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ABSTRACT

In a given partial wave, the strength of the centrifugal term to be incorporated into the WKBA solutions in different spatial regions can be adjusted so as to make the first order wave functions everywhere smooth and, in strong coupling, exactly reproduce Quantum Mechanics throughout the space. The relevant higher order approximations supply an absolute convergent series expansion of the exact scattering state.

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АННОТАЦИЯ

Принимая силу центробежного члена свободным параметром, путем итерации квазиклассического приближения разложение точной волновой функции рассеяния можно привести к сходящемуся ряду. В случае сильной связи ряд сокращается до одного члена.

Dolinszky T.: A Born sorok analogonja erős csatolások esetén. KFKI-1989-53/A

KIVONAT

A centrifugális tag erősségét szabad paraméternek tekintve, a kváziklasszikus közelítés iterálása útján az egzakt szórási hullámfüggvény konvergens sorfejtését állíthatjuk elő. Erős csatolás esetén a sor egyetlen taggá redukálódik.

1. Introduction

The standard WKB approach (WKBA) to the scattering theory of Quantum Mechanics (QM) relies on pairs of analytically introduced wave functions the character of which changes from the exponential to the trigonometric type and <u>vice versa</u> each time we cross a classical turning point (TP). In particular, such a pair $w_\lambda^\pm(r)$ of functions reads inside the nearest TP for scattering of a spinless particle of energy k^2 by the potential $q^2 V(r)$ in the partial wave of index l the following [1]

$$w_\lambda^\pm(r) \equiv \frac{k^{1/2}}{\mathcal{K}_\lambda^{1/2}(r)} \exp\left[\pm \int_{R_\lambda}^{r} d\varrho\, \mathcal{K}_\lambda(\varrho)\right], \quad [r < R_\lambda], \tag{1.1}$$

where

$$\mathcal{K}_\lambda^2(r) \equiv \left| k^2 - q^2 V(r) - \frac{\lambda^2}{r^2} \right| \tag{1.2}$$

and, at one's disposal,

$$\lambda^2 = \begin{matrix} \lambda_a^2 \equiv (l + 1/2)^2 \\ \lambda_b^2 \equiv l(l+1) \end{matrix} \quad \text{or} \tag{1.3}$$

(yet the former alternative preferred), while the definition of the nearest TP at $r = R_\lambda$ is given by

$$\mathcal{K}_\lambda(R_\lambda) = 0 . \tag{1.4}$$

Within the region indicated, the functions $w_\lambda^\pm(r)$ are considered to reproduce a pair of solutions of QM, under certain extreme conditions exactly, otherwise approximately.

The circumstances under which WKBA is expected to work best can easily be extracted from the Schroedinger type of differential equation solved by both $w_\lambda^{+}(r)$ and $w_\lambda^{-}(r)$ of eq.(1.1) and governed by the potential term [2]

$$(1.5) \qquad W_\lambda(r) = g^2 V(r) + \frac{5}{16}\frac{1}{K_\lambda^4}\left(\frac{dK_\lambda^2}{dr}\right)^2 - \frac{1}{4}\frac{1}{K_\lambda^2}\frac{d^2K_\lambda^2}{dr^2} + \frac{\lambda^2 - l(l+1)}{r^2} .$$

The r.h.s. of this equation consists of the physical potential plus an $O(1)$, $[g^2 \to \infty]$, contribution, on account of which WKBA should become exact in strong coupling, at least <u>inside</u> the nearest TP.

How to push $w_\lambda^{\pm}(r)$ <u>across</u> the singularity of $W_\lambda(r)$ at the TP [see eq. (1.4)], is just the Connection Problem. By insisting on the form (1.5) of the WKBA potential, solutions of the trigonometric type can be introduced beyond the TP, such superpositions of which should be found that correctly join to the inner solutions $w_\lambda^{\pm}(r)$. The only reasonable interpretation of the term 'correctly' is obviously that the overall $[r = 0 \to \infty]$ solution should reproduce exactly [e.g. for $g^2 \to \infty$] or approximately [e.g. for large g^2] <u>inside and beyond the TP one and the same QM wave function.</u> In QM, in turn, self-identity of a solution is ensured by its overall smoothness, a property that is automatic whenever $V(r)$ is continuous and can be extra postulated if the potential though developing some jumps remains bounded throughout. In WKBA, however, the potential $W_\lambda(r)$ of eq. (1.5) is singular at $r = R_\lambda$.

Hence, the smoothness postulate is here necessarily frustrated and ought to be substituted for by some virtually equivalent claim. How this issue has been dealt with by different semiclassical approaches is briefly outlined just below.

The standard treatment of the Connection Problem, the linear-turning-point approach (LTPA) combines the WKBA formalism near the TP with QM, whereby one realizes smoothness at two matching points (MP's), one taken inside, the other beyond the singularity. Yet, proper selection of the MP's is rather problematic. The nearer we put them to the TP the less realistic values of the WKBA enter the argument. The farther we set them the less reliable will be there our knowledge of the exact solution. What is more, however sophisticated our choice may be it still cannot heal an additional weakness of LTPA, namely its failing to <u>simultaneously</u> reproduce the physical behaviour for $r \to 0$ and $r \to \infty$. Whichever of the alternatives of eq.(1.3) is chosen for λ^2, $(l+1/2)^2$ or $l(l+1)$, the potentials of QM and WKBA <u>differ by the dominant term</u> $1/(4r^2)$ either for $r \to \infty$ or $r \to 0$, respectively, as is straightforward to extract from eq.(1.5).

The recently proposed [3] double-centrifugal-strength approach (DCSA) goes beyond the standard WKBA and succeeds in solving the Connection Problem in such a way that the relevant wave functions do become compatible with QM simultaneously for $r \to 0$ and $r \to \infty$. The clue to doing this rests in incorporating into the formalism both candidates of the centrifugal strength , λ_a^2 and λ_b^2 of eq.(1.3), working therefore with a pair of TP's, R_a and R_b . The crucial point is that for decreasing (repulsive) potentials R_b is always smaller than R_a . In consequence, any point R of the interval $r=(R_b, R_a)$ divides the space into two regions, I_a and I_b , so that I_a contains R_b [observe the indices] while I_b contains R_a . Therefore, the inner solutions $w_a^{\pm}(r)$ implying the strength λ_a^2 do not develop singularities in I_a nor do it the external solutions $w_b^{(j)}(r)$, $[j=1,2]$ of the strength λ_b^2 in the interval I_b . Smooth joining at $r=R$ of the solutions $w_a^{\pm}(r)$ to adequate superpositions of $w_b^{(j)}(r)$ yields smooth WKBA wave functions of correct physical behaviour both at the smallest and largest distances.

The present paper is meant to be a generalization of DCSA of ref .[3]. What is common in both methods is the presence of more than one strength of the centrifugal terms. What will be new here is, on the one hand, the inclusion of potentials that while being still repulsive near the origin yet develop an attractive tail. On the other hand, while ref.[3] has been restricted to 1st order WKBA ,the present considerations introduce higher order approximations to finally generate for the exact QM scattering wave function an infinite series expansion the convergence and truncation problem of which will also be examined in detail.

2. Redefining concepts of standard WKBA

The LTPA of the WKBA divides the space into domains in terms of the classical turning points and works then in all of them with the same centrifugal term. The decomposition of the space in the present approach is done, in turn, by means of the zeros of the physical potential and certain unperturbed turning points, and each of the new intervals will be supplied by adequately chosen centrifugal strength. The new system of intervals will be specified in this section.

The conditions imposed on the potentials $g^2 V(r)$ to be included in the discussions are the ones usual in the scattering theory of QM. In addition, we postulate that $V(r)$ should be smooth and subject to the following requirements :

$$[rV(r)] \to (+0), \quad [r \to 0], \tag{2.1}$$

and

$$[r^2 V(r)] \to (-0), \quad [r \to \infty], \tag{2.2}$$

as well as

$$V(z) = 0; \qquad V(r) \neq 0, \quad [r \neq z]. \tag{2.3}$$

Thus the potential is repulsive in the region $r = (0, z)$ and attractive beyond. Unperturbed TP's are defined in $r = a$ and b via λ_a^2 and λ_b^2 of eq. (1.3) by putting

$$a \equiv \frac{\lambda_a}{k}, \qquad b \equiv \frac{\lambda_b}{k}, \tag{2.4}$$

whence

$$(2.5) \qquad b < a .$$

We distinguish 3 types of scattering problems depending on whether

$$(2.6) \qquad \begin{array}{ll} \alpha. & b < z < a , \\ \beta. & b < a < z \\ \gamma. & z < b < a . \end{array}$$

The axis $r = (0, \infty)$ is divided into intervals I_m, $[m = 1, 2, M]$, with running coordinates r_m as implied in the inequalities

$$(2.7) \qquad d_{m-1} \leq r_m < d_m$$

and with endpoints d_m specified in Table 1 for the types α through γ of problems, separately.

Owing to eqs. (2.3), the potential has within each of the intervals I_m a unique sign,

$$(2.8) \qquad \sigma_m = sgn\, V(r_m) ,$$

by virtue of the single zero z of $V(r)$ being included in each list of the endpoints, see Table 1.

The character of the basis functions should depend, by definition, on the local sign of the potential as follows

(2.9a)

$$w_{mj}(r) \equiv \frac{k^{1/2}}{K_m^{1/2}(r)} \exp\left[(3-2j)\int_{d_m}^{r} d\varrho\, K_m(\varrho)\right], \quad [\sigma_m = 1; j = 1, 2],$$

and

(2.9b)
$$w_{mj}(r) \equiv (-1)^j \sqrt{2}\, \frac{k^{1/2}}{\mathcal{K}_m^{1/2}(r)} \sin\Big[\int_{d_{m-1}}^{r} d\varrho\, \mathcal{K}_m(\varrho) + (3-2j)\pi/4 \Big],$$
$$[\sigma_m = -1;\ j = 1,2]\ .$$

The functions $\mathcal{K}_m(r)$ involved are unspecified for the time being. It is straightforward to calculate the Wronskians

(2.10)
$$\mathcal{W}\{w_{m1}(r);\ w_{m2}(r)\} = -2k,$$
$$[\sigma_m = \pm 1]\ ,$$

the r- independence of which already suggests that the functions (2.9) of given m may solve both the same 2nd order differential equation. Indeed,

(2.11)
$$\Big[\frac{d^2}{dr^2} - \sigma_m \mathcal{K}_m^2(r) - \mathcal{D}_m(r)\Big] w_{mj}(r) = 0\ ,$$
$$[\sigma_m = \pm 1, \quad j = 1,2]\ ,$$

where

(2.12)
$$\mathcal{D}_m(r) = \frac{5}{16}\,\frac{1}{\mathcal{K}_m^4(r)}\left(\frac{d\mathcal{K}_m^2}{dr}\right)^2 - \frac{1}{4}\,\frac{1}{\mathcal{K}_m^2(r)}\,\frac{d^2\mathcal{K}_m^2}{dr^2}\ .$$

Notice that eqs.(2.11) work independently of the definition of $\mathcal{K}_m^2(r)$. Yet, henceforward we shall work with the choice

(2.13) $$\mathcal{K}_m^2(r) \equiv \sigma_m\left[-k^2 + q^2 V(r) + \frac{\lambda_m^2}{r^2}\right],$$

involving still the unspecified parameters λ_m^2. Each of the equations

$$(2.14)\qquad \mathcal{K}_m^2(r) = 0\,, \qquad [r = R_{m\mu}\,,\quad \mu = 1,2,M]\,,$$

has, in general, a set of (real) roots some of which may fall onto the actual interval I_m. The points $R_{m\mu}$ are, in fact, the TP's , candidates for causing singularities of the relevant basis functions of eq.(2.9). Appearance of active singularities can be avoided if we work with the functions $w_{mj}(r)$ exclusively in the interval I_m, and this I_m contains none of the TP's $R_{m\mu}$, $[\mu = 1,2,M]$. Appropriate choice of the free parameters λ_m^2 will satisfy this claim.

Incorporate eq.(2.13) into eq.(2.11) and obtain after rearrangement the Schroedinger type differential equation

$$(2.15)\quad \left[\frac{d^2}{dr^2} + k^2 - W_m(r) - \frac{l(l+1)}{r^2}\right] w_{mj}(r) = 0,$$

with the WKBA potential

$$(2.16)\quad W_m(r) \equiv g^2 V(r) + \Delta_m(r)\,,$$

where $\Delta_m(r)$ is the residual interaction in I_m ,expressed via eq.(2.12) as

$$(2.17)\quad \Delta_m(r) \equiv \mathcal{B}_m(r) + \frac{\lambda_m^2 - l(l+1)}{r^2}\,.$$

Overall potential and overall residual interaction is introduced by putting

$$
(2.18)\quad
\begin{aligned}
W(r) &\equiv W_m(r), \\
\Delta(r) &\equiv \Delta_m(r), \\
W(r) &= q^2 V(r) + \Delta(r) .
\end{aligned}
\qquad [d_{m-1} \le r \le d_m],
$$

Observe that both $W(r)$ and $\Delta(r)$ develop jumps at $r = d_{m-1}$ whenever our future choice of λ^2_{μ} will be different in the intervals I_{m-1} and I_m, see eq.(2.13).

In terms of $W(r)$, an overall WKBA differential equation can also be set up as follows

$$
(2.19)\quad \left[\frac{d^2}{dr^2} + k^2 - W(r) - \frac{l(l+1)}{r^2} \right] w(r) = 0,
$$

each solution of which can conveniently be labelled by the parameters that specify the solution in I_1, and can also be constructed in the rest of the intervals I_m as superpositions of the relevant basis $w_{mj}(r)$, $[j = 1, 2]$. An example of the notation is given here

$$
(2.20)\quad
\begin{aligned}
w^+(r) &\equiv w_{11}(r), \\
w^-(r) &\equiv w_{12}(r),
\end{aligned}
\qquad [0 \le r < d_1],
$$

$$
w^{\pm}(r) \equiv A^{\pm}_{m1} w_{m1}(r) + A^{\pm}_{m2} w_{m2}(r), \qquad [d_{m-1} \le r < d_m].
$$

Notice that the 'propagation' of the constants $A^{\pm}_{mj}$ from interval to interval should be prescribed by some adequate principle.

We are also interested in the behaviour of the residual interaction under extreme conditions. It is straightforward to extract from the definitions (2.18), (2.17) and (2.12) that

(2.21)

$$\Delta(r) \to \left[\lambda_1^2 - l(l+1) - \frac{1}{4}\right]\frac{1}{r^2} + \frac{1}{\lambda_1^2}\left[k^2 - q^2 V(0)\right], \quad [r \to 0];$$

$$\to \left[\lambda_m^2 - l(l+1)\right]\frac{1}{r^2} + \frac{3\lambda_m^2}{2k^2}\frac{1}{r^4}, \quad [r \to \infty];$$

$$\to \mathcal{D}^\infty(r) + \left[\lambda_m^2 - l(l+1)\right]\frac{1}{r^2}, \quad [q^2 \to \infty, \quad r = r_m],$$

where

(2.22)

$$\mathcal{D}^\infty(r) = \frac{5}{16}\frac{1}{V^2(r)}\left(\frac{dV}{dr}\right)^2 - \frac{1}{4}\frac{1}{V(r)}\frac{d^2V}{dr^2} .$$

So much about definitions and notation.

3. Physics postulates specify free parameters

In exact QM the scattering problem under discussion is governed by the Schroedinger equation

(3.1)

$$\left[\frac{d^2}{dr^2} + k^2 - q^2 V(r) - \frac{l(l+1)}{r^2}\right] u_l(r) = 0 .$$

Recall that after sect. 2 we left with the sets λ_m^2 and $A^{\pm}_{mj}$ of unspecified parameters. We seek the best choice of them so as to bring the differential equations of WKBA and QM, eqs. (2.19) and (3.1), as close tobether as possible. To this end we raise by physics a set of postulates listed below. Hence the present 'variable centrifugal-strength' approach to WKBA will be referred to by the abbreviation VCSA. The postulates [P1] through [P6] read as follows :

[P1] VCSA should reproduce the QM's small-distance behaviour
$u_\ell(r) \rightrightarrows (kr)^{1/2}\,(kr)^{\pm(\ell+1/2)}$, $[(kr)\to 0]$,
of the physical and nonphysical solutions, respectively.

[P2] VCSA should reproduce the QM's large-distance behaviour
$u_\ell(r) \to a_\ell\,\hat{j}_\ell(kr) + b_\ell\,\hat{n}_\ell(kr)$, $[(kr)\to\infty]$,
(observe the same index on both sides).

[P3] VCSA should be governed by a potential $W(q^2, r)$ that remains free from singularities at any fixed value of q^2 , just as QM is.

[P4] The VCSA's potential $W(q^2, r)$ should reproduce the QM's $q^2 V(r)$ at least to $O(q^2)$, $[q\to\infty]$.

[P5] The residual interaction $[q^2 V(r) - W(q^2, r)]$ should be minimized at each fixed r so far as is still compatible with above postulates, e.g. [P3].

[P6] The VCSA's wave functions $w^{\pm}(r)$ should be everywhere smooth, just as in QM are.

The propositions for selecting the values of the free parameters to meet postulates [P1] through [P6] are collected in Table 2 and eq.(3.2).-(3.3).

We extract from eqs.(2.20) that

$$(3.2) \qquad A^+_{11} = A^-_{12} = 1\,, \qquad A^+_{12} = A^-_{11} = 0\,.$$

Our proposition for the coefficients $A^{\pm}_{mj}$ is implied in the iteration scheme

$$(3.3) \quad A^{\pm}_{m+1,j} = \frac{1}{k}\Big\{ W_{d_m}\{v_{m1}\,;\,v_{m+1,2/j}\}\cdot A^{\pm}_{m1} + W_{d_m}\{w_{m2}\,;\,v_{m+1,2/j}\}\cdot A^{\pm}_{m2} \Big\}\,,$$

where the wave functions contained in the arguments of the Wronskians are to be taken from eqs.(2.9).

We are going now to check performance of our propositions in satisfying the postulates listed above.

[P1] - Table 2 suggests $\lambda_1^2 - \lambda_a^2$ for all the types α through γ of scattering , a choice that yields by eqs.(2.18), (2.1), (2.21) and (1.3) that

(3.4) $$[r\, W(r)] \to 0, \qquad [r \to 0].$$

It is obvious that this property ensures, indeed, QM-compatible small-distance behaviour and should be compared to its QM's analogue (2.1). Reinforced is this statement by incorporating the above value of λ_1^2 into eq.(2.9a) . After some calculation one obtains that

(3.5) $$w^{\pm}(r) \to const.(kr)^{1/2}(kr)^{\pm(l+1/2)}, \quad [r \to 0].$$

For the details of the derivation we refer to ref.[2] .

[P2] - Insertion of $\lambda_M^2 = \lambda_b^2$ of Table 2 (notice the index on the l.h. side) for problems of types α through γ furnishes by eqs.(2.18), (2.2), (2.21) and (1.3) that

(3.6) $$[r^2\, W(r)] \to 0, \qquad [r \to \infty] .$$

This property is, just as its QM analogue (2.2), sufficient for the QM-compatible large-distance behaviour to hold. Moreover, the above value of λ_M^2 provides by eqs. (2.20) and (2.9b) the phase shift of the VCSA as

(3.7)
$$\delta^{+}_{VCSA} = (\ell+\tfrac{1}{2})\tfrac{\pi}{2} - \int_{d_{M-1}}^{\infty} d\varrho\, \varrho \frac{d}{d\varrho} \mathcal{K}_b(\varrho) + [\sigma_M - d_{M-1}\cdot \mathcal{K}_b(d_{M-1})],$$

where

(3.8)
$$\sigma_M \equiv \arccos \frac{A^{+}_{M1}}{[(A^{+}_{M1})^2 + (A^{+}_{M2})^2]^{1/2}}$$

For comparison, we reproduce here the analogous formula of the LTPA as

(3.9)
$$\delta^{+}_{LTPA} = (\ell+\tfrac{1}{2})\tfrac{\pi}{2} - \int_{R_a}^{\infty} d\varrho\, \varrho \frac{d}{d\varrho} \mathcal{K}_a(\varrho) .$$

Observe that $\mathcal{K}_a(\varrho)$ and R_a involved here depend on the centrifugal strength $\lambda_a^2 = (\ell+\tfrac{1}{2})^2$ while the VCSA's phase shift of eq. (3.7) involves $\lambda_b^2 = \ell(\ell+1)$.

[P3] - The entire formalism is, by eqs. (2.9) and (2.12), singularity free whenever

(3.10)
$$\mathcal{K}_m^2(r_m) > 0, \quad [m = 1, 2, M],$$

i.e. if the TP's lie off the respective intervals I_m. We introduce the notation

(3.11)
$$y_m(r) \equiv k^2 - \frac{\lambda_m^2}{r^2}, \qquad \text{whence}$$

$$\mathcal{K}_m^2(r) = \sigma_m \left[y_m(r) - q^2 V(r) \right] .$$

Thus, [P3] is fulfilled if none of the curves $y_m(r)$ crosses $q^2 V(r)$ in I_m, $[m = 1, 2, M]$. Therefore, we conclude from Figs. 1-3 that condition (3.1o) is indeed realized by the choice of λ_m^ℓ contained in Table 2. Observe also that, in each case, a single, q -independent value of λ_m^ℓ works for all possible values of the coupling constant. This experience is, in fact, a crucial point of our argument to come.

[P4] - A simple comparison of eqs.(2.18) and (2.21) shows that but two isolated points, the origin $r = 0$ and $r = \bar{r}$, i.e. the zero of the physical potential, the postulate of strong-coupling exactness is everywhere fulfilled. At the origin this property breaks down by the dominance of the centrifugal term. At $r = \bar{r}$, the physical and WKBA potentials differ also by an $O(q^2)$, $[q \to \infty]$ term.

[P5] - The values of λ_1^ℓ and λ_M^ℓ that are exclusively acceptable from the physical point of view have already been uniquely fixed by postulates [P1] and [P2]. The postulate of optimum choice of λ_m^ℓ concerns therefore only the case $m = \ell$, type β and γ of scattering. In these cases the single free parameter in finding optimum is, by Table 2, ε^ℓ. Also, since we are primarily interested in strong coupling it is convenient to consider the limiting formula obtained by eqs. (2.21)-(2.22). as

$$(3.12)\qquad |\Delta_\ell(r_\ell)| \to \left| \mathcal{D}^\infty(r_\ell) + \frac{\lambda_\ell^2 - \lambda_i^2}{r_\ell^2} \right|, \quad [q \to \infty].$$

Particularly simple is optimization if, throughout the interval I_ℓ, the centrifugal contribution dominates the $\mathcal{D}^\infty$ - term [see eq. (2.22)], i.e. if

(3.13) $$r_2^2 \, | \mathcal{B}^{\infty}(r_2) | < | \lambda_2^2 - \lambda_b^2 | \, .$$

If so then one write by eq. (3.12) that

(3.14) $$| \Delta_2(r_2) | = sgn(\lambda_2^2 - \lambda_b^2) \cdot \Delta_2(r_2) \, .$$

On the other hand, eqs. (2.4) - (2.6), together with Table 2, imply that for the two types of scattering considered here

(3.15) $$\beta. \quad \lambda_a < kz, \qquad \lambda_b^2 < \lambda_2^2 = k^2 z^2 + \varepsilon^2 ;$$
$$\gamma. \quad kz < \lambda_a, \qquad \lambda_2^2 = k^2 z^2 - \varepsilon^2 < \lambda_b^2 \, .$$

Since $\mathcal{B}^{\infty}(r_2)$ of eq. (2.22) is independent of ε^2, eqs. (3.12) and (3.14) - (3.15) combine to yield

(3.16)
$$\frac{d\,|\Delta_2(\varepsilon^2, r_2)|}{d\,\varepsilon^2} \underset{q \to \infty}{\longrightarrow} \frac{d}{d\varepsilon^2}\left(\frac{k^2 z^2 + \varepsilon^2 - l(l+1)}{r_2^2}\right) = \frac{1}{r_2^2} \, , \quad [type\ \beta] ;$$
$$\underset{q \to \infty}{\longrightarrow} \left[- \frac{d}{d\varepsilon^2}\left(\frac{k^2 z^2 - \varepsilon^2 - l(l+1)}{r_2^2}\right)\right] = \frac{1}{r_2^2} \, , \quad [type\ \gamma] \, .$$

Thus, whenever condition (3.13) holds throughout I_2, optimum of λ_2^2 is attained by infinitesimal values of ε^2 for both types β and γ of scattering problems. Consider also Figs. 2-3.

[P6] - It is straightforward to check that eqs. (3.3) are just the formulae of smooth matching of $v^{\pm}(r_m)$ and $v^{\pm}(r_{m+1})$ of eqs. (2.20) at $r_m = d_m = r_{m+1}$. Thus the set $\{m\}$ of eqs. (3.3) ensures, indeed, overall smoothness of the solutions $v^{\pm}(r)$ along the entire r - axis.

Note that eqs.(3.3) have been derived with due regard to eq.(2.1o). Recall that the overall VCSA potential $W(r)$ is discontinuous at the endpoints of the intervals I_m. Nevertheless, smoothness of the solutions $v^{\pm}(r)$ is sufficient for their Wronskian to remain constant along the entire r- axis:

(3.17) $$\mathcal{W}\{v^{+}(r); v^{-}(r)\} = -2k, \quad [0 \leq r],$$

the numerical value of which has been taken from applying eq.(2.1o) for the case $m = 1$.

4. Series expansion of the exact scattering state

In this section we shall set up an integral equation for the exact scattering wave function in terms of the VCSA scattering problem as a reference system. By iteration, the integral equation generates a series the convergence of which will also be studied.

The Volterra type integral equation reads by eqs. (2.18), (3.1) and (2.19)

(4.1) $$u^{+}(r) = v^{+}(r) + \int_0^r dr' \, \mathcal{G}(r, r') \, \Delta(r') \, u^{+}(r'),$$

with the resolvent normalized by eq.(3.17) as

(4.2) $$\mathcal{G}(r, r') = \frac{1}{2k} \{ v^{+}(r) v^{-}(r') - v^{+}(r') v^{-}(r) \}.$$

That the possible solution of eq.(4.1) will, indeed, supply the <u>physical</u> scattering state is warranted by the small-distance behaviour of the inhomogeneous term as contained in eq.(3.5). The iteration scheme for solving eq.(4.1) is obviously the following

(4.3)
$$u^{(0)}(r) \equiv w^{+}(r),$$
$$u^{(s)}(r) \equiv \int_0^r dr'\, \mathcal{G}(r,r')\,\Delta(r')\,u^{(s-1)}(r'), \qquad [s \geq 1].$$

The r.h. side is decomposed into the contributions of the single intervals I_m as

(4.4)
$$u^{(s)}(r_m) = \sum_{\mu=1}^{m-1} \int_{d_{\mu-1}}^{d_\mu} dr_\mu\, g_{m\mu}(r_m, r_\mu)\,\Delta(r_\mu)\,u^{(s-1)}(r_\mu)$$
$$+ \int_{d_{m-1}}^{r_m} dr_m'\, g_{mm}(r_m, r_m')\,\Delta(r_m')\,u^{(s-1)}(r_m'),$$

where we used the notation

(4.5)
$$g_{m\mu}(r_m, r_\mu') \equiv \frac{1}{2k} \sum_{i,j=1}^{2} B^{mi}_{\mu j}\, w_{mi}(r_m)\, w_{\mu j}(r_\mu').$$

The coefficients involved are obtained by means of eq.(2.2o) as

(4.6)
$$B^{mi}_{\mu j} \equiv A^{+}_{mi}\, A^{-}_{\mu j} - A^{+}_{\mu j}\, A^{-}_{mi}.$$

We are looking for majorizing the series to be developed by iterating eq.(4.1). We first rewrite the definition (2.13) by eq.(3.1o)

$$(4.7) \qquad \mathcal{K}_m^2(r_m) = \left| \left[k^2 - \frac{\lambda_m^2}{r_m^2} \right] - q^2 V(r_m) \right| .$$

Hence we shall extract a number of inequalities, which, however, can also be checked by Figs. 1-3. The notation to be used below rests on eqs.(2.7) and Table 1. Accordingly, we find for problems of type α that

$$(4.8) \qquad \begin{aligned} &0 < \left[\frac{\lambda_1^2}{z^2} - k^2 \right] < \mathcal{K}_1^2(r_1) < \frac{C_\alpha^2}{r_1^2} , \\ &0 < \left[k^2 - \frac{\lambda_2^2}{z^2} \right] < \mathcal{K}_2^2(r_2) < \left[k^2 + q^2 \max |V(r_2)| \right], \end{aligned}$$

with C_α^2 being a suitably chosen constant. In a similar way, one obtains for case β that

$$(4.9) \qquad \begin{aligned} &0 < q^2 \min V(r_1) < \mathcal{K}_1^2(r_1) < \frac{C_\beta^2}{r_1^2} , \\ &0 < \left[k^2 - \frac{\lambda_2^2}{z^2} \right] < \mathcal{K}_2^2(r_2) < \left[\frac{\lambda_2^2}{a^2} + q^2 \max V(r_2) \right], \\ &0 < \left[k^2 - \frac{\lambda_3^2}{z^2} \right] < \mathcal{K}_3^2(r_3) < \left[k^2 + q^2 \max |V(r_3)| \right], \end{aligned}$$

where C_β^2 is again an adequate constant. The corresponding relationships are for case γ the following

$$(4.10) \qquad \begin{aligned} &0 < \left[\frac{\lambda_1^2}{z^2} - k^2 \right] < \mathcal{K}_1^2(r_1) < \frac{C_\gamma^2}{r_1^2} , \\ &0 < \left[k^2 - \frac{\lambda_2^2}{z^2} \right] < \mathcal{K}_2^2(r_2) < \left[k^2 + q^2 \max |V(r_2)| \right], \\ &0 < \left[k^2 - \frac{\lambda_3^2}{a^2} \right] < \mathcal{K}_3^2(r_3) < \left[k^2 + q^2 \max |V(r_3)| \right]. \end{aligned}$$

Notice the arguments r_m of the functions $\mathcal{K}_m(r_m)$ throughout the set of the above inequalities. Hence we conclude that each of the functions $\mathcal{K}_m(r)$ remains bounded and nonzero in its 'eigeninterval' I_m. Consequently, for $m \geq 2$ both the trigonometric and the exponential type basis functions $w_{mj}(r)$ of eq.(2.9) can be majorized in I_m by suitable constants as

(4.11) $$|w_{mj}(r_m)| < \beta^2_{mj}, \quad [\sigma_m = \pm 1;\ j = 1,2].$$

As to interval I_1, we refer to the small-distance formula (3.5), which can also be slightly modified to read

(4.12) $$w_{1j}(r_1) \to \mathrm{const}\cdot \eta_j(r_1), \qquad [r_1 \to 0],$$

where

(4.13) $$\eta_j(r) \equiv \left(\frac{kr}{1+kr}\right)^{\frac{1}{2}+(3-2j)(l+\frac{1}{2})}, [j=1,2].$$

Notice that the definition (2.20) has also been incorporated in eq.(4.12). The relationships (4.11)-(4.12) can be combined to the set of inequalities

(4.14) $$|w_{mj}(r_m)| < \mathrm{const}\cdot \eta_j(r_m), \quad [j=1,2;\ m=1,2,M].$$

By monotonity, one extracts from eq.(4.13) that

(4.15) $$\eta_2(r_m)\,\eta_1(r_\mu) \leq \eta_2(r_\mu)\,\eta_1(r_m), \qquad [r_\mu \leq r_m].$$

We infer from the last two statements that

(4.16) $$|w_{mi}(r_m)\,w_{\mu j}(r_\mu)| < \mathrm{const}\cdot \eta_2(r_\mu)\,\eta_1(r_m),$$
$$[r_\mu \leq r_m],$$

working for all possible pairs of i and j . The definition (4.5) leads then to the majorization of the partial resolvent as

(4.17)
$$|q_{m\mu}(r_m, r'_\mu)| \leq \text{const.}\ \eta_2(r'_\mu)\ \eta_1(r_m),\quad [r'_\mu \leq r_m].$$

The number of intervals is M, the finiteness of which permits working with an overall constant in the set $\{m\}$ of eqs.(4.17) throughout the r-axis. The insertion of inequality (4.17) into eq.(4.4) yields

(4.18)
$$|u^{(s)}(r)| < \text{const.}\ \eta_1(r) \int_0^r dr'\, \eta_2(r')\, |\Delta(r')|\, |u^{(s-1)}(r')|,$$

or, identically,

(4.19)
$$\eta_1(r)^{-1}\, |u^{(s)}(r)| \leq$$
$$\text{const.} \int_0^r dr'\, \frac{r'}{1+kr'}\, |\Delta(r')|\, \eta_1(r')^{-1}\, |u^{(s-1)}(r')| .$$

We can write by virtue of the definitions (2.20) and (4.3) as well as the inequality (4.14) that

(4.20)
$$\eta_1(r)^{-1}\, |u^{(0)}(r)| < c = \text{const.}$$

Iteration of the inequality (4.19) leads therefore by some algebra to

(4.21)
$$\eta_1(r)^{-1}\, |u^{(s)}(r)| < \text{const.}\ \frac{1}{s!}\left[c \int_0^r dr'\, \frac{r'}{1+kr'}\, |\Delta(r')|\right]^s .$$

On the other hand, the centrifugal strength is given in the external region for all types of scattering problems by

(4.22) $$\mu^2 = l(l+1).$$

Hence we see by the relevant asymptotical formula of eqs. (2.21) that

(4.23) $$\Delta(r) \to const. \frac{1}{r^4}, \quad [r \to \infty],$$

on account of which the integral contained in eq.(4.21) is bounded in the variable r along the entire r-axis. We conclude then from eq.(4.21) by analysis that

(4.24)
$$\sum_{s=0}^{\infty} |u^{(s)}(r)| < const. \left(\frac{kr}{1+kr}\right)^{l+1} \exp\left[c\int_0^r dr' \frac{r'}{1+kr'} |\Delta(r')|\right].$$

This inequality is, however, equivalent to stating that the series

(4.25) $$u(r) \equiv \sum_{s=0}^{\infty} u^{(s)}(r)$$

is absolute convergent. Remember that the set of functions $u^{(s)}(r)$ has been introduced by iteration in terms of eqs.(4.3). Boundedness of the argument of the exponential function in the formula (4.24) suggests also uniform convergence. The sum $u(r)$ of eq. (4.25) is by analysis the unique solution of the integral equation (4.1) and provides thus the exact scattering state of QM.

5. Increasing the coupling constant

Up to this point we have been interested in producing an expansion of the scattering solution of the Schroedinger equation at fixed strength of the physical potential. The present section is devoted to studying of how the single terms of this expansion behave in strong coupling.

As we have already pointed to, the increase of the coupling constant does not, in fact, induce active TP's, i.e. such ones that would give rise to singularities of the VCSA potential or wave functions.

The second step of the present argument is to find the strong-coupling form of the iteration scheme (4.3). From eq.(4.7) we extract the limits

$$\text{(5.1)} \quad \begin{aligned} \varkappa_m^2(r_m) &\to q^2 |V(r_m)| \,, \quad [q^2 \to \infty,\ r_m \neq z] \,, \\ &\to k^2 - \frac{\lambda_m^2}{r_m^2} \,, \quad [q^2 \to \infty,\ r_m = z] \,, \end{aligned}$$

by means of which we consider the higher order contributions $u^{(n)}(r)$ to eq.(4.25) in the interval I_1. Equations (4.3) read for $p = 1$

$$\text{(5.2)} \quad u_1^{(1)}(r) = \frac{1}{2k} \int_0^{r_1} dr' [w^+(r)\, w^-(r') - w^+(r')\, w^-(r)]\, \Delta(r')\, w^+(r') \,,$$

where eq.(4.2) has also been incorporated. If we insert here eq.(5.1) integration by parts furnishes the strong-coupling leading term, by substituting $d_0 = 0$

(5.3)
$$u^{(1)}(t_1) \to \Big[-\frac{1}{2} \int_{d_0}^{t_1} dt' \, \frac{\Delta(t')}{K_1(t')} \Big] \, u^{(0)}(t_1), \qquad [q^2 \to \infty].$$

Owing to eqs.(4.3) and (5.3), one finds the strong-coupling form of $u^{(2)}(t_1)$ by simultaneously replacing in eq.(5.2)

(5.4) $u^{(1)}(t_1)$ by $u^{(2)}(t_1)$

and

$\Delta(t')$ by $\Big[-\frac{1}{2} \Delta(t') \int_{d_0}^{t'} dt'' \, \frac{\Delta(t'')}{K_1(t'')} \Big]$

to obtain

(5.5)
$$u^{(2)}(t_1) \to \Big[\Big(-\frac{1}{2}\Big)^2 \int_{d_0}^{t_1} dt' \, \frac{\Delta(t')}{K_1(t')} \int_{d_0}^{t'} dt'' \, \frac{\Delta(t'')}{K_1(t'')} \Big] u^{(0)}(t_1).$$

Further iteration procedure is straightforward and provides eventually by analysis

(5.6)
$$u^{(s)}(t_1) \to \frac{1}{s!} \Big[-\frac{1}{2} \int_{d_0}^{t_1} dt' \, \frac{\Delta(t')}{K_1(t')} \Big]^s \cdot u^{(0)}(t_1), \quad [q^2 \to \infty],$$

whence

(5.7)
$$\frac{u^{(s)}(t_1)}{u^{(s-1)}(t_1)} \to \frac{1}{s} \Big[-\frac{1}{2} \int_{d_0}^{t_1} dt' \, \frac{\Delta(t')}{K_1(t')} \Big], \qquad = O(q^{-1}), \qquad [q^2 \to \infty].$$

We have employed here the limit (5.1) as well as the estimation

(5.8)
$$\Delta(t_1) = O(1), \qquad [q^2 \to \infty],$$

obtained from eqs.(2.21). So much about the innermost interval.

The treatment beyond becomes the more complicated the farther we get from the origin. To avoid extensive calculations, we restrict the succeeding discussion to the type α of scattering problems, where there are only two intervals involved, i.e. where $M=2$. Using the notations (4.3) and (2.20), the 1st approximation to the scattering wave function reads in interval I_2

(5.9)
$$u^{(0)}(r_2) = A^+_{21}\, w_{21}(r_2) + A^+_{22}\, w_{22}(r_2)\,.$$

Remember that the basis $w_{2j}(r)$ is in case α of the trigonometric type. The first iteration of eq.(5.9) is to be done in terms of eqs.(4.4)-(4.6) with the result

(5.10)
$$\begin{aligned} u^{(1)}(r_2) = {} & u^{(1)}(z) + \\ \frac{1}{2k}\Big\{ & A^+_{21}\Big[B^{21}_{22}\, w_{21}(r_2)\int_z^{r_2} dr'\, w_{22}(r')\,\Delta(r')\,w_{21}(r') + \\ & \qquad B^{22}_{21}\, w_{22}(r_2)\int_z^{r_2} dr'\, w_{21}(r')\,\Delta(r')\,w_{21}(r')\Big] \\ & + A^+_{22}\Big[B^{21}_{22}\, w_{21}(r_2)\int_z^{r_2} dr'\, w_{22}(r')\,\Delta(r')\,w_{22}(r') + \\ & \qquad B^{22}_{21}\, w_{22}(r_2)\int_z^{r_2} dr'\, w_{21}(r')\,\Delta(r')\,w_{22}(r')\Big]\Big\}\,. \end{aligned}$$

In strong coupling, eqs.(2.9b) and (5.1) combine to

(5.11)
$$w_{2j}(q^2, r_2) \to \frac{k^{1/2}}{K_2^{1/2}(r_2)}\, \sin\Big[q\int_z^{r_2} d\varrho\, V^{1/2}(\varrho) + (3-2j)\pi/4\Big]\,.$$

Hence we conclude that, for $q^2 \to \infty$, the contributions of the integrands containing squares of $w_{2j}(r_2)$ is the respective contribution of the products $w_{21}(r_2)w_{22}(r_2)$ times a factor of $O(q)$, $[q \to \infty]$. Retaining the dominant terms we can write

(5.12)

$$\mu^{(1)}(r_2) \to \mu^{(1)}(z) + \frac{1}{4}\left[A^{+}_{22}\, B^{21}_{22}\, w_{21}(r_2) + A^{+}_{21}\, B^{22}_{21}\, w_{22}(r_2)\right] \cdot \int_{z}^{r_2} dr' \, \frac{\Delta(r')}{\mathcal{K}_2(r')} \, , \quad [q^2 \to \infty] \, .$$

On the other hand, eqs. (3.2)-(3.3) furnish for $m = 1$ that

$$(5.13) \quad \begin{aligned} A^{+}_{21} &= \frac{1}{2k}\, W_2\{w_{11}\, ; w_{22}\}\, , \\ A^{-}_{21} &= \frac{1}{2k}\, W_2\{w_{12}\, ; w_{22}\}\, , \\ A^{+}_{22} &= \frac{1}{2k}\, W_2\{w_{11}\, ; w_{21}\}\, , \\ A^{-}_{22} &= \frac{1}{2k}\, W_2\{w_{12}\, ; w_{21}\}\, . \end{aligned}$$

Furthermore, the definitions (2.9) inserted in eqs. (5.13) supply

$$(5.14) \quad \begin{aligned} A^{+}_{21} &= \frac{1}{2k}\left[\alpha_{12} - \beta_{12} - \beta_{21}\right] , \\ A^{-}_{21} &= \frac{1}{2k}\left[\alpha_{12} - \beta_{21} + \beta_{12}\right] , \\ A^{+}_{22} &= \frac{1}{2k}\left[\alpha_{12} + \beta_{21} - \beta_{12}\right] , \\ A^{-}_{22} &= \frac{1}{2k}\left[\alpha_{12} + \beta_{21} + \beta_{12}\right] , \end{aligned}$$

where

(5.15)

$$\alpha_{12} \equiv \frac{1}{2}\left[- \frac{\mathcal{K}_2^{(1)}(z)}{\mathcal{K}_1^{1/2}(z)\,\mathcal{K}_2^{1/2}(z)} + \frac{\mathcal{K}_1^{(1)}(z)}{\mathcal{K}_1^{1/2}(z)\,\mathcal{K}_2^{1/2}(z)}\right] ; \quad \beta_{\mu\nu} \equiv \left(\frac{\mathcal{K}_\mu(z)}{\mathcal{K}_\nu(z)}\right)^{1/2} .$$

Observe that

$$(5.16)\qquad \alpha_{12} = O(q^2), \quad \beta_{\mu\nu} = O(1), \quad [q^2 \to \infty],$$

whence

$$(5.17)\qquad B^{21}_{22} \to -4\,\beta_{12}\,\beta_{21} \quad , \quad [q^2 \to \infty], \\ = -4 .$$

One has by the definition (4.6) that

$$(5.18)\qquad B^{22}_{21} = -B^{21}_{22} .$$

Incorporation of eqs.(5.17)-(5.18) into eq.(5.12) yields

$$(5.19)\qquad u^{(1)}(r_2) \to u^{(1)}(2) - \\ [A^{+}_{22}\, w_{21}(r_2) - A^{+}_{21}\, w_{22}(r_2)] \cdot \int_2^{r_2} dr' \, \frac{\Delta(r')}{K_2(r')} .$$

On the other hand, eqs.(5.14) and (5.16) imply that

$$(5.20)\qquad \begin{aligned} A^{+}_{21} &= \alpha_{12} + O(1), \\ A^{+}_{22} &= -\alpha_{12} + O(1), \end{aligned} \qquad [q^2 \to \infty],$$

on account of which eq.(5.19) can be rewritten as

$$(5.21)\qquad u^{(1)}(r_2) \to [A^{+}_{21}\, w_{21}(r_2) + A^{+}_{22}\, w_{22}(r_2)] \cdot \int_2^{r_2} dr' \, \frac{\Delta(r')}{K_2(r')} \\ = O(q^{3/2}) \quad , \quad [q^2 \to \infty],$$

since the trigonometric type basis $w_{2j}(r_2)$, $[j = 1, 2]$, is $O(q^{-1/2})$, $[q \to \infty]$. Omission of the contribution

(5.22)
$$u^{(1)}(r) = \frac{k^{1/2}}{K_1^{1/2}(r)} = O(q^{-1/2}), \qquad [q^2 \to \infty],$$

from eq.(5.21) is also justified. A comparison of eqs. (5.9) and (5.21) leads, after putting $d_1 = r$, to

(5.23)
$$u^{(1)}(r_2) \to \left[\int_{d_1}^{r_2} dr' \, \frac{\Delta(r')}{K_2(r')} \right] u^{(0)}(r_2) .$$

This relationship is, for the interval I_2, the counterpart of eq. (5.3) working in interval I_1. Just as eq. (5.3) had led to eq. (5.7) so does eq. (5.23) lead to

(5.24)
$$\frac{u^{(s)}(r_2)}{u^{(s-1)}(r_2)} \to \frac{1}{s} \int_{d_1}^{r_2} dr' \, \frac{\Delta(r')}{K_2(r')} = O(q^{-1}), \qquad [q^2 \to \infty] .$$

Equations (5.7) and (5.24) can be combined to the overall statement that

(5.25)
$$\frac{u^{(s)}(r)}{u^{(s-1)}(r)} = O(q^{-1}), \qquad [q^2 \to \infty] .$$

This result can be reworded by saying that the truncation error caused by cutting off the series (4.25) decreases by a factor of $O(q^{-1})$ once one includes in the expension one additional term.

6. Discussion

The proper position of the variable-centrifugal -strength approach among the semiclassical approximations can be found by comparing it to the standard WKBA, i.e. to the linear-turning-point approximation. In doing so, our guide will be the set of postulates [P1] through [P6] raised by physics argumentation in sect. 3. We restrict the present comparison to scattering problems that develop one classical turning point and a single zero of the potential.

As to postulate [P1], overall freedom from singularities is realized both in LTPA and VCSA, though by different means. Still common is in both approaches that different parts of the physical space are supplied by different type model interactions. Thus, in an LTPA that works with the centrifugal strength $\lambda^2 = \lambda_a^2$ of eq. (1.3) there are 3 intervals on the r - axis the central of which contains the turning point $r = R_a$ and is governed by the linear potential $q^2 V^{(1)}(R_a)\cdot(r - R_a)$. In the first and third intervals the WKBA potential $W_a(r)$ involving λ_a^2 enters the Schroedinger equation. In VCSA, in turn, decomposition of the space is done by the unique zero of the physical potential along with the unperturbed turning points fixed by the strengths λ_a^2 and λ_b^2. Each of the successive intervals is supplied by a WKBA potential $W_m(r)$

involving a strength λ_m^2 that is adjusted so as to keep the perturbed turning point R_m off just I_m. The singularities generated by the turning points are thus made 'dummy'.

Fulfilment of the postulates [P2] and [P3] should be examined together. In particular, in LTPA - depending on the choice of the centrifugal strength, $(l+1/2)^2$ or $l(l+1)$, either the small- or the large- distance behaviour is compatible with Quantum Mechanics. In VCSA, in turn, both the $r \to 0$ and the $r \to \infty$ limits are simultaneously correct.

As regards postulate [P4], realization of the physical strong-coupling limit in LTPA encounters the difficulty that upon increasing q^2 sooner or later there will be no reasonable choice of the matching points available. Namely, near the turning point R_a the physical and the linear potentials differ by a term $1/2\, q^2 \cdot V^{(n)}(R_a) \cdot (r-R_a)^2$ while the physical and the WKBA potentials deviate by $5/16 \cdot (r-R_a)^{-2}$, as is easy to extract from eqs. (2.17) and (2.12). The larger q^2 becomes the more hopeless is to keep both correction terms below acceptable limits within and around the matching points, respectively. In VCSA, in turn, elimination of the singularities has been achieved by means of q - independent intervals and centrifugal strengths, which thus work even in the limit $q^2 \to \infty$.

To find the 'best' 1st order approximation, postulated by [P5], there are, in fact, very few free parameters available within the approaches considered. In LTPA, one disposes of the positions of the matching points with due regard to the correction terms given explicitely just above. It seems to be a rather complicated task to account for simultaneously of the effects of both types of residual interactions involved. In VCSA, the free parameter is ε^2, fixing the only problematic centrifugal strength. For a large class of potentials specified by eq.(3.12) the best choice is obviously an infinitesimal ε^2. For the rest of the interactions, optimization seems to be also straightforward.

Overall smoothness of the solutions, postulate [P6], is exactly fulfilled in both procedures under discussion. The only remark to this point is that this property is realized in Quantum Mechanics for smooth potentials spontaneously while in both LTPA and VCSA it can be and must be prescribed.

The final issue of this Discussion concerns VCSA generating an absolute convergent series expansion of the QM's exact scattering wave function, no counterpart of which is known and probably not even possible in LTPA. The series developed in the present paper proved to be a strong-coupling expansion, which converges the faster the larger q^2 is and reduces to a single term for $q^2 \to \infty$

The same limit seems to cause in LTPA serious difficulties in the turning point region.

It is well known that the various Born series of the scattering theory are, in fact, weak coupling expansions. In particular, the Fredholm type Born series is on and off the energy shell [4] a pure power expansion involving nonnegative powers of q^2 but fails to work with certainty above a critical value of the coupling constant. Uniqueness theorem of power series does not exclude, in principle, the existence of a relevant expansion that would consist of nonpositive powers of q^2 at the expense of converging exclusively beyond the convergence radius of the Born series. However, the expansion proposed in the present paper is, at fixed values of the coupling constant, not a pure power series. Nevertheless, in strong coupling it becomes a product of the first order VCSA wave function times a series of inverse-power-like $O(q^{-n})$ terms.

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Table 1

Type	M	d_0	d_1	d_2	d_3
α	2	0	z	∞	–
β	3	0	a	z	∞
γ	3	0	z	a	∞

Endpoints d_m of the intervals $I_m = (d_{m-1}, d_m)$ the r - axis is decomposed into which for the different types of scattering problems [see eqs. (2.3) - (2.6)] .

Table 2

Type	λ_1^2	λ_2^2	λ_3^2
α	λ_a^2	λ_b^2	—
β	λ_a^2	$k^2 z^2 + \varepsilon^2$	λ_b^2
γ	λ_a^2	$k^2 z^2 - \varepsilon^2$	λ_b^2

The strengths λ_m^2 of the centrifugal terms vary from interval to interval [see eqs. (2.3)-(2.6) and (2.13)].

Figure Captions

Fig.1 - Type α of scattering problems: $b < z < a$; $M = 2$ [see eqs.(2.4)-(2.7)]. The parameters λ_m, $[m = 1, 2]$, the numerical values of which are taken from Table 2 label the curves $y_m(r)$ of eq.(3.11) which cross the curve $q^2 V(r)$ at the turning points R_m. Observe that each of the points R_m lies outside the respective interval I_m, however large q^2 is chosen. The functions $K_m^2(r_m)$ of eqs.(2.13) and (3.11) are seen to be positive, proving thereby the inequality (3.10).

Fig.2 - Type β of scattering problems: $b < a < z$; $M = 3$. Upon increasing q^2 from zero to infinity the turning points R_1, R_2 and R_3 start at the resp. points a, c and b to get asymptotically to $r = z$, alike. Observe that meanwhile none of the points R_m, $[m = 1, 2, 3]$ enters the respective interval I_m.

Fig.3 - Type γ of scattering problems: $z < b < a$; $M = 3$. While q^2 varies from 0 to ∞ the turning points R_1 and R_3 move exclusively in I_2 but R_2 moves in the interval I_1.

AX3642

Fig. 1

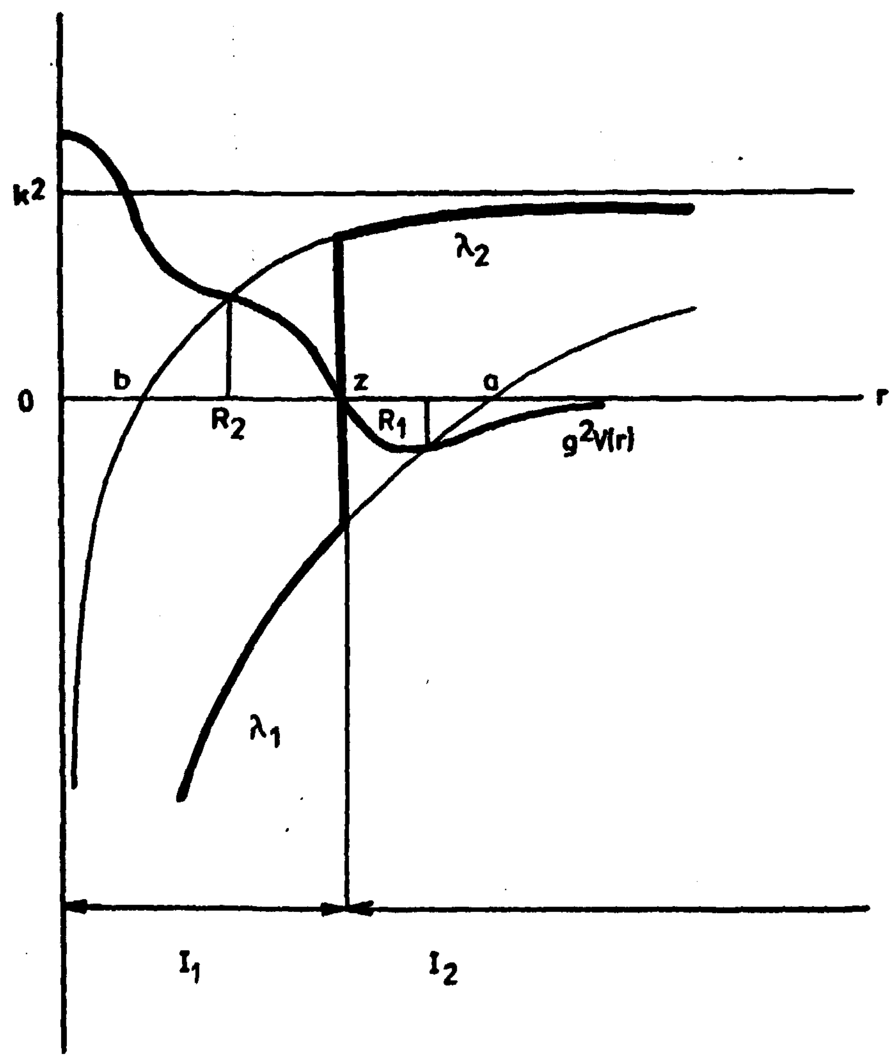

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Fig. 2

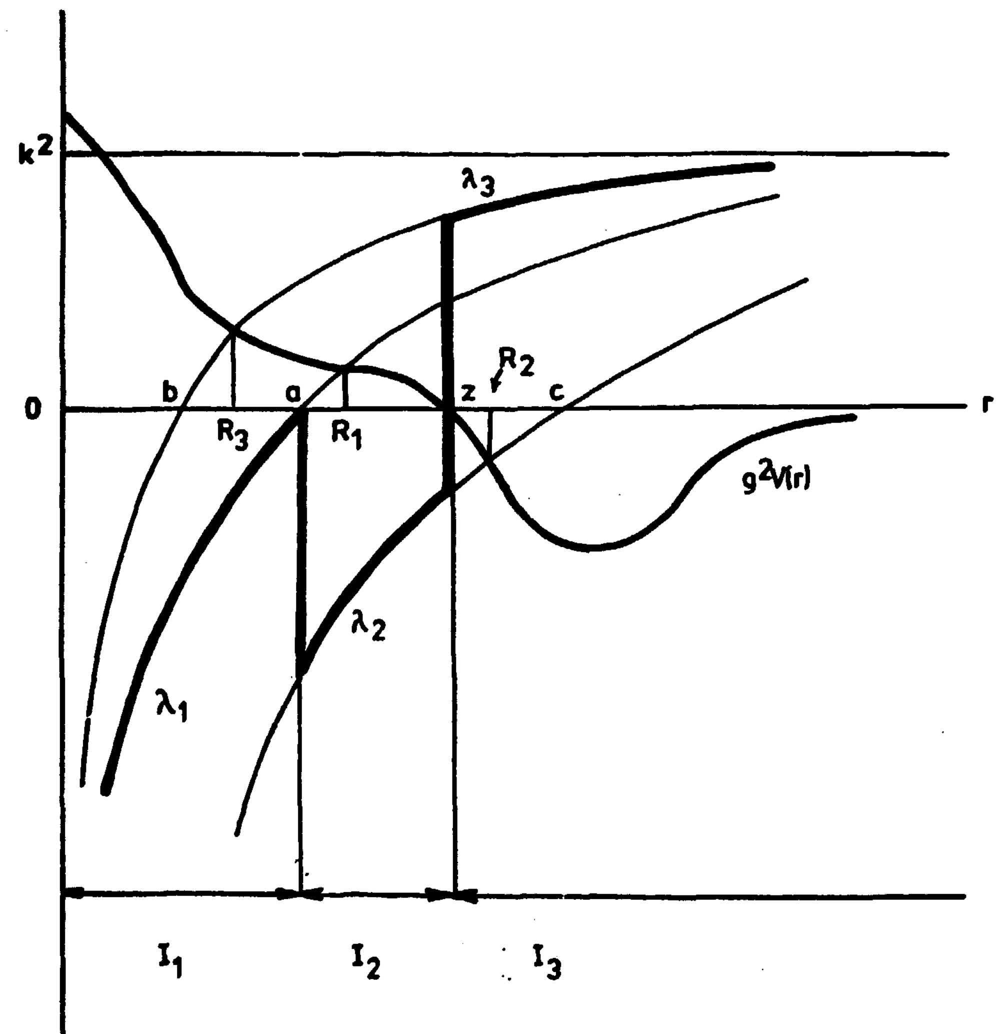

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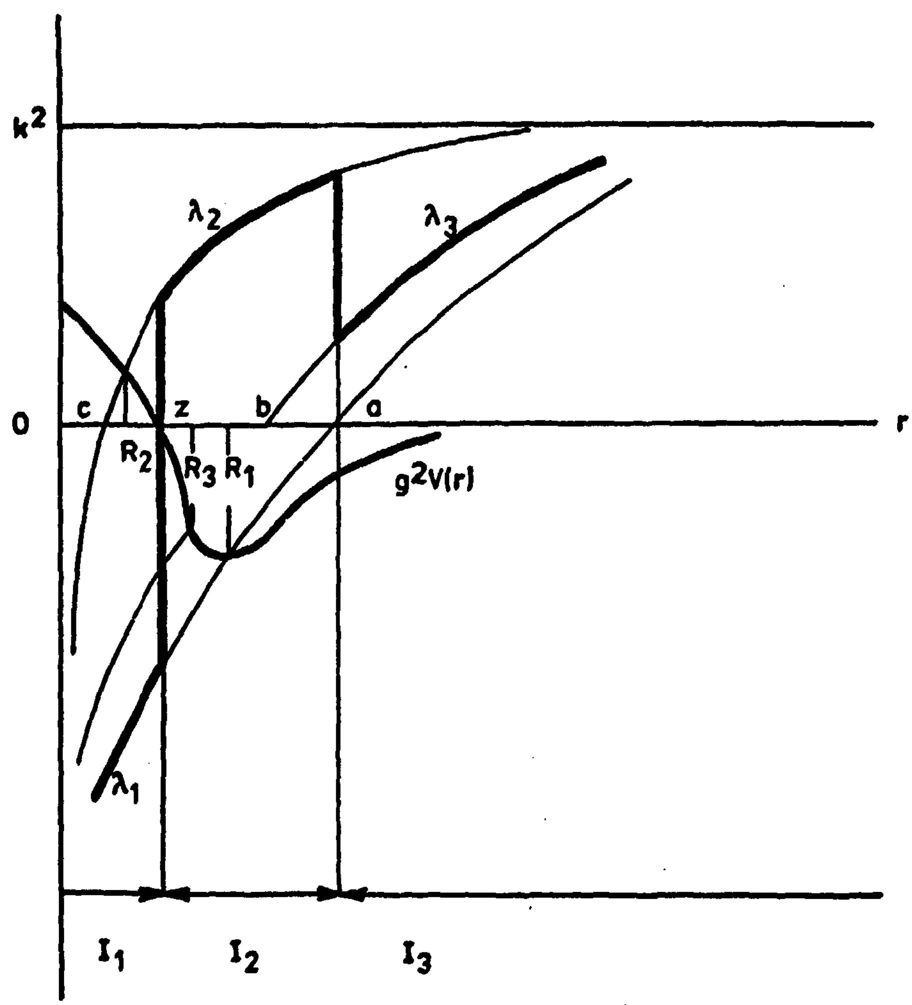
Fig. 3



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