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ABSTRACT

In this paper we study some properties of monogenic functions taking values in a Clifford algebra and depending on several parameters. It is proved that the Hartogs extension theorems are valid for these functions and for the multi-monogenic functions, which contain solutions of many important systems of partial differential equations in Theoretical Physics.

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1. INTRODUCTION

In [1] the theory of functions,taking values in a Clifford algebra was studied. It is proved, that many important properties of holomorphic functions in one complex variable may extend to monogenic functions, which are solutions of the generalized Cauchy-Riemann operator and take a very important place in Theoretical Physics.

Following this way, in [2] Brackx and Pincket introduced a version of biregular functions taking values in a Clifford algebra and prove some properties of holomorphic functions on several complex variables for these functions.

This is a generalization of the theory of monogenic functions in higher dimensions.

The purpose of this paper is at first to study some properties of monogenic functions depending on parameters, and then apply the obtained results to generalize, in a new way, the theory of monogenic functions in higher dimensions.

In the third section we prove the global real-analyticity of a function defined by an integral formula (Lemma 3.2). Applying this lemma we can prove the extension theorems for monogenic functions with parameters (theorems 3.1 and 3.2). Notice that this is the generalization of the results in [3].

In the fourth section we introduce a version of *multi-monogenic* functions, which are the other generalizations of monogenic functions in higher dimensions, and prove the Hartogs extension theorems for these functions. This contains the results in [4] as special cases.

2. PRELIMINARIES

Let $\{e_1, ..., e_n\}$ be an orthonormal basis of R^n, then by $\mathcal{A}$ we denote the real Clifford algebra constructed by means of this basis. This means that the product in $\mathcal{A}$ is determined by the relations

$$e_i e_j + e_j e_i = -2\delta_{ij}$$

and that a general element of $\mathcal{A}$ is of the form

$$a = \sum_\alpha a_\alpha e_\alpha ; a_\alpha \in R$$

where N = 1,...,n ;

$$\alpha = \{\alpha_1, ..., \alpha_h\}; 1 \leq \alpha_1 \prec ... \prec \alpha_h \leq n$$

$$e_\alpha = e_{\alpha_1} ... e_{\alpha_h}.$$

An involution in $\mathcal{A}$ is given by

$$\overline{a} = \sum_\alpha a_\alpha \overline{e}_\alpha$$

where

$$\overline{e}_\alpha = \overline{e}_{\alpha_h} ... \overline{e}_{\alpha_1}; \overline{e}_j = -e_j j = 1, ..., n.$$

For $m \leq n, R^{m+1}$ is naturally imbedded in $\mathcal{A}$. Hence

$$x = (x_o, x_1, ..., x_m) \in R^{m+1}$$

will be identified with

$$x = x_o + \vec{x} = x_o e_o + \sum_{j=1}^m x_j e_j,$$

where $e_o = e_\emptyset = 1$ is the identity of $\mathcal{A}$. Then $\overline{x} = x_o - \vec{x}$.

The inner product is defined by

$$(a, b)_o = 2^n \sum_\alpha a_\alpha b_\alpha, (b = \sum_\alpha b_\alpha e_\alpha).$$

Hence a norm is defined by

$$\|a\|_o = 2^{n/2} (\sum_\alpha a_\alpha^2)^{1/2},$$

which turn $\mathcal{A}$ into a Banach Algebra of dimension 2^n.

For the other definitions we refer the reader to [1].

In the sequel we consider the functions f(x,t) of

$$x = (x_o, x_1, ..., x_m) \in \Omega \subseteq R^{m+1}$$

and

$$t = (t_o, t_1, ..., t_k) \in \mathcal{T} \subseteq R^{k+1}$$

and taking values in $\mathcal{A}; m, k \leq n$.

Those functions are of the form

$$f(x, t) = \sum_\alpha f_\alpha(x, t) e_\alpha,$$

where components $f_\alpha(x, t)$ are real-valued functions of $(x, t) \in \Omega \times \mathcal{T} \subset R^{m+k+2}(x, t)$.

Suppose that f(x,t) is (left) monogenic on $x \in \Omega$ by each fixed $t \in \mathcal{T}$, namly f satisfies the system

$$D_x f = 0 \qquad (2.1)$$

where

$$D_x = \sum_{j=o}^m e_j \frac{\partial}{\partial x_j}$$

is the generalized Cauchy-Riemann operator.

Then f(x,t) is (real) analytic on $x \in \Omega$ by each fixed $t \in \mathcal{T}$, (see [1,3]).

Such f is said to be a *monogenic function with parameter*. (Note that t is considered as parameter).

3. MONOGENIC FUNCTIONS WITH PARAMETER

Let $\Omega, \mathcal{T}$ be domains in $R^{m+1}(x)$ and $R^{k+1}(t)$ respectively, f(x,t) is monogenic on $x \in \Omega$ (by fixed $t \in \mathcal{T}$) and (real) analytic on $t = (t_o, t_1, ..., t_k) \in \mathcal{T}$ by fixed $x \in \Omega$, then we have

Lemma 3.1:

If f = o in a non-empty open subset $\sigma \subset \Omega \times \mathcal{T}$, then $f \equiv 0$ in $\Omega \times \mathcal{T}$.

Proof:

We may without restriction assume that

$$\sigma = \Delta_x \times \Delta_t,$$

where Δ_x and Δ_t are polydiscs in $R^{m+1}(x)$ and $R^{k+1}(t)$ respectively.

By fixed $t \in \mathcal{T}$, since f is monogenic it is (real) analytic on $x \in \Omega$. From the Identity theorem (for real analytic functions) it follows that f = 0 for all $x \in \Omega$. Applying the same theorem for f with respect to the variable t we have $f \equiv 0$ in $\Omega \times \mathcal{T}$. Q.e.d.

Lemma 3.2:

Let Σ be an open neighbourhood of $\partial(\Omega \times \mathcal{T})$ and f(x,t) be (real) analytic in all variables $(x_o, ..., x_m; t_o, ..., t_k)$ for $(x,t) \in \Sigma$. Then

$$F(x,t) = \frac{1}{\omega_{m+1}} \int_{\partial\Omega} \frac{\overline{u} - \overline{x}}{\mid u - x \mid^{m+1}} d\sigma_u f(u,t) \qquad (3.1)$$

is (real) analytic in (x,t) for $(x,t) \in \Omega \times \mathcal{T}$;

where

$$u = u_o e_o + u_1 e_1 + ... + u_m e_m$$

$$x = x_o e_o + x_1 e_1 + ... + x_m e_m$$

$$d\sigma_u = \sum_{j=0}^{m} (-1)^j e_j d\hat{u}_j$$

with

$$d\hat{u}_j = du_o \wedge ... \wedge du_{j-1} \wedge du_{j+1} \wedge ... \wedge du_m$$

and ω_{m+1} is the surface area of the unit sphere S^{m+1} in R^{m+1}.

Proof:

First, note that F(x,t) is defined for all $(x,t) \in \Omega \times \mathcal{T}$.

Take arbitrarily a point $(a,b) \in \Omega \times \mathcal{T}$. We show now that F can be expressed as sum of a power series, which converges normally in a neighbourhood of (a,b).

Let $u^o \in \partial\Omega$ be an arbitrary point. Because of the hypothesis f(x,t) is defined and (real) analytic in an open neighbourhood of the point $(u^o, b) \in \Sigma$. Hence there exist $\rho_{u^o} \succ o$ and $\rho_b \succ 0$ such that

$$f(u,t) = \sum_{\mu,\nu=0}^{\infty} c_{\mu,\nu}(u - u^o)^{\mu}(t - b)^{\nu} \qquad (3.2)$$

for

$$u \in B_m(u^o, \rho_{u^o}); t \in B_k(b, \rho_b);$$

where $B_m(u^o, \rho_{u^o})$ and $B_k(b, \rho_b)$ are balls in R^{m+1} and R^{k+1} respectively;

$$(u - u^o)^{\mu} = (u_o - u^o_o)^{\mu_o} ... (u_m - u^o_m)^{\mu_m}$$

$$(t - b)^{\nu} = (t_o - b_o)^{\nu_o} ... (t_k - b_k)^{\nu_k}$$

$$\mu = (\mu_o, ..., \mu_m); \nu = (\nu_o, ..., \nu_k) : \mu_i \geq 0, \nu_j \geq 0$$

$c_{\mu\nu}$ are Clifford constants.

The series converges normally in

$$B^o_m(u^o, \rho_{u^o}) \times B^o_k(b, \rho_b)$$

and

$$B_m(u^o, \rho_{u^o}) \times B_k(b, \rho_b) \subset \Sigma.$$

Set

$$\sigma_{u^o} = B_m(u^o, \rho_{u^o}) \cap (\partial\Omega),$$

then the system

$$\partial := \{\sigma_{u^o}; u^o \in \partial\Omega\}$$

is an open covering of $\partial\Omega$.

Since $\partial\Omega$ is compact, we can choose a finite subcover $\{\sigma_1, ..., \sigma_p\}$ from ∂ (of $\partial\Omega$).

Now set

$$\Gamma_1 = \sigma_1; \Gamma_2 = \sigma_2 \setminus \sigma_1, ..., \Gamma_p = \sigma_p \setminus (\Gamma_1 \cup ... \cup \Gamma_{p-1}),$$

then

$$\partial\Omega = \Gamma_1 \cup ... \cup \Gamma_p,$$

$$\Gamma_q \cap \Gamma_l = 0; 1 \leq q \neq l \leq p,$$

and

$$F(x,t) = \sum_{q=1}^{p} F_q(x,t), \tag{3.3}$$

where

$$F_q(x,t) = \frac{1}{\omega_{m+1}} \int_{\Gamma_q} \frac{\overline{u} - \overline{x}}{(\mid u - x \mid)^{m+1}} d\sigma_u f(u,t). \tag{3.4}$$

Because of the definition of Γ_q, we see that if $u \in \Gamma_q$, then

$$u \in B_m(u^{(q)}, \rho_q)$$

where

$$u^{(q)} \in \partial\Omega; \rho_q \succ 0.$$

Hence for those u we have

$$f(x,t) = \sum_{\mu,\nu=0}^{\infty} c_{\mu\nu}^{(q)} (u - u^q)^\mu (t-b)^\nu, \tag{3.5}$$

and the last series converges normally in

$$B_m(u^{(q)}, \rho_q) \times B_k^o(b, \rho_b).$$

On the other hand, if x sufficiently close to a , we have

$$\frac{\overline{u} - \overline{x}}{(\mid u - x \mid)^{m+1}} = \sum_{\beta=0}^{\infty} d_\beta (x-a)^\beta, \tag{3.6}$$

where $\beta = (\beta_o, \beta_1, ..., \beta_m)$

$$(x-a)^\beta = (x_o - a_o)^{\beta_o} ... (x_m - a_m)^{\beta_m},$$

d_β are Clifford constants and the series converges normally.

From (3.4) , (3.5) and (3.6) it follows:

$$F_q(x,t) = \sum_{\beta,\nu=0}^{\infty} c_{\beta\nu}^{(q)} (x-a)^\beta (t-b)^\nu \tag{3.7}$$

and the last series converges normally in a sufficiently small neighbourhood of (a,b).

This means that F_q can be presented as sum of a power series, which converges normally in a neighbourhood of (a,b).

Since F is the sum of a finite number of F_q, it has the same properties as F_q.

So for each point $(a,b) \in \Omega \times \mathcal{T}$ an open neighbourhood was found in which F(x,t) may be developed into a convergent multiple power series; hence F is (real) analytic in $\Omega \times \mathcal{T}$. Q.e.d.

Theorem 3.1:

Suppose that f(x,t) is a given (real) analytic function in Σ and f is (left) monogenic on variable x by each fixed t. Then there exists a unique function F(x,t) defined in $\Omega \times \mathcal{T}$ with the same properties and

$$F(x,t) = f(x,t)$$

in Σ.

Remark:

Roughly speaking, this theorem means that every monogenic function f (depending real-analytically on parameter t) in an open neighbourhood of $\partial(\Omega \times \mathcal{T})$, can be extended *monogenically* in the whole of $\Omega \times \mathcal{T}$.

Proof of Theorem 3.1:

Let F(x,t) be a function defined by the formula (3.1), where f(x,t) is the given function. Then because of the Lemma 3.2, F(x,t) is defined and (real) analytic for all $(x,t) \in \Omega \times \mathcal{T}$.

On the other hand, by definition, F is monogenic with respect to x, for $x \in \Omega$ by fixed $t \in \mathcal{T}$, (see[1]).

When t is sufficiently close to $\partial\mathcal{T}$ and $x \in \partial\Omega$ we have $(x,t) \in \Sigma$ and the right hand side of (3.1) becomes the generalized Cauchy integral formula for the given function f(x,t). Hence F(x,t) = f(x,t) for such t. From the Uniqueness theorem (for real analytic functions) it follows that

F(x,t) = f(x,t) in Σ.

Thus F(x,t) is the required extension of f(x,t) in $\Omega \times \mathcal{T}$. From Lemma 3.1 it follows that the extension F of f is unique. Q.e.d.

Let K be a compact subset of Ω and $\mathcal{U}$ be an open neighbourhood of the set

$$(K \times \mathcal{T}) \cup (\Omega \times \partial\mathcal{T}).$$

Then we have

Theorem 3.2:

Let f(x,t) be a given function in $\mathcal{U}$, which satisfies the hypothesis as in theorem 3.1. Then there exists a unique function F(x,t) with the same properties in $\Omega \times \mathcal{T}$, such that

$F(x,t) = f(x,t)$ in $\mathcal{U}$.

Proof:

We define the function F(x,t) again by formula (3.1). So F(x,t) is defined in the whole of $\Omega \times \mathcal{T}$ and monogenic with respect to x by fixed $t \in \mathcal{T}$. By repeating the proof of Lemma 3.1 we can show that F is (real) analytic in $\Omega \times \mathcal{T}$.

Further we need only repeat the proof of theorem 3.1 and see that F is the required extension of f and this extension is unique. Q.e.d.

4. APPLICATION

In the sequel Ω will denote an open set of the Euclidean space

$$R^{m_1+1} \times ... \times R^{m_l+1} := R^M$$

where

$$1 \preceq m_j \preceq n; j = 1, ..., l; M = m_1 + ... + m_l + l.$$

We consider functions f defined in Ω and taking values in the Clifford algebra $\mathcal{A}$.

Then

$$f = \sum_{\alpha \subseteq N} f_\alpha(x^{(1)}, ..., x^{(l)}) e_\alpha,$$

where

$$x^{(j)} = (x_o^{(j)}, x_1^{(j)}, ..., x_{m_j}^{(j)}) \in R^{m_j+1}; j = 1, ..., l.$$

We introduce the generalized Cauchy-Riemann operators

$$D_{x^{(j)}} := \sum_{i=0}^{m_j} e_i$$

Definition 4.1:

A function $f; \Omega \longrightarrow \mathcal{A}$ is called *multi-monogenic* in Ω iff

$$f \in c^1(\Omega; \mathcal{A})$$

and satisfies the system

$$D_{x^{(j)}} f = 0; j = 1, ..., l. \qquad (4.1)$$

Note that in case l = 1 the multi-monogenic functions reduce to (left) monogenic functions.

Thus the multi-monogenic functions are natural generalizations to higher dimension of the (left) monogenic functions. In case l = 2 the multi-monogenic functions are quite different from the biregular functions, whose theory was studied by F.Brackx and W.Pincket (see[2]).

Remark 4.1:

Let f be multi-monogenic in Ω; and

$$B(a,r) = B_1(a^{(1)}, r_1) \times \dots \times B_l(a^{(l)}, r_l),$$

where $B_j(a^{(j)}, r_j)$ is the ball in R^{m_j+1} with center $a^{(j)} = (a_o^{(j)}, \dots, a_{m_j}^{(j)})$ and the radius $r_j \succ 0$.

Suppose that $\overline{B(a,r)} \subset \Omega$.

By repeated use of the Cauchy's integral formula for monogenic functions (see[1]) we obtain

$$f(x) = \frac{1}{\omega_{m_1+1}} \dots \frac{1}{\omega_{m_l+1}} \int_{\partial_o B} \frac{\overline{u}^{(1)} - \overline{x}^{(1)}}{| u^{(1)} - x^{(1)} |^{m_1+1}} d\sigma_{u^{(1)}} \dots \frac{\overline{u}^{(l)} - \overline{x}^{(l)}}{| u^{(l)} - x^{(l)} |^{m_l+1}} d\sigma_{u^{(l)}} \quad (4.2)$$

for $x \in B^o(a, r)$;

where

$$u^{(j)} = u_o^{(j)} e_o + \dots + u_{m_j}^{(j)} e_{m_j}$$

$$u = (u^{(1)}, \dots, u^{(l)}) \in \partial_o B;$$

$$\partial_o B = \partial B_1 \times \dots \times \partial B_l;$$

ω_{m_j+1} are the surfaces areas of the unit spheres

$$S^{m_j+1} \subset R^{m_j+1}(x^{(j)})$$

and $d\sigma_{u^{(j)}}$ are defined as in **3**.

Remark 4.2:

Let $f \in c^2(\Omega; \mathcal{A})$ be a multi-monogenic function, then

$$\Delta_{x^{(j)}} f = \overline{D}_{x^{(j)}} . D_{x^{(j)}} f = 0$$

for j = 1,...,l.

Hence

$$\Delta f = (\Delta_{x^{(1)}} + \dots + \Delta_{x^{(l)}}) f = 0.$$

this means that f is harmonic in Ω. Then f is (real) analytic and the Uniqueness Theorem is valid for such functions f.

Now let

$$\Omega = \Omega_1 \times \dots \times \Omega_l \subset R^M$$

be a polycylinder, where Ω_j are domains in $R^{m_j+1}(x^{(j)})$; and Σ be an open neighbourhood of $\partial\Omega$.

Then we have

Theorem 4.1:

For every multi-monogenic function f in Σ, there exists an unique multi-monogenic function F in $\Omega \cup \Sigma$ such that F = f in Σ.

Roughly speaking, every multi-monogenic function in an open neighbourhood of the boundary of a polycylinder can be extended multi-monogenically in the whole of the polycylinder.

Proof:

If we consider f as a monogenic function of variable $x^{(1)} \in \Omega_1$ and depending on $x^{(2)}, \dots, x^{(l)}$ as parameter, then from theorem 3.1 it follows that there exists a function $F(x^{(1)}, \dots, x^{(l)})$, which is defined in $\Omega \cup \Sigma$, monogenic in respect to $x^{(1)}$, (real) analytic in respect to all variables in Ω and F = f in Σ.

For j = 2,...,l we set

$$F_j(x^{(1)}, \dots, x^{(j)}, \dots, x^{(l)}) = D_{x^{(j)}} F.$$

Then F_j is analytic in respect to $x^{(j)}$ in Ω_j (by fixed other variables). If $x^{(j)}$ is close to $\partial\Omega_j$, then

$$x = (x^{(1)}, \dots, x^{(j)}, \dots, x^{(l)}) \in \Sigma$$

and hence

$$F = f; F_j = D_{x^{(j)}} f = 0.$$

From the Uniqueness Theorem for (real) analytic functions it follows that

$$F_j = D_{x^{(j)}} F = 0$$

for all $x^{(j)} \in \Omega_j$.

Thus F is a multi-monogenic function in Ω and is the required extension of f. The uniqueness of the extension follows from the Uniqueness Theorem. Q.e.d.

Applying theorem 3.2 we can prove the second Hartogs extension theorem as follows.

Let K_1 be a compact subset of Ω_1 and $\mathcal{U}$ be an open neighbourhood of the set

$$(K_1 \times' \Omega) \cup (\Omega_1 \times \partial'\Omega),$$

where

$$'\Omega = \Omega_2 \times \ldots \times \Omega_l.$$

Then we have

Theorem 4.2:

For every multi-monogenic function f in $\mathcal{U}$ there exists a unique multi-monogenic function F in $\Omega = \Omega_1 \times' \Omega$ such that $F = f$ in $\mathcal{U}$.

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References

[1] F.Brackx, R.Delanghe and F.Sommen: *Clifford Analysis* Research Notes in Mathematics 76, Pitman Books Ltd., London 1982.

[2] F.Brackx and W.Pincket: *Two Hartogs theorems for nullsolutions of overdetermined systems in Euclidean space.* Complex Variables, vol. 5 , 1985.

[3] Le Hung Son: *Ein fortsetzungssatz fuer raeumliche holomorphe funktionen.* Math. Nachr.106, 1982, p.121-128.

[4] Le Hung Son: *Extension problem for functions with values in a Clifford algebra.* Preprint der TH Darmstadt Nr. 957, January 1986.

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