IC/90/61

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

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INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1990 MIRAMARE - TRIESTE

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International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
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ABSTRACT

We perform a semiclassical expansion in the Wheeler–DeWitt equation, in powers of the gravitational constant. We then show that quantum gravitational fluctuations can provide a correction to the wave–functions which are solutions of the Schrödinger equation for matter. This also implies a correction to the expectation values of quantum mechanical observables.

MIRAMARE – TRIESTE
March 1990

* Submitted for publication.
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1. Introduction

The Wheeler-DeWitt equation

$$\left\{-\frac{\hbar^2}{M_0^2}G_{ijkl}\frac{\delta^2}{\delta h_{ij}(x)\delta h_{kl}(x)} - M_0^2\sqrt{h(x)}\ {}^3R + H_m(\phi,\frac{\delta}{\delta\phi})\right\}\Psi(\phi,h) = 0 \tag{1}$$

is obtained[1] by applying the rules of canonical quantization to general relativity, and is a possible candidate for the correct quantum theory of gravity. Here, $M_0^2 = c^3/16\pi G$, and $h_{ij}(x)$ and ${}^3R$ are respectively the 3-metric on and curvature of a spatial hypersurface, and H_m is the Hamiltonian for a generic matter field ϕ. This equation has the virtue that both classical general relativity and quantum field theory in a classical curved space are obtainable from it as approximations, in the following sense. If the wave-functional $\Psi(\phi,h)$ of (1) is re-written as $\Psi(\phi,h) = \exp[iS(\phi,h)/\hbar]$ and the functional $S(\phi,h)$ expanded in a power series in the Planck's constant $\hbar$, one obtains in the leading order the Hamilton-Jacobi equation for classical general relativity, as in the usual WKB approximation. From here, one can arrive at the classical Einstein equations.[2]

On the other hand, if one expands the functional $S(\phi,h)$ as a power series in the gravitational constant G, one obtains at the leading order, the Hamilton-Jacobi equation for source free gravity (no coupling to matter), and at the next order, the functional Schrodinger equation for the quantum field ϕ propagating in a classical curved space.[3–5] At this stage it is also possible to introduce a source term in the Einstein euations (back-reaction) under appropriate conditions.[5] The fact that both classical general relativity and the Schrodinger equation for matter are contained in the Wheeler-DeWitt equation as approximations lends support to the belief that this equation may be the correct description of quantum gravity. In this letter, we carry out the expansion of $S(\phi,h)$ in powers of G to one order higher than has been done before, because this allows us to see how quantum gravitational fluctuations can possibly modify the solutions of the Schrodinger equation for matter fields.

It is convenient to first work with a toy-model in non-relativistic quantum mechanics[4], which consists of two interacting particles Q and q, and is described by the action

$$A(q,Q) = \int dt\left[\frac{1}{2}M\dot{Q}^2 - MV(Q) + \frac{1}{2}m\dot{q}^2 - u(q,Q)\right]. \tag{2}$$

The corresponding stationary Schrödinger equation

$$\left\{-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial Q^2}+MV(Q)-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}+u(q,Q)\right\}\psi(q,Q)=E\psi \tag{3}$$

is analogous to a minisuperspace version of (1) when we identify Q with the metric, q with matter, M with M_0^2, and in (3) set E as zero. More generally, the results obtained by a semiclassical expansion of (3) hold for (1) as well, because going from (3) to (1) essentially involves converting ordinary derivatives to functional derivatives.

In (3), defining $\psi(q,Q)=\exp[iS(q,Q)/\hbar]$ and setting $E=0$ leads to

$$-\frac{i\hbar}{2M}S''+\frac{S'^2}{2M}+MV(Q)-\frac{i\hbar}{2m}S_{qq}+\frac{S_q^2}{2m}+u(q,Q)=0 \tag{4}$$

where a prime and the subscript q respectively denote partial derivatives w.r.t. Q and q. Next, we expand $S(q,Q)$ as a power series in M:

$$S(q,Q)=MS_0(q,Q)+S_1(q,Q)+M^{-1}S_2(q,Q)+O(1/M^2). \tag{5}$$

Substituting this expansion in (4), and taking the limit $M\to\infty$ yields the following equations at successive orders in M:

$$O(M^2)\ :\ (\partial S_0/\partial q)=0, \tag{6}$$

$$O(M)\ :\ \frac{1}{2}MS_0'^2+MV(Q)=0, \tag{7}$$

$$O(M^0)\ :\ S_0'S_1'=\frac{i\hbar}{2}S_0''+\frac{i\hbar}{2m}S_{1qq}+\frac{S_{1q}^2}{2m}+u(q,Q), \tag{8}$$

$$O(M^{-1})\ :\ S_0'S_2'=\frac{i\hbar}{2}S_1''+\frac{i\hbar}{2m}S_{2qq}-\frac{S_{1q}S_{2q}}{m}-\frac{S_1'^2}{2}. \tag{9}$$

Eqn.(7) is the Hamilton-Jacobi equation for the uncoupled Q-mode, and is analogous to the Hamilton-Jacobi equation for source free Einstein equations.

By defining $f(q,Q)=\sqrt{S_0'}\exp(iS_1/\hbar)$ Eqn.(8) can be rewritten as

$$i\hbar S_0'\frac{\partial f}{\partial Q}=-\frac{\hbar^2}{2m}\frac{\partial^2 f}{\partial q^2}+u(q,Q)f. \tag{10}$$

This equation can be interpreted as the Schrodinger equation for the matter wave-function $f(q,Q)$ in a classical background Q obtained by solving (7), after one identifies the time-evolution operator $(\partial/\partial t)$ with $(S_0'\partial/\partial Q)$. The analogue of this equation in the gravitational case is the functional Schrödinger equation for the wave-functional f of the matter field in a classical curved space. (For a detailed discussion see Refs.3-5). The wave-function at this order of approximation is

$$\psi(q,Q)=\frac{1}{\sqrt{S_0'}}e^{iMS_0(Q)/\hbar}.f(q,Q), \tag{11}$$

and since Eqn.(7) allows us to interpret MS_0' as the classical momentum of the Q-mode, the wave-function (11) has the form of the WKB wave-function for gravity, multiplied by the wave-function for matter.

The essence of the expansion in (5) is that it is equivalent to the usual WKB approximation for the *gravitational half* of the system, while leaving the matter half quantum mechanical. This expansion has such a property because the constants $\hbar$ and M appear only in the ratio $\hbar/M$ in the gravitational half of the system, as may be seen from (3). Thus, so far as Q is concerned, the limits $\hbar\to 0$ and $M\to\infty$ are equivalent. This can be attributed to the scaling of $V(Q)$ with M, hence to the scaling of the gravitational action with G, and ultimately to the equivalence principle. Since the Schrödinger equation (10) for matter is already recovered at $O(M^0)$ in this expansion it is interesting to investigate what happens at the next order, that is, $O(M^{-1})$.

3. Corrections to the matter wave-function

To analyse Eqn.(9) we first rewrite $MS_0' = P$, and then substitute for S_0' and S_1 in terms of P and f, respectively, to get

$$\begin{aligned} \frac{P}{M}S_2' - \frac{3}{8}\hbar^2\frac{P'^2}{P^2} + \frac{\hbar^2}{4}\frac{P''}{P} \\ - \frac{i\hbar}{2m}S_{2qq} - \frac{i\hbar}{m}\frac{f_q}{f}S_{2q} - \frac{\hbar^2}{2}\frac{f''}{f} + \frac{\hbar^2}{2}\frac{f'}{f}\frac{P'}{P} = 0. \end{aligned} \tag{12}$$

Next, we define the function $g(q,Q) = S_2(q,Q) - S_{20}(Q)$, where $S_{20}(Q)$ is taken to be a solution of the equation

$$\frac{P}{M}S_{20}' - \frac{3}{8}\hbar^2\frac{P'^2}{P^2} + \frac{\hbar^2}{4}\frac{P''}{P} = 0. \tag{13}$$

It is useful to carry out such a separation in (12) because Eqn.(13) is precisely the second-order equation in the usual WKB approximation for a one-particle quantum mechanical system.[6] The corresponding equation in the gravitational case will be the second order WKB equation for source-free gravity. The function $g(q,Q)$ then satisfies the equation

$$i\hbar\frac{P}{M}g' = -\frac{\hbar^2}{2m}g_{qq} - \frac{\hbar^2}{m}\frac{f_q}{f}g_q + \frac{i\hbar^3}{2}\frac{f''}{f} - i\hbar^3\frac{P'}{2P}\frac{f'}{f}. \tag{14}$$

The wave-function correct to order M^{-1} can then be written as

$$\psi(q,Q) = \psi^{(2)}_{WKB}(Q).f \ \exp ig/M\hbar, \tag{15}$$

where

$$\psi^{(2)}_{WKB}(Q) = \frac{1}{\sqrt{P/M}}e^{(i/\hbar)\int PdQ} \ . \ e^{iS_{20}/M\hbar} \tag{16}$$

is the second order gravitational WKB wave-function. Looking at (15), it is reasonable to think of the function $\exp(ig/M\hbar)$ as providing a correction to the usual matter wave-function f which satisfies the Schrödinger Eqn.(10), where g is to be obtained by solving (14). Since in the Wheeler-DeWitt equation, M will be replaced by $(c^3/16\pi G)$, the corrected matter wave-function

$$\chi(q,Q) = fe^{ig/M\hbar} \tag{17}$$

will involve the gravitational constant (while, of course, f does not). Hence, if one were to compute expectation values of usual quantum mechanical matter observables using $\chi(q,Q)$, instead of f, these expectations would involve a correction dependent on G.

How does one interpret the wave-function (15)? The wave-function in (11), which is accurate to order M^0, can be said to describe a nearly classical gravitational field, and a quantum field propagating in this background. In (15), $\psi^{(2)}_{WKB}$ describes fluctuations around the classical gravitational field. It is then reasonable to think of $\chi(q,Q)$ as incorporating the leading order effect of gravitational fluctuations on the usual quantum mechanical wave-function f. However we can no longer think of the mode q as evolving in a classical background Q. We also note that by using (17) to express g in terms of χ and f, and by using Eqn.(14), we can obtain a non-linear equation for χ. However, it appears simpler to work with the two different equations for f and g.

To solve Eqn.(14) we have to first specify $P(Q)$ by solving (7), and $f(q,Q)$ by solving (10), and in general (14) is difficult to solve. Here we consider the relatively simpler case of a harmonic oscillator $u(q,Q) = m\omega^2q^2/2$, with ω independent of Q. In this case, (10) is the usual Schrödinger equation for the harmonic oscillator, with the time operator $(\partial/\partial t)$ defined as before. We will not need to specify $V(Q)$. We choose $f(q,Q)$ to be the ground state of the oscillator,

$$f(q,Q) = e^{-iE\int dQ.M/P(Q)}.f_0(q) \equiv \phi(Q)f_0(q) \tag{18}$$

with $f_0(q) = Nexp(-q^2/4\sigma^2), N^2 = 1/\sqrt{2\pi\sigma^2}, \sigma^2 = \hbar/2m\omega$. In this case, (14) is separable because f_q/f depends only on q, while f''/f and f'/f depend only on Q. We define $g(q,Q) = g_1(Q) + g_2(q)$ and get

$$g_1' = i\hbar EM^2\frac{P'}{P^3} - \frac{E^2M^3}{2P^3}, \tag{19}$$

$$g_2(q) = i\alpha\int_{-\infty}^{q}\frac{dq'}{f_0^2(q')} + i\beta \equiv i\alpha F(q) + i\beta \tag{20}$$

where α and β are constants, and we choose them to be real, so that we may obtain a normalisable wave-function and finite expectation values. We fix β by demanding that $\chi(q,Q)$ in (17) be normalisable over q, so that

$$e^{-2\beta/M\hbar}\int_{-\infty}^{\infty}dq.e^{-2\alpha F(q)/M\hbar} = 1. \tag{21}$$

This determines β in terms of α, but there seems to be no way of fixing α from the theory.

The expectation value of the Hamiltonian $h(q, Q)$ of the oscillator in the state $\chi(q, Q)$ is given by

$$< h >= \int_{-\infty}^{\infty} dq\ \chi^*(q, Q) \left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + \frac{1}{2}m\omega^2 q^2 \right\} \chi(q, Q) \tag{22}$$

which can be simplified to get

$$< h >= e^{i(\sigma_1 - \sigma_1^*)/M\hbar} \left[\frac{1}{2}\hbar\omega - \frac{\alpha^2}{2mM^2N^4} \frac{\int_{-\infty}^{\infty} dq\ \exp[\frac{q^2}{2\sigma^2} - \frac{2\alpha F(q)}{M\hbar}]}{\int_{-\infty}^{\infty} dq\ \exp[-\frac{q^2}{2\sigma^2} - \frac{2\alpha F(q)}{M\hbar}]} \right]. \tag{23}$$

We find that $< h >$ has two kinds of corrections imposed on its usual value, $\hbar\omega/2$. The first is the time-independent shift as in the second term in the bracket. The ratio of the two integrals in this term is very nearly unity, because near $q = \pm\infty$, the behaviour of each integral is dominated by $F(q)$. Thus the time-independent shift is nearly $(\alpha^2/2mM^2N^4) = \pi\alpha^2\hbar/2m^2\omega M^2$. The constant α has dimensions of $(\text{mass})^2/\text{time}$, and since it is same for all stationary states of the oscillator, it could possibly have the value $m^2\omega$, in which case the shift would be $(\pi m^2/M^2).(\hbar\omega/2)$, thus depending on the ratio of the squares of the two masses. Moreover, as m approaches $M/\sqrt{\pi}$, the ground state energy approaches zero, because of the correction. The second correction, indicated by the exponent outside the brackets in (23), destroys the constancy of the ground state energy, but this time dependence will arise only at $O(M^{-1})$.

If the above analysis is repeated for the Wheeler-DeWitt equation, Eqn.(14) will be replaced by a functional equation for the functional $g(\phi, h)$, and (17) would be the corrected wave-functional for matter, with M replaced by M_0^2. Moreover, (10) will be replaced by the functional Schrödinger equation, which will reduce to (10) in the non-relativistic one-particle limit. In a similar manner, the functional equation replacing (14) will reduce to (14) in the non-relativistic limit. Hence (14) describes the non-relativistic limit of the $O(M^{-1})$ approximation to the Wheeler-DeWitt equation. In particular, the example of the harmonic oscillator that we have worked out above, corresponds to an oscillator in flat space-time, and the gravity induced correction to the zero point energy will be $(\pi\alpha^2\hbar/2m^2\omega M_0^4)$. Thus quantum gravitational fluctuations induce corrections in flat-space quantum mechanics as well. A detailed investigation of these results is in progress.

We end with a series of comments on the above results, their implications and possible generalisations:

(i) In evaluating the expectation values, one should integrate the full wave-function (15) over q and Q, and not just integrate $\chi(q, Q)$ over q. However, this would not affect the shift in $(\hbar\omega/2)$ that we have found above.

(ii) Perhaps it is possible to fix α from theory, though it is not clear how.

(iii) In principle, the expansion of $S(q, Q)$ in powers of G could be carried to all orders, and when summed, would provide the effective correction induced by gravity.

(iv) The normalisation of the matter wave-function has to be carried out afresh at every successive order in M.

(v) Although Eqn.(7) corresponds to source free Einstein equations, it is possible to derive Einstein equations with a source $< T_{ik} >$ under appropriate conditions,[5] via an expansion in powers of G.

(vi) In those cases where (3) can be solved exactly, one would like to know how the solution (15) arises as an approximation to the exact solution.

(vii) It appears important to investigate the equation corresponding to (14) in interacting quantum field theory, and find the effect of gravitational fluctuations. It is possible that these effects can be found by integrating over just the matter degrees of freedom, and without facing the difficult question of integrating over the gravitational modes.

ACKNOWLEDGEMENTS

I am grateful to D.W. Sciama for encouragement, and to T. Padmanabhan for many useful and detailed discussions on semiclassical gravity. It is a pleasure to thank Y. Anini, S. Mohanty and S. Sonego for helpful conversations. I also thank Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at ICTP, Trieste.

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