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On the Spherical Symmetry of Static Perfect Fluids in General Relativity

R. Beig, W. Simon
Institut für Theoretische Physik
Universität Wien

Abstract

We present a theorem which establishes uniqueness, in particular spherical symmetry, of a wide class of general relativistic, static perfect-fluid models provided there exists a spherically symmetric model with the same equation of state and surface potential. The method of proof, which is inspired by recent work of Masood-ul-Alam, is illustrated by demonstrating uniqueness of a class of solutions due to Buchdahl which correspond to an extreme case of the inequality on the equation of state required by our theorem.

While the uniqueness of static black holes is by now fairly well understood, the same cannot be said about the analogous issue in the case of perfect fluids. The black-hole case can be proven in two different ways, namely either by integrating suitably chosen divergence-identities between the horizon and infinity [1] or, under less restrictive assumptions, by applying a suitable conformal transformation and (a corollary to) the positive-mass theorem [2]. For perfect fluids, best results to date are, on one hand, the result due to Künzle and Savage [3] which shows that there do not exist non-spherical models close to a given spherical one. On the other hand, there is the recent work of Masood-ul-Alam [4]. He proceeds in two steps which correspond to, and combine, the two above-said methods of proof in the black hole case. In each step he needs to impose conditions on the equation of state $\rho = \rho(p)$. Unfortunately one of these conditions seems unnatural mathematically and physically over-restrictive (e.g. it is violated for the Harrison-Wheeler equation of state [5] already at densities $\rho = 10^6$ gcm^{-3}). We think that our work cures these drawbacks. Our main new input is a conformal technique applied to a divergence identity in the presence of perfect fluids.

We write the static metric as $ds^2 = -V^2dt^2 + g_{ab}dx^a dx^b$ $(V > 0)$ and consider g_{ab} and V as fields on the 3-manifold M subject to Einstein's equations

$$\Delta V = 4\pi V(\rho + 3p), \qquad R_{ab} = V^{-1}D_aD_bV + 4\pi(\rho - p)g_{ab} \tag{1}$$

where D_a, $\Delta = D_aD^a$ and the Ricci tensor R_{ab} are formed from g_{ab}. From (1) there follows the equation of hydrostatic equilibrium

$$\frac{dp}{dV} = -V^{-1}(\rho + p). \tag{2}$$

We assume that there is only "one body", i.e., we take the open set $Q \subset M$ occupied by the fluid and its boundary ∂Q both to be connected and $\bar{Q} = Q \cup \partial Q$ to be compact (the "n-body case" [6] could be treated in the same manner). Metric and potential are smooth off ∂Q and satisfy the standard junction conditions across ∂Q, in particular $p_{|\partial Q} = 0$ and $V_{|\partial Q} = V_s = constant$ ($0 < V_s < 1$) [7]. We also assume that $\rho > 0$, $p > 0$ on Q. Furthermore ρ is related to p by an equation of state $\rho(p)$ which is smooth in $(0, \infty)$ with $d\rho/dp \geq 0$. When $\rho(0) > 0$, i.e. when the density drops to zero discontinuously on ∂Q, we assume smoothness also at $p = 0$, whereas, when $\rho(0) = 0$, we instead require that $\int_0^p [\rho(s) + s]^{-1} ds < \infty$ for finite p, so that equ. (2) can be integrated. (The last requirement entails that $\lim_{p \to 0} d\rho/dp = \infty$ when $\rho(0) = 0$.) The global conditions on M – which include the requirement of asymptotic flatness – are as follows: After adding a point Λ, $\tilde{M} = M \cup \{\Lambda\}$ becomes a compact manifold on which $\lim_{\to\Lambda} V = 1$. Furthermore, $\tilde{g}_{ab} = (\frac{1-V}{2})^4 g_{ab}$ is a smooth metric and $\sigma = (\frac{1-V}{1+V})^2$ a smooth function on $\tilde{M} \setminus \partial Q$ with $\sigma_{|\Lambda} = D_a\sigma_{|\Lambda} = 0$. (These apparently very stringent asymptotic properties are known to follow from rather weak fall-off conditions on V and g_{ab} [8]). We also assume the existence of a "spherical reference" (SR-) model, i.e.:

Assumption: There exists a model $(\mathbf{R}^3, \bar{g}_{ab}, \bar{V})$ with the same equation of state $\rho(p)$ and the same surface potential V_s as the given one, which is spherically symmetric.

It is generally believed ("fluid ball conjecture") that the above assumptions, perhaps with some further restrictions on $\rho(p)$, imply uniqueness in the sense that (M, g_{ab}, V) is related to $(\mathbf{R}^3, \bar{g}_{ab}, \bar{V})$ by a diffeomorphism. We have proved the following

Theorem: Uniqueness, in the above sense, holds provided the equation of state satisfies

$$I \equiv \frac{1}{5}\kappa^2 + 2\kappa + (\rho + p)\frac{d\kappa}{dp} \leq 0 \qquad (p > 0) \tag{3}$$

where $\kappa \equiv (\rho + p)(\rho + 3p)^{-1} d\rho/dp$.

As to the range of validity we first note that, in the nonrelativistic limit $p \ll \rho$, equ. (3) is satisfied for polytropic fluids $\rho = Ap^{n/(n+1)}$ (A and $n \geq 0$ are const.) provided $n \leq 5$. Recall that for $n \geq 5$ the fluid region of the spherical solution necessarily extends to infinity [9].

Equ. (3), together with the boundary conditions, implies

$$K \equiv p(\rho + p)\frac{d\rho}{dp} - \frac{5}{6}\rho^2 \leq 0 \qquad (p > 0) \tag{4}$$

which is precisely the condition required for applying the positive-mass theorem [4]. Conversely, equality in (4) implies equality in (3).

To get an idea of how the proof works, and of the role of conditions (3,4), it is instructive to consider matter satisfying $K \equiv 0$, which is equivalent to $p(\rho) = \frac{1}{6}\rho^{6/5}(\rho_0^{1/5} - \rho^{1/5})^{-1}$ ($\rho < \rho_0 = const.$). For fixed $\rho_0 > 0$, the spherically symmetric solutions corresponding to this equation of state form a one-parameter family (parametrized, e.g., by the mass m) which was found by Buchdahl [10]. Like their Newtonian analogues, the polytropic fluids with index 5, these solutions are not very physical since the matter region extends to infinity and since they are unstable against radial perturbations. Note also that they have a lower – rather than an upper – limit for the mass, namely $\frac{16}{3}\pi\rho_0 m^2 > 1$.

We now show uniqueness for matter satisfying $K \equiv 0$. For this purpose the preceding conditions on Q are modified in the sense that the fluid region Q is all of M, in particular unbounded and asymptotically flat. The "SR-model" is defined to be the Buchdahl solution with the same ρ_0 and the same mass m as the given one, which includes the requirement that $\frac{16}{3}\pi\rho_0 m^2 > 1$ holds. We outline two proofs, applying each of the methods which work in the black hole-case.

Define $W = g^{ab}D_a V D_b V$. When $\bar{W} = \bar{W}(\bar{V})$ is the corresponding quantity in the SR-model, define a function W_0 on M by $W_0 = \bar{W}(V)$. Explicitly,

$$W_0 = (1 - V^2)^4 \left[\frac{1}{16m^2} - \frac{1}{3}\pi\rho_0\left(\frac{1-V}{1+V}\right)^2\right]. \tag{5}$$

From the definition of the SR model, and from (2), it follows that the functions $\rho(V)$ and $p(V)$ are identical with those of the given solution. Using this fact and standard formulas [11], one finds an elliptic equation for $W - W_0$ which, after making the substitution

$$\tilde{W} - \tilde{W}_0 = \left(\frac{1-V^2}{2}\right)^{-4}(W - W_0)$$
$$\tilde{g}_{ab} = V^{-2}\left(\frac{1-V^2}{2}\right)^4 g_{ab} \tag{6}$$

acquires a non-negative r.h. side, namely

$$\tilde{\Delta}(\tilde{W} - \tilde{W}_0) = \frac{1}{4}\tilde{W}^{-1}\tilde{B}_{abc}\tilde{B}^{abc} + \frac{3}{4}\tilde{W}^{-1}\tilde{D}_a(\tilde{W} - \tilde{W}_0)\tilde{D}^a(\tilde{W} - \tilde{W}_0). \tag{7}$$

Here $\tilde{B}_{abc} \equiv 2\tilde{D}_{[c}\tilde{R}_{b]a} - \frac{1}{2}\tilde{g}_{a[b}\tilde{D}_{c]}\tilde{R}$ is the "Bach-tensor" or "conformal tensor" of $\tilde{g}_{ab}$, and $\tilde{R} = \tilde{R}_a{}^a$. Next observe that $\tilde{W} - \tilde{W}_0$ and $\tilde{g}_{ab}$ extend smoothly to $\tilde{M} = M \cup \{\Lambda\}$ which is a compact manifold without boundary. Thus, from the maximum principle [12] applied to equ. (7) and from $\tilde{W} - \tilde{W}_{0|\Lambda} = 0$, there follows $\tilde{W} - \tilde{W}_0 = 0$ and $\tilde{B}_{abc} = 0$ on $\tilde{M}$, whence g_{ab} is conformally flat. Now uniqueness can be shown using known arguments [13].

The following alternative proof is more powerful since it works without the "SR-model". The metric $g'_{ab} = (\frac{1+V}{2})^4 g_{ab}$ has vanishing mass and, from $K \equiv 0$, vanishing scalar curvature $R(g') = 0$. A corollary to the positive-mass theorem [14] implies that g'_{ab} is isometric to the standard metric on $\mathbb{R}^3$. Straightforward integration of (1) now shows that (g_{ab}, V) has to be a unique Buchdahl solution.

For the general case, with a fluid of finite extension, we have to combine the above two methods (compare Ref. [4]). Consider the function $\bar{W}(\bar{V})$ of the SR-model and view, as before, $W_0 = \bar{W}(V)$ as a function on M. W_0 is defined at first only for $V \geq \bar{V}_{\min}$ (the minimum of $\bar{V}$). Extend W_0 to all of M by requiring

$$\Sigma \equiv \frac{dW_0}{dV} - \frac{8\pi}{3}V(\rho + 3p) + \frac{4}{5}V^{-1}W_0\kappa \tag{8}$$

to vanish for $V < \bar{V}_{\min}$. (For the Buchdahl solutions, Σ vanishes everywhere.) The generalization of the conformal substitution (6), which we give here in the case $\rho(0) = 0$ only, reads

$$\begin{aligned} \tilde{W} - \tilde{W}_0 &= \Psi^{-4}(W - W_0) \\ \tilde{g}_{ab} &= \Psi^4 V^{-2} g_{ab} \end{aligned} \tag{9}$$

where $\Psi = \Psi(V)$ is determined by

$$\Psi^{-1}\frac{d\Psi}{dV} = \frac{1}{4}W_0^{-1}[\frac{dW_0}{dV} - \frac{8\pi}{3}V(\rho + 3p)] \tag{10}$$

up to a constant factor. As before, $\tilde{M} = M \cup \Lambda$ is compact, and $\tilde{W} - \tilde{W}_0$ and $\tilde{g}_{ab}$ are smooth near Λ. Again, we find an elliptic equation with non-negative r.h. side, namely

$$\begin{aligned} &[\tilde{\Delta} - \frac{20\pi}{3}V^3 W_0^{-2}\Sigma(\rho + 3p)\tilde{W}](\tilde{W} - \tilde{W}_0) = \\ &= \frac{1}{4}\tilde{W}^{-1}\tilde{B}_{abc}\tilde{B}^{abc} + \frac{3}{4}\tilde{W}^{-1}\tilde{D}_a(\tilde{W} - \tilde{W}_0)\tilde{D}^a(\tilde{W} - \tilde{W}_0) - \frac{4}{5}\tilde{W}^2 J \end{aligned} \tag{11}$$

where $J = 0$ for $V \geq \bar{V}_{\min}$ and $J = I$ for $V < \bar{V}_{\min}$. It can be shown that $I \leq 0$ implies $\Sigma \geq 0$, and that $W_0^{-2}\Sigma$ is finite on $V = \bar{V}_{\min}$ where $W_0 = 0$. Furthermore, $\tilde{W} - \tilde{W}_0$ is sufficiently smooth across ∂Q and on the set where $V = \bar{V}_{\min}$, such that appropriate versions of the maximum principle [12] can be applied. It follows that either $\tilde{W} - \tilde{W}_0 \leq 0$ or $\tilde{W} - \tilde{W}_0 = const. > 0$. The second case leads to a Buchdahl solution which is, however, ruled out by the present boundary conditions. $\tilde{W} - \tilde{W}_0 \leq 0$ implies, in particular, that $\bar{V}_{\min} \leq V_{\min}$ since W_0 is negative for $V < \bar{V}_{\min}$.

The second step proceeds as in Ref. [4]. One performs another conformal transformation $g_{ab}' = \Phi^4(V)g_{ab}$, with $\Phi = \frac{1}{2}(1 + V)$ in the vacuum region, such that the Ricci scalar is of the form $R(g') = P(V)(W_0 - W)$, where $P(V)$ is non-negative by virtue of $K \leq 0$. Moreover, g_{ab}' has vanishing mass. From the positive-mass theorem we again infer that g_{ab}' is flat and complete the proof as before.

Note that the condition $I \leq 0$ plays a threefold role as it controls the signs of Σ and J in equ. (11) and that of P in step two.

We end up with some remarks on the two crucial assumptions in our theorem. The existence of the SR-model $(\mathbf{R}^3, \bar{g}_{ab}, \bar{V})$ seems indispensable at present. Conceivably one could derive this requirement provided V_s lies in some given range dependent on the equation of state. (There cannot be a theorem without at least a positive lower limit for V_s [15].) One would then replace the SR-assumption by a condition on V_s or m. Secondly there is the question whether the condition $I \leq 0$ rules out cases of physical interest. We have checked $I \leq 0$ for the ideal degenerate neutron gas and found it to be valid for densities up to $4 \cdot 10^{15}$ gcm^{-3}, which is roughly the critical density where instability sets in [16]. Similarly, $I \leq 0$ holds for (the analytical fits to) the Harrison-Wheeler equation of state [5] up to densities of order 10^{15} gcm^{-3}, except for the region in which "neutron drip" occurs. Our method of proof suggests that the theorem might still hold provided that $I \leq 0$ is valid only in some sufficiently large part of the star which contains the centre. Whether one can obtain a theorem which covers equations of state of realistic neutron stars would probably require rather involved considerations.

The details of the proof of our theorem will be published elsewhere.

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