_h \leq n.$$

The product in $\mathcal{A}$ is determined by the relations

(4.2) $\quad e_i e_j + e_j e_i = -2\delta_{ij}; i = 1...n$

$(e_1, ..., e_n)$ being an orthogonal basis of R^n.

In the following Ω will denote an open subset of the Euclidean space $R^p \times R^q$, where $1 \leq p, q \leq n; 1 \leq p + q \leq n$.

We consider functions f defined on Ω and taking values in $\mathcal{A}$. Those are of the following form

$$f : \Omega \longrightarrow \mathcal{A}$$

$$(x, y) \longrightarrow f(x, y) = \sum_A f_A(x, y) e_A; x = (x_1, ..., x_p); y = (y_1, ..., y_q)$$

We denote by $c^l(\Omega; \mathcal{A}); c^\infty(\Omega; \mathcal{A})$ the set of functions f such that $f_A \in c^l(\Omega), f_A \in c^\infty(\Omega)$, respectively.

next we introduce the generalized Cauchy-Riemann operators

$$D_x := \sum_{i=1}^{p} e_i \frac{\partial}{\partial x_i}; D_y := \sum_{j=1}^{q} e_j \frac{\partial}{\partial y_j}$$

and consider the system

(4.3) $\quad D_x f = f D_y = 0$

It should be noticed at once that in this system, the operator D_x acts from the left and the operator D_y acts from the right upon f. However the system (4.3) is a special case of (1.7). A regular solution of (4.3) is

called a *biregular* function. For a detailed survey of the theory of biregular functions we refer the reader to [2,3].

Let us now consider the system:

$$(4.4) \quad \begin{cases} D_x f = \varphi(x, y) \\ f D_y = \psi(x, y) \end{cases}$$

where $\varphi, \psi \in c^\infty(\Omega; \mathcal{A})$.

Lemma 4.1:

Assume that

$\Omega = \Omega_1 \times \Omega_2$, where Ω_1 and Ω_2 are domains in $R^p(x)$ and $R^q(y)$, respectively; and that

(4.5) $\quad \varphi(x, y) D_y = D_x \psi(x, y)$.

Then the system (4.4) has a solution $f \in c^\infty(\Omega; \mathcal{A})$.

(For the proof of this lemma we refer the reader to [10].)

In view of the lemma 4.1, the Problem 1 for the system (4.3) can be solved as follows.

From the given data (of the Problem 1) $u^{(\alpha)} \in \tilde{R}(V_\alpha; \mathcal{A})$ we set

$$g^{(\alpha\beta)} = u^{(\alpha)} - u^{(\beta)}$$

then

$$g^{(\alpha\beta)} \in R(V_\alpha \cap V_\beta; \mathcal{A})$$

where $\tilde{R}(V_\alpha; \mathcal{A})$ is the set of biregular functions with singularities in V_α and $R(V_\alpha \cap V_\beta; \mathcal{A})$ is the set of biregular functions in $V_\alpha \cap V_\beta$.

From the definition of $g^{(\alpha\beta)}$ it follows immediately

$$g^{(\alpha\beta)} + g^{(\beta\alpha)} = 0$$

in $V_\alpha \cap V_\beta$ and

$$g^{(\alpha\beta)} + g^{(\beta\gamma)} = g^{(\gamma\alpha)} = 0$$

in $V_\alpha \cap V_\beta \cap V_\gamma$.

Now define $h^{(\beta)}$ by formula (3.6) we have

$$h^{(\beta)} \in c^\infty(V_\beta; \mathcal{A}).$$

However, on $V_\alpha \cap V_\beta$ we have

$$D_x(h^{(\beta)} - h^{(\alpha)}) = 0$$

$$(h^{(\beta)} - h^{(\alpha)})D_y = 0,$$

so the functions

$$\varphi = D_x h^{(\beta)}$$

and

$$\psi = h^{(\beta)} D_y$$

on V_β are well defined on Ω and $\varphi, \psi \in c^\infty(\Omega; \mathcal{A})$.

By definition, φ and ψ satisfy the condition (4.5). from Lemma 4.1 it follows that the system

$$\begin{cases} D_x W = \varphi \\ W D_y = \psi \end{cases}$$

has a solution $W \in c^\infty(\Omega; \mathcal{A})$.

Let

$$g^{(\alpha)} = h^{(\alpha)} - W$$

then we have

$$g^{(\beta)} - g^{(\alpha)} = h^{(\beta)} - h^{(\alpha)} = g^{(\alpha\beta)}.$$

Also

$$D_x g^{(\alpha)} = D_x h^{(\alpha)} - D_x W = 0$$

$$g^{(\alpha)} D_y = h^{(\alpha)} D_y - W D_y = o$$

on every $\alpha \in I$.

Thus we obtain

Theorem 4.1:

The Problem 1 can be solved for system (4.3) on domain $\Omega = \Omega_1 \times \Omega_2$ as in Lemma 4.1.

4.2.a. As an application of the theorem 3.2 let us consider the Beltrami-system (1.4) in a domain Ω of C^n.

Suppose that

$$B_{jk}(z) \in c^\infty(\Omega)$$

and this system is elliptic, integrable in the sense that

(4.6) $\quad [L_i, L_j] = L_i L_j - L_j L_i = 0$

where

$$L_j := \frac{\partial}{\partial \bar{z}_j} - \sum_{k=1}^{n} B_{jk}(z) \frac{\partial}{\partial z_k}.$$

In view of the theorem Newlander-Nirenberg (see [16]), we can introduce in Ω a complex analytic structure such that Ω becomes an analytic manifold, which is denoted by M_Ω. Then a regular solution W(z) of (1.4) in Ω is holomorphic on M_Ω. Hence we have

$$R(\Omega) \approx H(M_\Omega)$$

$$\tilde{R}(\Omega) \approx \mathcal{M}(M_\Omega)$$

where $H(M_\Omega)$ and $\mathcal{M}(M_\Omega)$ are the spaces of holomorphic and meromorphic functions in M_Ω respectively.

Note that $\tilde{R}(\Omega)$ is the set of solutions with such singularities, as those are defined in [7].

From [9] we have

(4.7) $\quad H^p(\Omega; R) \approx H^p(M_\Omega; \mathcal{O}), p = 0, 1, 2, \ldots$

where $H^p(\Omega; R)$ is the p^{th}-cohomology group of Ω with values in the sheaf R of germs of regular solutions and $H^p(\Omega; \mathcal{O})$ is the p^{th}-cohomology group of M_Ω with values in the sheaf $\mathcal{O}$ of germs of holomophic function on M_Ω. (see [9]).

Definition 4.1:

A domain $\Omega \subset C^n$ is called a *non extended domain* for system (4.1) if whenever we are given open connected subsets $U \subset V \subset C^n$ with $U \subset \Omega$ such that $W|_U$ extends (as a solution) to V for all $W \in R(\Omega)$, then $V \subset \Omega$.

If Ω is a non extended domain then M_Ω is a Stein manifold (see [9]). By virtue of (4.7) and of the theorem Cartan B (see [6],p.182) we obtain

(4.8) $\quad H^p(\Omega; R) = 0; p \geq 1.$

Hence the condition (3.11) holds. Thus we are able to apply the theorem 3.2 and obtain

Theorem 4.2:

For the elliptic, integrable system (1.4), the Problem 1 can be solved on the non extended domains of this system.

b. Let us now solve the Problem 2 for the Beltrami-system (1.4).

Suppose that the assumptions in **4.1.a** for system (1.4) are satisfied. If in the sequence (3.13) we set C = Z, then this sequence becomes exact. Here Z is the sheaf of integers, R and R^* are defined as in (3.13) for system (1.4).

If Ω is a non extended domain, then we have (4.8) and hence the assumption (3.15) of theorem 3.3 is fulfilled. In this case the assumption (3.17) has the form

(4.9) $\quad H^2(\Omega, Z) = 0.$

Applying the theorem 3.3 we have

Theorem 4.3:

Suppose that the system (4.1) satisfies all assumptions of the theorem 4.2. Then the Problem 2 can be solved on the non extended domain Ω,which satisfies the condition (4.9).

4.3. Now we consider The Problems 1 and 2 for the Vekua-system (1.3) in case $B_j(z) = 0; j = 1, ..., n$. Then this system reduces to

(1.3)' $\quad \frac{\partial W}{\partial \bar{z}_j} = A_j(z)W; j = 1, ..., n.$

Assume that

all $A_j(z) \in c^\infty(G)$,

where G is a domain of C^n.

Suppose that G is a domain of holomorphy. Then the necessary and sufficiently condition for the existence of a non-trivial solution of (1.3)' is

(4.10) $\quad \frac{\partial A_j}{\partial \bar{z}_k} = \frac{\partial A_k}{\partial \bar{z}_j}$

for j,k = 1,...,n.(see[11]).

Assume that (4.10) holds, then the system

(4.11) $\quad \frac{\partial \omega}{\partial \bar{z}_j} = A_j(z); j = 1, ..., n.$

is solvable in $c^\infty(G)$.

Let Ω be a subdomain of G and $W \in \mathcal{V}(\Omega)$, where $\mathcal{V}(\Omega)$ is the space of regular solutions of (1.3)' in Ω. Then the function

(4.12) $\quad \Phi = We^{-\omega_o} \in H(\Omega),$

where $\omega_o \in c^\infty(G)$ is a solution of (4.11).

Now define the maps

$$T : \mathcal{V}(\Omega) \longrightarrow H(\Omega)$$

and

$$\tilde{T} : \tilde{\mathcal{V}}(\Omega) \longrightarrow M(\Omega)$$

by

$$W \longrightarrow \Phi = W.e^{-\omega_o}$$

Note that here $\tilde{\mathcal{V}}(\Omega)$ is the set of solutions W(z) with singularities such that

$$\Phi(z) = W(z)e^{-\omega_o} \in M(\Omega),$$

and $H(\Omega), M(\Omega)$ are defined as in **3.3** .

It may easily be verified that the above defined maps T and $\tilde{T}$ satisfy the assumptions of theorem 3.4. Applying this theorem we obtain

Theorem 4.4:

For the system (1.3)', whose coefficients satisfy (4.10), the Problem 1 can be solved on the domains of holomorphy.

Applying the theorem 3.5 we have

Theorem 4.5:

Suppose that all assumptions of theorem 4.4 are fulfilled. Then the Problem 2 can be solved on the domain Ω, which satisfies the condition (3.21).

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