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BRST Formulation of the Gupta-Bleuler Quantization Method

Z. Haslewicz [*], *J. Kowalski-Glikman* [•], *J. Lukierski* [*],
J.W. van Holten [†]

Abstract

In this paper we show, how an algebra of mixed firts and second class constraints can be transformed into an algebra of the Gupta-Bleuler type, consisting of holomorphic and anti-holomorphic constraints. We perform its quantization by BRST methods. We construct a second-level BRST operator Ω by introducing a new ghost sector (the second-level ghosts), in addition to the ghosts of the standard BRST operator. We find an inner product in this ghost sector such that the operator Ω is hermitean. The physical states, as defined by the non-hermitean holomorphic constraints, are shown to be given in terms of the cohomology of this hermitean BRST charge.

[*] *Institute for Theoretical Physics, University of Wroclaw, ul. Cybulskiego 36, Wroclaw, Poland*
[•] *CHEAF, PO Box 41882, 1009 DB Amsterdam, The Netherlands, bitnet t18@nikhefh.nikhef.nl*
[†] *NIKHEF-H PO Box 41882, 1009 DB Amsterdam, The Netherlands, bitnet t18@nikhefh.nikhef.nl*

1 Introduction

BRST-methods [1, 2] have become a powerful tool in the quantization of constrained systems. Not only do they provide a method for establishing the consistency of the covariant quantization procedure of gauge theories , such as used in the path-integral formulation of Yang-Mills theory [3], they also have become important in the elucidation of the canonical structure of constrained systems [4, 5, 6, 7] like relativistic point-particles and strings [8]. In spite of all the past successes of the method, there still doesn't exist a completely unified and unique prescription for the application of BRST-methods to arbitrary constrained systems and surprises keep turning up when studying new systems with new types of gauge-invariances.

A case which is of considerable interest with a view to the covariant formulation of the superparticle [9] and superstring [10], is the BRST-quantization of systems with mixed first and second class constraints [11]-[14]. It was found recently, that a very elegant solution to the BRST-quantization of such systems is possible by separating the constraints into sets of holomorphic and anti-holomorphic first class constraints after the method of Gupta and Bleuler [15] and subsequent extension of the BRST-invariance to a local symmetry of the theory [16, 17]. Moreover, for Yang-Mills theories and gravity it has been shown, that local BRST-invariance implies the same constraints on normalizable states as rigid BRST-invariance [18], and therefore the theory with local BRST-invariance describes the same physical system.

In the following sections we present an improved and more detailed treatment of systems with mixed first and second class constraints along the lines of refs. [16, 17, 18]. We begin with a discussion of the algebra of Gupta-Bleuler type holomorphic and anti-holomorphic first class constraints and its equivalence to an algebra of mixed first and second class real constraints in sect.2. If the states of a quantum theory satisfy holomorphic first-class constraints, the corresponding BRST operator Q is complex, and *not* hermitean. Henceforth we call this and the hermitean conjugate operator $Q^{\dagger}$ the first level BRST operators. In order to apply the standard BRST-methods to a system with Gupta-Bleuler type of constraints, we introduce in the next section a nilpotent second-level BRST operator Ω and proceed to show that there exists an inner product in the ghost sector with respect to which this operator is hermitean. An additional hermitean operator can be constructed which commutes with Ω, representing the total ghost number $\mathcal{N}$, and which can be used to impose restrictions on the Ω-cohomology classes. The main result of this paper is contained in sect. 4, where we show that the restricted cohomology classes of Ω reduce to those of the first level BRST operator Q and its hermitean conjugate. Some technical details have been collected in two appendices.

2 Holomorphic first-class constraints

In this paper we consider quantum mechanical systems described by some extended pseudo-Hilbert space [1], in which the physical states form a subset characterized completely by a set of N constraints:

$$G_A|phys> = 0, \qquad A = 1, ..., N. \tag{2.1}$$

These constraints will be *complex* in general: it is not necessary to require hermiticity of the operators G_A. However, consistency of the constraints requires the $\{G_A\}$ to have a closed (first-class) commutator algebra:

$$[G_A, G_B\} = f_{AB}^{\ \ C}\, G_C. \tag{2.2}$$

The notation $[\,,\,\}$ denotes a graded commutator, allowing both bosonic and fermionic types of constraints. The structure functions $f_{AB}^{\ \ C}$ may be operator-dependent and will also be complex in general. The graded commutator implies that they have the symmetry property

$$f_{AB}^{\ \ C} = -\,(-)^{AB}\, f_{BA}^{\ \ C}. \tag{2.3}$$

In addition, the Jacobi identity requires the structure functions to satisfy

$$(-)^{AC}\left[f_{AB}^{\ \ D}\, f_{DC}^{\ \ E} - \left[G_A, f_{BC}^{\ \ E}\right\}\right] + (\text{cyclic permutations in } ABC) = 0. \tag{2.4}$$

As a consequence of the constraints (2.2) the adjoint operators $G_A^\dagger$ annihilate the dual physical states (the bra-states):

[1] We will use the name 'pseudo-Hilbert space' instead of more precise 'indefinite inner product space' (cf. [26])

$$< phys|G^\dagger_A = 0. \tag{2.5}$$

Obviously they form a first-class algebra as well:

$$\left[G^\dagger_A, G^\dagger_B\right\} = G^\dagger_C \, \bar{f}^C{}_{AB}, \tag{2.6}$$

where the adjoint structure functions are related to the original ones by

$$\bar{f}^C{}_{AB} = (-)^{(A+B+C)C} \, f^{*\,C}_{AB}. \tag{2.7}$$

However, the graded commutator of the constraints G_A with their adjoints $G^\dagger_A$ does not need to vanish on the physical states or their duals. Therefore this commutator will generally have the form

$$\left[G_A, G^\dagger_B\right\} = 2\, Z_{AB}, \tag{2.8}$$

where the 'central charge' Z_{AB} is not a linear combination of constraints and its matrix elements in the physical sector of the pseudo-Hilbert space do not vanish identically:

$$< phys(j)|Z_{AB}|phys(i) > \equiv Z^{ji}_{AB} \neq 0. \tag{2.9}$$

Of course, in special cases the matrix elements Z^{ji}_{AB} *can* vanish; the equation above only states that this is not *necessary*, and the analysis which follows is applicable to this more general situation. Note that Z_{AB} is not defined uniquely; it may be shifted by terms of the form

$$Z_{AB} \rightarrow Z'_{AB} = Z_{AB} + \lambda_{AB}^{\;\;C}\, G_C + G^\dagger_C \, \bar{\lambda}^C{}_{AB}, \tag{2.10}$$

which is achieved by making some arbitrary linear transformation on the G_A. However, the matrix elements Z_{AB} in the physical space of states are defined uniquely and the maximal rank of these physical matrix elements is called the *rank* of Z_{AB}. A constraint algebra of the type presented above is called an algebra of *holomorphic* and *anti-holomorphic* first-class constraints on the ket- and bra-states respectively.

Since the system of quantum constraints considered above is the most general one possible, any classical system which is quantized subject to constraints is described by a holomorphic constraint algebra with this structure.

Special cases exist in which the generators of the constraints are real, or in which the 'central charge' Z_{AB} vanishes, but we do not restrict ourselves to such special cases. This implies, that in general the classical counterpart of the quantum algebra of constraints, eqs. (2.2)-(2.8), may be an algebra of Poisson brackets for mixed real first- *and* second-class constraints. The relation between the complex algebra (2.2) and systems of real mixed classical constraints is discussed below. Here we would like to stress that one of the motivations for using the complex form of the constraints with the first-class algebra (2.2) is, that quantization of the classical system can proceed without the necessity of first having to eliminate all second-class constraints. In fact, because of the first-class nature of the holomorphic constraints we were able to introduce the quantum constraints directly in a way which is analogous to the Gupta-Bleuler quantization of electrodynamics, without having to discuss a corresponding classical system at all. Thus we find an alternative for the Dirac quantization procedure [25] and also that of Faddeev and Shatashvili, Batalin, Fradkin, Fradkina and Niemi [19], [20, 21] [22] which neither requires the elimination of second-class constraints nor the introduction of new variables at the classical level. Another point of interest to be mentioned here is that the phenomenon of 'class-mutation' which takes place in our treatment (and also in the BFFFS procedure), where second-class constraints can be treated as first-class, may find useful application in the quantization of anomalous systems.

The aim of the following sections of this paper is to extend the BRST formalism for dealing with constrained systems to the case of holomorphic first-class constraints, eqs. (2.2)-(2.8). This was achieved already in part in ref. [17]. The main aspect which was not discussed there in any detail was the definition of the inner-product and the construction of a norm. These problems are solved in the present paper and it will turn out that the solution has rather important consequences for the cohomology properties of the BRST operator. However, before doing this, we first discuss how holomorphic first-class algebra's may arise starting from a classical system with real first- and second-class constraints.

Let us consider a classical mechanical system described by real generalized co-ordinates $q^a, a = 1, \ldots, N$[3], and conjugate momenta p_a, with Poisson-brackets

$$\{q^a, p_b\} = \delta^a_b. \tag{2.11}$$

For ease of presentation we restrict ourselves here to the case of a purely bosonic system, the generalization to the case of fermionic degrees of freedom being obvious. Now suppose that the motion of the system is subject to a set of (real) constraints $G_i(q,p) = 0$, $i = 1, \ldots, k$, where $k \leq 2N$. Then the dynamics of the system is restricted to a constraint hyper-surface $\mathcal{P}$ in the extended phase space $\mathcal{M}$. The Poisson brackets of the constraints, evaluated on the constraint surface $\mathcal{P}$, define a skew-symmetric matrix c_{ij}:

$$\{G_i, G_j\}\,|_{\mathcal{P}} = c_{ij}, \tag{2.12}$$

which must be of rank $2m \leq k$ on $\mathcal{P}$. In the extended phase space (away from $\mathcal{P}$) the structure of the constraint brackets then is

$$\{G_i, G_j\} = c_{ij}^{\ \ k}\, G_k + c_{ij}. \tag{2.13}$$

The co-efficients $c_{ij}^{\ \ k}$ are allowed to be non-constant functions of the phase-space co-ordinates. The matrix c_{ij} is anti-symmetric and has $l = k - 2m$ zero-modes (at least one, if k is odd), and one can always find a subset of constraints (or point-wise linear combinations of constraints) denoted by $G_\alpha, \alpha = 1, \ldots, l$, with a closed (first-class) algebra of Poisson brackets:

$$\{G_\alpha, G_\beta\} = c_{\alpha\beta}^{\ \ \gamma}\, G_\gamma. \tag{2.14}$$

In technical language, the Poisson brackets of these constraints generate Hamiltonian vector fields tangent to the constraint surface $\mathcal{P}$, thus defining the gauge orbits. If we disregard a *maximal* set of first-class constraints, we are left with $2m$ real second-class constraints G_r. The algebra of these second-class constraints now has the general form

$$\begin{aligned} \{G_r, G_s\} &= c_{rs} + c_{rs}^{\ \ t}\, G_t + c_{rs}^{\ \ \alpha}\, G_\alpha, \\ \{G_\alpha, G_r\} &= c_{\alpha r}^{\ \ t}\, G_t + c_{\alpha r}^{\ \ \beta}\, G_\beta, \end{aligned} \tag{2.15}$$

[3]The generalisation to systems with infinite, and even continuous, range of a can be made without difficulty.

where the matrix c_{rs} is invertible on $\mathcal{P}$.

In order to obtain a holomorphic form of the constraint algebra analogous to eq.(2.2), it must be possible to polarize the matrix c_{rs}. This means first of all, that we introduce a complex structure, a real $2m \times 2m$ matrix J (possibly co-ordinate dependent) with the property

$$J^2 = -1. \tag{2.16}$$

Any such matrix can be used to construct complex projection operators $P_\pm$:

$$P_\pm = \frac{1}{2}(1 \pm iJ). \tag{2.17}$$

These projection operators have all the properties of the chiral projectors $(1 \pm \gamma_5)/2$ in the Dirac theory of spinors. They allow us to pass from $2m$ real second-class constraints G_r to m complex (holomorphic) constraints, which are the first m linearly independent components of

$$G_r^{(-)} = (P_-)_r{}^s\, G_s, \tag{2.18}$$

whilst the anti-holomorphic constraints are constructed from the first m independent components of

$$G_r^{(+)} = (P_+)_r{}^s\, G_s. \tag{2.19}$$

Note that half of the components $G_r^{(\pm)}$ now can be expressed in terms of the other half; indeed:

$$iJG^{(\pm)} = \pm G^{(\pm)}. \tag{2.20}$$

In order to obtain the structure of a holomorphic first-class algebra for the full set of the $(k-m)$ constraints

$$\{G_A\}_{A=1}^{k-m} = \{G_r^{(-)}\}_{r=1}^{m} \cup \{G_\alpha\}_{\alpha=1}^{k-2m}, \tag{2.21}$$

we must impose some extra conditions on the complex structure J. In particular, if we require holomorphic first-class properties for the G_A this leads to the conditions

$$\begin{aligned}
\left\{G_r^{(-)}, G_s^{(-)}\right\} &= f_{rs}{}^t G_t^{(-)} + f_{rs}{}^\alpha G_\alpha, \\
\left\{G_\alpha, G_r^{(-)}\right\} &= f_{\alpha r}{}^t G_t^{(-)} + f_{\alpha r}{}^\beta G_\beta.
\end{aligned} \tag{2.22}$$

From this we can infer that J must satisfy a symplectic condition with respect to c_{rs}:

$$J_r{}^t c_{ts} + J_s{}^t c_{rt} = 0. \tag{2.23}$$

In addition J must also satisfy differential relations such as

$$\{G_\alpha, J_r{}^s\} = [\, f_\alpha, J\,]_r{}^s, \tag{2.24}$$

where the commutator on the right-hand side is to be interpreted as an ordinary matrix commutator with respect to the indices (r, s). Another consequence of these results is, that the part of Z_{AB} which is non-vanishing on the constraint hypersurface $\mathcal{N}$ is

$$Z_{rs} = \frac{1}{4}\left(\delta_r^t - iJ_r{}^t\right) c_{ts}. \tag{2.25}$$

The general conditions for the existence of this complex structure J are not known to us and this problem needs further investigation. Obviously, if it is not possible to absorb all second-class constraints in holomorphic first-class ones, one must still eliminate such special second-class constraints first by Dirac's procedure, before the passage to the holomorphic structure can be made. However, the fact that for bosonic systems the algebra of the second-class constraints can always be transformed locally to the algebra of a (q,p)-system, with $\{q,p\} = 1$, indicates that in this case the complex structure always exists. Then it is no longer necessary to remove second-class constraints before quantization, as the whole analysis above, eqs. (2.11)-(2.25), can be repeated at the quantum level (replacing Poisson-brackets by commutators). For fermionic systems, the situation is different[3]. Indeed it is mostly in this area that the general applicability of our scheme still needs

[3] The only example known to us of a system for which a construction of holomorphic constraints seems impossible without additional second-class constraints is a system with an odd number of fermionic second-class constraints, such as the spinning (super)particle in odd dimensions.

further investigation. On the other hand, the first non-trivial application of our procedure was the quantization of the 4D-superparticle [16], which does happen to be a system with fermionic constraints. Therefore we are confident that our procedure should work also in most cases with fermionic symmetries.

3 The inner product and the holomorphic BRST-operator

The canonical BRST-procedure consists of constructing a nilpotent operator Q in such a way that the constraints (2.1) on the physical states become the solution of the cohomology problem defined by this BRST operator Q. Explicitly:

$$Q^2 = 0, \tag{3.1}$$

and the physical states are characterized by the conditions

$$Q|phys> = 0, \tag{3.2}$$

with two states being identified if they differ by a BRST-exact term:

$$|phys> \sim |phys'> \Leftrightarrow |phys'> = |phys> + Q|*>, \tag{3.3}$$

where $|*>$ is an arbitrary state. In order to construct such an operator, we must introduce ghost operators c^A and their conjugate momenta π_A:

$$\left\{\pi_B, c^A\right\} = \delta_B^A. \tag{3.4}$$

For the constraint algebra (2.2),(2.4) a nilpotent operator Q can be constructed of the form

$$Q =: c^A G_A : + \frac{1}{2}(-)^A : c^A c^B f_{BA}{}^C \pi_C : + \ldots \tag{3.5}$$

where the trailing dots denote additional terms which are required if the set of constraints is reducible and/or open. If the constraint algebra is complex, there also exists an independent hermitean conjugate BRST operator $Q^\dagger$:

$$Q^\dagger =: c^{\dagger A} G^\dagger_A : + \frac{1}{2}(-)^A \; : \pi^\dagger_C f^C{}_{AB} c^{\dagger B} c^{\dagger A} : + \ldots, \tag{3.6}$$

with the property that

$$< phys|Q^\dagger = 0. \tag{3.7}$$

To construct the operator $Q^\dagger$ above, we introduce the new set of ghost variables $(c^\dagger, \pi^\dagger)$, which anticommute with their adjoint operators (c, π) and have the fundamental (anti)commutator

$$\left[c^{A\dagger}, \pi^\dagger_B \right\} = \delta^A_B. \tag{3.8}$$

Both Q and $Q^\dagger$ are nilpotent, as may be checked by explicit calculation. The full algebra of there operators reads [17]

$$\begin{aligned} Q^2 = Q^{\dagger 2} &= 0, \\ \left\{ Q, Q^\dagger \right\} &= 2Z, \\ [Z, Q] = \left[Z, Q^\dagger \right] &= 0, \end{aligned} \tag{3.9}$$

where Z is defined by

$$Z =: c^A Z_{AB} c^{\dagger B} : + \ldots \tag{3.10}$$

It is well known that the non-trivial solutions of the condition (3.2) – on the states that are ghosts free – satisfy the physical state conditions (2.1). We take this as an established fact. However, having physical states is not sufficient: we also need an inner product to obtain matrix elements and probabilities. In particular physical quantities are obtained as matrix elements of observables, an observable being defined as a non-trivial BRST-invariant operator with zero ghost-number:

$$[Q, O) = 0. \tag{3.11}$$

There is an ambiguity in any observable of the form $\Delta O = [Q, *)$, but this should be immaterial when we compute matrix elements between physical states; indeed, for *real* Q this is obvious:

$$< phys'|[Q,*\}|phys >= 0, \qquad (3.12)$$

because Q annihilates both the physical ket- and bra-states. However, in the general case of complex constraints, this is no longer true, since Q and $Q^\dagger$ are independent and Q does not annihilate the bra-states.

Another difficulty connected with the complex BRST-algebra is, that BRST-exact states are not spurious; they have non-vanishing norm:

$$\|Q|\chi>\|^2 =< \chi|Q^\dagger Q|\chi > \neq 0. \qquad (3.13)$$

and they neither decouple from physical states.

A solution to this problem was presented in refs. [16], [17]. It consists of regarding the algebra (3.9) as another holomorphic constraint algebra on the physical states and constructing a corresponding *real* BRST operator, which may be called a second-level BRST operator:

$$\Omega = s^\dagger Q + Q^\dagger s + \gamma Z - 2s^\dagger s\beta, \qquad (3.14)$$

where $(s^\dagger, s)$ are commuting scalar ghosts and (γ, β) is a pair of real conjugate Fermi-type ghost operators:

$$\left[s, s^\dagger\right] = 0, \qquad (3.15)$$

$$\{\gamma, \beta\} = 1. \qquad (3.16)$$

These properties, together with the algebra (3.9) suffice to prove that

$$\Omega^2 = 0. \qquad (3.17)$$

In the next section we verify that the cohomology classes of this operator define the cohomology classes of the operators Q and $Q^\dagger$ and hence the physical states. Before doing so, we first turn to a discussion of the ghost system.

In order to define a quantum theory we must introduce the pseudo-Hilbert space of wave functions, on which the operators defined above act. This space has the natural structure of a tensor product of spaces corresponding to the action of the real ghost operators, (γ, β), first and second level ghost operators, $(c, c^\dagger; s, s^\dagger)$ and the operators corresponding to the classical gauge system, like the constraints G and $G^\dagger$. We assume that this last part of the space of states is known and turn to the discussion of the ghost sectors.

From the theory of Clifford algebras we know, that the only irreducible representation of the real fermionic ghost system (γ, β) is 2-dimensionional, a typical element of the representation space being given by the ordered pair of functions $\Psi = (a, b)$ of the other variables. It can be equipped naturally with an indefinite inner-product

$$(\Psi_1, \Psi_2) = a_1^* b_2 + b_1^* a_2, \tag{3.18}$$

where the multiplication on the right-hand side is to be understood as an inner product. The representation of the anticommutation relation (3.16) can be realised in terms of a single real Grassmann variable[4] η, with γ being the multiplication operator with respect to η, and β the derivative. An arbitrary state in this realisation is of the form

$$\Psi = a + \eta\, b, \tag{3.19}$$

with

$$\begin{aligned} \gamma\,\Psi &= \eta\,\Psi = \eta\, a, \\ \beta\,\Psi &= \frac{\partial \Psi}{\partial \eta} = b. \end{aligned} \tag{3.20}$$

Finally, the inner-product (3.18) is realised by the bilinear Grassmann-integral expression

$$(\Psi_1, \Psi_2) = \int d\eta\; \Psi_1^* \Psi_2. \tag{3.21}$$

Equivalently, one can write

$$(\Psi_1, \Psi_2) = \left(a_1^* \frac{\partial}{\partial \eta} + b_1^* \right) (a_2 + \eta b_2)\, |_{\eta=0}. \tag{3.22}$$

[4]Here and below we deliberately use different letters to denote operators and wave function variables

The state space for the complex bosonic ghosts operators $(s, s^\dagger)$ is specified by their commutation relation (3.15) and an inner-product, which is taken to be indefinite. The ghosts operators being bosonic, their representation spaces are infinite dimensional and states consist of analytic functions of two complex variables (z, z^*) in involution[5], $\Psi = \sum_{m,n=0}^{\infty} a_{nm} z^n z^{*m}$. This state space can now be endowed with an inner-product of the form

$$(\Psi_1, \Psi_2) = \sum_{m,n=0}^{\infty} a_{nm}^* a_{mn}, \tag{3.23}$$

where again the multiplication on the right hand side should be understood as an inner product. The natural realisation of this bosonic ghost space is the space of polynomials in ghost variables with complex co-efficients. Such a representation space results for example from the holomorphic representation of Fock-space[6]. In this representation the inner-product (3.23) can be realised straightforwardly.

We define a representation of the operators $(s, s^\dagger)$ on the space described above in terms of multiplication and differentiation with respect to the variables (z, z^*) respectively:

$$s \rightarrow \frac{\partial}{\partial z}, \qquad s^\dagger \rightarrow z^*. \tag{3.24}$$

These operators certainly commute, as required by eq.(3.15). In order to prove that they are mutually adjoint we construct the following basis for the ghost states:

$$\Psi_{n,m}(z, z^*) = z^n z^{*m}. \tag{3.25}$$

Hence for a general state Ψ we can write

$$\Psi = \sum_{m,n=0}^{\infty} a_{nm} \Psi_{n,m}. \tag{3.26}$$

The action of the operators $(s, s^\dagger)$ on the basis elements is

[5]Cf. [27]. One should not mix the notion of involution denoted by a star with the complex conjugation denoted by a bar.

[6]A discussion can be found in [27].

$$s\Psi_{n,m} = n\Psi_{n-1,m}, \qquad s^\dagger\Psi_{n,m} = \Psi_{n,m+1}. \tag{3.27}$$

The inner product (3.23) can now be realised by taking

$$\begin{aligned}(\Psi_{k,l}, \Psi_{n,m}) &= \left(\frac{\partial}{\partial z}\right)^l \left(\frac{\partial}{\partial z^*}\right)^k \Psi_{n,m}|_{z=z^*=0} \\ &= k!\, l!\, \delta_{l,n}\, \delta_{k,m}.\end{aligned} \tag{3.28}$$

It is then easy to check that

$$(s\,\Psi_{k,l}, \Psi_{n,m}) = \left(\Psi_{k,l}, s^\dagger\,\Psi_{n,m}\right), \tag{3.29}$$

which proves that $(s, s^\dagger)$ as defined in (3.24), (3.27) are adjoint operators indeed. A peculiarity of our construction -and of the holomorphic representation of Fock space in general- is that taking the adjoint is not the same as involution of the variables (z, z^*). Nevertheless one can show that the eigenvalues of self-adjoint operators in a space of this type are real.

For the first-level ghost operators $(c^A, c^{\dagger A})$ a similar construction is possible[7]. In the space of anticommuting ghosts we define the basis elements to be strings of ghosts, to wit

$$W^I_J(g, g^*) = g^{i_1} g^{i_2} \ldots g^{i_k} g^*_{j_1} g^*_{j_2} \ldots g^*_{j_l}, \tag{3.30}$$

where I, J are ordered multi-indices ($I = (i_1, \ldots\ldots, i_k)$ and $|I| = k$). With the definition (3.30) we define the inner product $(\bullet, \bullet)$ so as to satisfy the following requirements

$$(W^I_J, W^{I'}_{J'}) = \begin{cases} (-1)^{|I||J|} \delta^I_{J'} \delta^{I'}_J & \text{if } |I| = |J'| \text{ and } |J| = |I'| \\ 0 & \text{otherwise} \end{cases} \tag{3.31}$$

One can then easily check that with respect to the inner product (3.31)

[7] In the construction presented below we assume that all the first-level ghost operators are anticommuting. For the commuting first-level ghost operators the inner product and the representation of the commutation relations is analogous to those presented above.

$$c^A \to g^A, \qquad \pi_A \to \frac{\partial}{\partial g^A},$$
$$c^{\dagger A} \to \frac{\partial}{\partial g^*_A}, \qquad \pi^\dagger_A \to g^*_A. \tag{3.32}$$

The explicit form of the new BRST operator Ω, eq.(3.14), now becomes:

$$\Omega = \frac{\partial}{\partial z}Q + Q^\dagger z^* + \eta Z - 2z^*\frac{\partial}{\partial z}\frac{\partial}{\partial \eta}. \tag{3.33}$$

where

$$Q = g^A G_A + \frac{1}{2}(-1)^A g^A g^B f_{BA}{}^C \frac{\partial}{\partial g^C} + \ldots, \tag{3.34}$$
$$Q^\dagger = G^\dagger_A \frac{\partial}{\partial g^*_A} + \frac{1}{2}(-1)^A g^*_C f^C{}_{AB} \frac{\partial}{\partial g^*_B}\frac{\partial}{\partial g^*_A} + \ldots. \tag{3.35}$$

By construction the operator Ω is self-adjoint with respect to the inner product defined above.

In addition to the second-level BRST operator Ω we also need another operator of bosonic type. For obvious reasons it is known as the total ghost number $\mathcal{N}$:

$$\mathcal{N} = N_c + N_c^\dagger + N_s + N_s^\dagger - 2a, \tag{3.36}$$

where the individual terms take the form

$$N_s = s\pi_s = z\frac{\partial}{\partial z}, \qquad N_s^\dagger = \pi_s^\dagger s^\dagger = z^*\frac{\partial}{\partial z^*}, \tag{3.37}$$

and similarly for the other ghost variables, such that

$$[N_s, s] = s,$$
$$\left[N_c, c^A\right] = c^A. \tag{3.38}$$

Note that in the space defined by the basis $\{\Psi_{n,m}\}$, eq.(3.25), the spectrum of eigenvalues of the operators $N_{s,c}$ consists of the non-negative integers and that $\mathcal{N}$ commutes with Ω.

The real number a in (3.36) is a constant yet to be determined, reflecting the operator ordering ambiguities. It should be stressed that different choices of the parameter a lead in general to the *different quantum theories*. We will see this phenomenon in the derivation of the cohomology of Ω below.

4 The cohomology of Ω

In this section we calculate the cohomology of the operator Ω defined in previous section, eq.(3.33). The representation space, $\mathcal{H}$ possesses a four-fold grading in the ghost variables. It is useful therefore to restrict our representation space to contain only the zero total ghost number subspace of the extended space of states

$$\mathcal{H}_{RED} = \{\Psi \in \mathcal{H} : \mathcal{N}\Psi = 0\}. \tag{4.1}$$

In order to define this restriction in the language of cohomology, we extend the operator Ω to the operator Ω^{ext} in a natural way[8],

$$\Omega^{ext} = \Omega + \alpha\,\mathcal{N}, \tag{4.2}$$

such that the cohomology of Ω^{ext} is isomorphic to the cohomology of Ω subject the condition (4.1). Details of the derivation of this result are presented in Appendix A.

Let us proceed with the calculation of the cohomology of our BRST operator Ω. Expanding Ψ in ghosts (z, z^*, η) in the standard way we have

$$\Psi = \sum_{m,n\geq 0} \psi^0_{n,m} z^n z^{*m} + \eta \sum_{m,n\geq 0} \psi^1_{n,m} z^n z^{*m}. \tag{4.3}$$

The derivation of the cohomology of Ω on the states (4.3) is given in Appendix B; here we present only the final result:

$$\Psi_{coh} \in \frac{Ker\Omega}{Im\Omega} \Longleftrightarrow$$

[8] The appearance of the Ω^{ext} operator may seem to be a little ad hoc. However in exactly the same way as the operator Ω is an operator obtained by gauging of the algebra (3.9) in the sense of ref. [18], the Ω^{ext} is a BRST operator obtained by gauging the algebra which the operators $Q, Q^\dagger, Z$ and $N_e + N_e^\dagger$ satisfy

$$
\begin{aligned}
\Psi_{coh} &= \sum_{n\geq 0}\psi^0_{n,0}z^n + \eta\sum_{m\geq 0}\psi^1_{0,m}z^{*m} \\
&+ \eta\sum_{n\geq 0}\frac{1}{2(n+1)}Q^\dagger\psi^0_{n,0}z^{n+1},
\end{aligned} \tag{4.4}
$$

where the coefficient functions satisfy the following conditions (δ is a remnant of the cohomology equivalence for Ω)

$$
\begin{aligned}
Q\psi^0_{n,0} &= 0, \\
\delta\psi^0_{n,0} &= Q\phi^0_{n,0},
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
Q^\dagger\psi^1_{0,m} &= 0, \\
\delta\psi^1_{0,m} &= Q^\dagger\phi^1_{0,m},
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
Q^\dagger Q\psi^0_{0,0} &= 0, \\
\delta\psi^0_{0,0} &= Q\phi^0_{0,0},
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
Q^\dagger\psi^1_{0,0} &= 0, \\
\delta\psi^1_{0,0} &= Q^\dagger Q\phi^1_{0,0}.
\end{aligned} \tag{4.8}
$$

Let us analyze the above conditions. From (4.5) we see that the states $\psi^0_{n,0}, n\geq 1$ are the cohomology classes of the operator Q. The states $\psi^1_{0,m}, m\geq 1$, (4.6), are the homology classes of $Q^\dagger$ ($Q^\dagger$ is a homology operator as it decreases the grading).

The interpretation of the states $\psi^0_{0,0}$ and $\psi^1_{0,0}$ is less obvious. Observe that these states are not defined in terms of cohomologies: $Q^\dagger Q$ in (4.7) is not a nilpotent operator, and in the BRST-transformation rule for $\psi^1_{0,0}$, (4.8), $Q^\dagger$ is restricted only to ImQ, instead of being the $Q^\dagger$-image of the whole space. However, it follows from eq. (4.7) that $Q\psi^0_{0,0}$ is a spurious state:

$$
(Q\psi^0_{0,0}, Q\psi^0_{0,0}) = (\psi^0_{0,0}, Q^\dagger Q\psi^0_{0,0}) = 0. \tag{4.9}
$$

Now we turn to the condition (4.1). From the requirement $\mathcal{N}\Psi = 0$ we find that

$$
\begin{aligned}
(n_g + n_{g^*} + n - 2a)\psi^0_{n,0} &= 0 && (4.10)\\
(n_g + n_{g^*} + m - 2a)\psi^1_{0,m} &= 0 && (4.11)
\end{aligned}
$$

where n_g and n_{g^*} are eigenvalues of N_c and $N_c^\dagger$ in the given states and are positive integers or zero.

We already observed that the states $\psi^0_{n,0}$ are restricted to be the cohomology of Q. To find the value of a we assume therefore that the space these states includes the physical states i.e. those which are independent of the variables g and g^*, so they have a vanishing first-level ghost number. From (4.10) it follows therefore that for these particular states $n = 2a$.

The next requirement is that the physical states must not be spurious i.e. they cannot have a vanishing product with Ψ_{eoA}. The only ψ^1 state, which has a nonvanishing product with the physical state $\psi^0_{2a,0}$ is the state $\psi^1_{0,2a}$, which is also by (4.11) first-level ghost free. From (4.10) and (4.11) we see also that since n, m, n_g, n_{g^*} are positive integers or zero, $\psi^0_{n,0}$ and $\psi^1_{0,m}$ vanish for $n, m > 2a$. Therefore a measures the lenght of the Q and $Q^\dagger$ cohomology and homology sequences, and we may define our quantum theory to be the one with the simplest possible choice, namely[9] $a = \frac{1}{2}$.

Our final result is therefore

$$
\begin{aligned}
\Psi_{eoA} &= \psi_0 + z\psi_1 + \eta(\psi_0^* + z^*\psi_1^*)\\
&+ \eta(\frac{1}{2}Q^\dagger\psi_0 z + \frac{1}{4}Q^\dagger\psi_1 z^3), && (4.12)
\end{aligned}
$$

where

$$
Q\psi_1 = 0 \longrightarrow G_A\psi_1 = 0, \qquad (4.13)
$$

[9] If we choose $a = 0$ instead, the resulting theory would be trivial, as the conditions (4.7),(4.8) are satisfied identically. On the other hand, in the case of bosonic constraints only, so that the first level ghosts are all anticommuting, one can choose the parameter a big enough, such that the states $\psi^i_{0,0}$, $i = 1,2$ vanish by virtue of eqs. (4.9)-(4.10). In any case the choice made above provides us with the simplest theory. Observe also that in our construction the problem of higher cohomology never arises.

and hence ψ_1 is a physical state, as defined in section 2.

The state ψ_1^* is annihilated by $Q^\dagger$ identically, and therefore it is defined to be a state in the physical Hilbert space modulo the relation

$$\delta\psi_1^* = Q^\dagger\phi. \tag{4.14}$$

Yet ϕ must be of the form $\phi = \phi^A g^{*A}$ and therefore the space of the states ψ_4 is the subspace of the physical Hilbert space, consisting of all the states which are not a $G_A^\dagger$ divergence, $G_A^\dagger\phi^A$. These states are clearly adjoint to the physical states (4.13). The interpretation of the states ψ_0 and ψ_3 is not clear to us at the moment and it deserves further study. It may well happen that by the further extension of the operator Ω (for example by adding the anti-BRST operators) one may end up with the zero-ghost cohomology only.

The result of this chapter can be summarized in the diagram

$$\begin{array}{ccccc} V_0 & \xleftarrow{Q} & V_1 & \xleftarrow{Q} & 0 \\ Q^\dagger Q \downarrow & & & & \\ V_0^* & \xrightarrow{Q^\dagger} & W_1^* & \xrightarrow{Q^\dagger} & 0. \end{array} \tag{4.15}$$

In this diagram we denote the space of wave functions ψ_i by V_i and similarly for starred wave functions. Therefore V_0 is the space of functions linear in first-level ghosts and V_1 is first-level ghost free. It follows from the discussion above that V_i and V_i^* are adjoint with respect to the inner product defined in sect. 3.

In conclusion, the result of this section is that the cohomology of the real operator Ω on the reduced space of states $\mathcal{H}_{RED}$ is given by the cohomologies of the complex operators Q and $Q^\dagger$. For the zero ghost level, the non-trivial cohomology on the spaces V_1, V_1^* can be interpreted as the physical state space and its adjoint. This result is consistent with what one expects from a Gupta-Bleuler type of procedure. Indeed, as in QED, we obtained the condition which involves only the holomorphic part of the set of constraints acting on physical states. The result obtained above should be compared with the calculation of the cohomology of Ω on the unrestricted space of states presented in [17]. There we found that the cohomology was the direct sum of the cohomologies of Q and $Q^\dagger$. In that case, however the states have unrestricted dependence of the first level ghosts g and g^*, which was a rather unpleasant feature. This shows how much the cohomology depends on the vector space in which the BRST operator acts. In fact we have seen this phenomenon also in this section: different choices of the parameter a lead to different cohomologies of the operator Ω. Indeed, a is a measure of the lenght of the sequence, or in other words it is an upper bound for the first-level ghost number.

5 Conclusions

In comparison with our earlier work (see [16], [17], [18]) we introduce in this paper an additional structure: the inner product (3.22), (3.28) in the ghost sector, implying unconventional hermicity relations (3.24), (3.32). These relations introduce the difference between involution '*' and hermitean conjugation '†'. Such a difference implies further that the cohomology of the hermitean operator Ω, which is *not* real, i.e. not invariant under involution, is described by the cohomologies of the complex operator Q and its *hermitean* conjugate $Q^\dagger$, representing at the zero ghost level the physical states which satisfy a *complex* condition (2.1) and its adjoint.

In this paper we did not discuss in detail the problems that arise if the set of constraints is open and/or reducible. It is obvious however, that in such a case one can follow the standard prescription [23],[24] and extend the BRST operators with ghosts-for-ghosts. The definition of the second-level BRST operator Ω is unchanged and therefore our description of its cohomology would still be valid in this case.

We would like to mention that another powerful method for quantization of the systems with second class constraints has been proposed recently [19], [20], [21]. In this method one extends the classical phase space in such a way that one can interprete the second class constraints as gauge fixing conditions. In contrast, we extend the ghost sector by introducing the second-level ghosts, keeping the set of physical variables unchanged. The advantage of our method is, that in the Gupta-Bleuler approach all manipulations can be performed already on the quantum level, where the structure of the constraints (including the anomalous second class part) is in principle known. Therefore we expect that our method can readily be applied to the BRST quantization of anomalous theories like chiral QED or string theory away from the critical dimension.

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7 Appendix A

In this appendix we introduce an extended second-level BRST charge Ω^{ext} by gauging in addition to the generators $Q, Q^\dagger, Z$ also the total ghost-number $\mathcal{N}$ defined in sect. 3. Corresponding to this operator we introduce a new anti-commuting ghosts α and we define a nilpotent, hermitean operator

$$\Omega^{ext} = \Omega + \alpha\,\mathcal{N}. \tag{A.1}$$

As usual we relate the physical states to the cohomology complex of the operator Ω^{ext}, as follows:

$$\Omega^{ext}\Psi = 0, \qquad \Psi \sim \Psi + \Omega^{ext}\Phi. \tag{A.2}$$

Expanding Ψ in α we get

$$\Psi = \psi_1 + \alpha\psi_2. \tag{A.3}$$

The first condition (A.2) then becomes

$$\begin{aligned} \Omega\psi_1 &= 0, && \text{(A.4)} \\ \mathcal{N}\psi_1 - \Omega\psi_2 &= 0. && \text{(A.5)} \end{aligned}$$

The functions ψ_i (i = 1,2) belonging to the same cohomology class of Ω^{ext} are related as follows

$$\begin{aligned} \psi_1' &= \psi_1 + \Omega\phi_1, && \text{(A.6)} \\ \psi_2' &= \psi_2 + \mathcal{N}\phi_1 - \Omega\phi_2. && \text{(A.7)} \end{aligned}$$

We solve the cohomology problem (A.4 - A.7) in two steps. First we observe from (A.7) that by an appropriate choice of ϕ_1 we can make

$$\mathcal{N}\psi_2 = 0. \tag{A.8}$$

Since $\mathcal{N}$ commutes with Ω, it follows from eq.(A.5) that

$$\mathcal{N}\psi_1 = 0. \tag{A.9}$$

After that it is still allowed to make gauge transformations (A.6,A.7) with gauge functions ϕ_i satisfying

$$\mathcal{N}\phi_i = 0. \tag{A.10}$$

We infer that all components of the solutions of eq.(A.2) satisfy the conditions

$$\Omega\psi_i = 0, \qquad \mathcal{N}\psi_i = 0, \tag{A.11}$$

still subject to gauge transformations

$$\psi_i \to \psi_i + \Omega\phi_i, \tag{A.12}$$

provided $\mathcal{N}\phi_i = 0$.

The final result may therefore be expressed in the following way

$$\mathcal{H}(\Omega^{ext}) = \bigoplus_{1}^{2}(\mathcal{H}(\Omega) \cap Ker(\mathcal{N})). \tag{A.13}$$

We observe that the cohomology complex of Ω^{ext} is two-fold degenerate. In the main text we use only one of the functions ψ_i as the representation of the Hilbert space on which the operator Ω acts.

8 Appendix B

In this appendix we present the derivation of the cohomology of the operator Ω. The generic state of the pseudo-Hilbert space is given by

$$\Psi = \sum_{n,m\geq 0} \psi^0_{n,m} z^n z^{*m} + \gamma \sum_{n,m\geq 0} \psi^1_{n,m} z^n z^{*m}. \tag{B.1}$$

In the first step let us try to gauge away some part of Ψ by using the fact that $\Psi \sim \Psi + \Omega\Phi$. Explicitly

$$
\begin{aligned}
\Omega\Phi &= \sum_{n,m\geq 0} \{nQ\phi^0_{n,m} z^{n-1} z^{*m} + Q^\dagger \phi^0_{n,m} z^n z^{*m+1}\} \\
&+ \gamma \sum_{n,m\geq 0} Z\phi^0_{n,m} z^n z^{*m} \\
&- \gamma \sum_{n,m\geq 0} \{nQ\phi^1_{n,m} z^{n-1} z^{*m} + Q^\dagger \phi^1_{n,m} z^n z^{*m+1}\} \\
&- 2 \sum_{n,m\geq 0} n\phi^1_{n,m} z^{n-1} z^{*m+1}. \qquad \text{(B.2)}
\end{aligned}
$$

Examining the last line of the above expression, we see that one can find $\phi^1_{n,m}$ such that the gauge fixed Ψ is of the form

$$
\Psi = \sum_{n\geq 0} \psi^0_{n,0} z^n + \gamma \sum_{n,m\geq 0} \psi^1_{n,m} z^n z^{*m}. \qquad \text{(B.3)}
$$

Now let us turn to the condition that the state Ψ is Ω-closed: $\Omega\Psi = 0$. Expanding this equation in γ we have

$$
\begin{aligned}
&\sum_{n\geq 0} \{nQ\psi^0_{n,0} z^{n-1} + Q^\dagger \psi^0_{n,0} z^* z^n\} \\
&- 2 \sum_{n,m\geq 0} n\psi^1_{n,m} z^{n-1} z^{*m+1} = 0, \qquad \text{(B.4)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{n\geq 0} Z\psi^0_{n,0} z^n - \sum_{n,m\geq 0} \{nQ\psi^1_{n,m} z^{n-1} z^{*m} \\
&+ Q^\dagger \psi^1_{n,m} z^n z^{*m+1}\} = 0. \qquad \text{(B.5)}
\end{aligned}
$$

Comparing terms with the same powers of z^* we learn from the eqs. (B.4), (B.5) that

$$Q\psi^{0}_{n,0} = 0, \qquad n \geq 1, \tag{B.6}$$

$$Q^{\dagger}\psi^{1}_{0,m} = 0, \tag{B.7}$$

$$Q^{\dagger}Q\psi^{0}_{0,0} = 0, \tag{B.8}$$

$$\psi^{1}_{n+1,0} = \frac{1}{2(n+1)}Q^{\dagger}\psi^{0}_{n,0}, \tag{B.9}$$

$$\psi^{1}_{n,m} = 0 \qquad n, m \neq 0. \tag{B.10}$$

We must still take into account the residual gauge invariance of the gauge fixed states (B.3). Demanding that the form of (B.3) remains unchanged under $\delta\Psi = \Omega\Phi$ we see from (B.2) that the only gauge transformations left are given by

$$\delta\psi^{0}_{n,0} = (n+1)Q\phi^{0}_{n+1,0}, \tag{B.11}$$

$$\delta\psi^{1}_{n,0} = \frac{1}{2}Q^{\dagger}Q\phi^{0}_{n,0}, \tag{B.12}$$

$$\delta\psi^{1}_{0,m} = Q^{\dagger}\phi^{1}_{m}, \quad m \geq 1, \tag{B.13}$$

where ϕ_m is a combination of $\phi^{1}_{0,m}$ and $\phi^{0}_{0,m}$. Equations (B.6)-(B.8) and (B.11)-(B.13) provide us with the desired description of the cohomology of Ω.

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