GRASSMANN, SUPER-KAC-MOODY
AND
SUPER-DERIVATION ALGEBRAS

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Abstract: We study the cyclic cocycles of degree one on the Grassmann algebra and on the super-circle with N supersymmetries (i.e. the tensor product of the algebra of functions on the circle times a Grassmann algebra with N generators). They are related to central extensions of graded loop algebras (i.e. super-Kač-Moody algebras). The corresponding algebras of super-derivations have to be compatible with the cocycle characterizing the extension; we give a general method for determining these algebras and examine in particular the cases $N = 1, 2, 3$. We also discuss their relations with the Ademollo et al. algebras, and examine the possibility of defining new kinds of super-conformal algebras, which, for $N > 1$, generalize the $N = 1$ Ramond-Neveu-Schwarz algebra.

1. Introduction

Supersymmetries in physics have brought the study of graded commutative algebras and graded Lie algebras to the forefront of the arena of theoretical physics in the last 10 years and have also triggered a revival of their interest in pure mathematics. Infinite dimensional Lie algebras and in particular Kač-Moody algebras have also shown their usefulness in quantum field theory, in particular in the realm of two dimensional models and in string theory. The study of supersymmetric string theories leads therefore to the introduction of infinite dimensional super-algebras (from now on, we will use mostly the adjective "super" rather than the traditional "Z_2-graded" qualifier used by most mathematicians). As it is usually the case in physics, "matter-fields" take their values in a representation space of an algebra (or of a group). What happens here is that most interesting representations happen to be projective representations (they transform correctly only up to a phase); this means that they are genuine representations of a central extension of the algebra. This situation was already recognized in the early days of quantum field theory (presence of Schwinger terms) and has been illustrated many times under different names since. For example, it is well known that central extensions of loop algebras are described by a so called "central charge", the result being a Kač-Moody algebra and it is also known that the algebra of derivations of such an algebra (the Virasoro algebra) can itself be described as the central extension of the algebra of diffeomorphisms of the circle. When one incorporates supersymmetries, i.e. Grassmann generators into the game, one could expect a similar situation. However, several new phenomena appear.

First, the number of independent "central charges" that one can add is no longer equal to one as it was the case previously and this number also depends upon the number of Grassmann generators. In other words, if we start with a given loop algebra and supersymmetrize it by adding N Grassmann generators, it will still be possible to construct several inequivalent graded Lie algebras (super-Kač-Moody algebras), each one characterized by the choice of a different central charge. Technically, such a "central charge" is described by the choice of a cocycle and the space of these cocycles is a linear space; the inequivalent extensions can therefore be described by choosing a basis in the space of these cocycles. In the first part of this article, we will describe this procedure in detail and relate these cocycles to the cyclic cohomology (see sect. 2) of the "super-circle" (this denotes the tensor product of the Grassmann algebra with N generators times the (commutative) algebra of functions on the circle). As a by-product – or a necessary tool ! – we will also describe the Hochschild and cyclic cohomologies of these two algebras. A result of independent interest will be the explicit expression of a basis for the cyclic one-cocycles of a Grassmann algebra using the Berezin integral. We will study explicitly the cases $N = 1, 2$ and 3 and we will also give general formulae for an arbitrary N.

Another phenomenon, which is a new feature of the supersymmetric case appears when one studies derivations of functions on the super-circle (the analogue of the algebra of vector fields of the circle). Indeed, an algebra of derivations for a central extension of a loop algebra has to be compatible – of course – with the cocycle defining the extension. This,

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in turns, is equivalent to select a subalgebra of all (super)derivations of super-functions of the super-circle itself. This procedure has already been followed in [1] where a particular central extension in the case $N = 1$ was investigated. Actually, this case corresponds to the usual $N = 1$ superconformal algebra. We will devote the second part of the present article to the study of the above mentioned compatibility equations and more generally to the study of (graded) derivations on the central extensions on the super-circle. The particular cases $N = 1$ and $N = 2$ will be treated explicitly (see sect. 3).

A class of algebras of particular interest in superstring theories are the so called "super-conformal" (or Ademollo et al.) algebras [2]. In the same way the $N = 1$ super-conformal (or Ramond-Neveu-Schwarz (RNS)) algebra can be defined as the particular algebra of super-derivations corresponding to a central extension on the $N = 1$ super-circle, one may wonder whether the other members of the family of the super-conformal algebras of ref. [2] can be characterized in the same way. We will see in sect. 4 that this is not the case; in particular we will show that there is no non trivial cocycle compatible with the $N \geq 2$ super-conformal algebra from the $N \geq 2$ super-circle. Such a negative result will lead us to construct for $N > 1$ new types of extended algebras, associated with super-Kač-Moody ones and generalizing the $N = 1$ RNS algebra.

2. Central extensions of the loop algebras with N supersymmetries

The organization of this section is the following. We start by recalling the definition of loop algebras and super-loop algebras. Then we show in sect. 2.1.2. how central extensions of these Lie algebras (i.e. Kač-Moody and super-Kač-Moody algebras) are determined by Lie algebra cocycles (or graded Lie algebra cocycles) of degree two which, in turn, are fully determined by cyclic cocycles of degree one on the algebra $C(S^1)$ of functions on the circle (or on the super-circle). We recast these results in the general framework of "Non Commutative Geometry": this paragraph 2.1.3. may be skipped in first reading. In sect. 2.1.4., we recall the description of cyclic cocycles of degree one on $C(S^1)$ and we describe explicitly in sect. 2.1.5. cyclic cocycles of degree one on the Grassmann algebra G_N. Finally, we show how the results of the above sections allow us to compute easily the cyclic cocycles of degree one on the supercircle $A_N = C(S^1) \otimes G_N$.

In sect. 2.2., we gather explicit results concerning cyclic cocycles for A_N in the case $N = 1, 2, 3$.

2.1. General discussion

2.1.1. Loop algebras and super loop algebras

A loop algebra $L(LieG)$ can be defined as the space of loops in a Lie algebra $Lie(G)$ with the commutator

$$[\xi,\eta](t) = [\xi(t),\eta(t)], \quad \xi,\eta \in L(LieG), \quad t \in S^1 \tag{1}$$

This Lie algebra can be identified with the Lie algebra $Lie(LG)$ of the loop group LG, itself defined [3][4] as the group of smooth maps from S^1 into a finite-dimensional Lie group G. As a particular case, we can take $G = U(1)$, then $Lie(G) = R$ and the loop algebra $L(LieG)$ can be identified with the space of periodic functions. This algebra is also the commutative algebra of functions on the circle S^1; it will be denoted [1] by $C(S^1)$ and will play an important role in the following discussion. Elements of $L(LieG)$ can be written as linear combinations of elements of the kind $X^\alpha(t)\lambda_\alpha$ where $X^\alpha(t)$ is a function on the circle, i.e. an element of $C(S^1)$ and $\{\lambda_\alpha\}$ denotes a basis in the Lie algebra $Lie(G)$.

In this paper, we are mostly interested in the supersymmetric case. In particular, we will replace the circle (or better the commutative algebra $C(S^1)$) by the super-circle A_N which is the (non-commutative) algebra $C(S^1) \otimes G_N$ where G_N is the Grassmann algebra with N generators. More generally, to any loop algebra K we can associate a super-loop algebra K_N by allowing the functions $X^\alpha(t)$ to belong to the super-circle A_N rather than to the commutative algebra $C(S^1)$.

2.1.2. Kač-Moody and Super Kač-Moody algebras

Kač-Moody algebras are central extensions of loop algebras. At the level of Lie algebras, a central extension of such a loop algebra can be written as $Lie(\hat{L}G) = Lie(LG) \oplus R$ with the Lie bracket

$$[(\xi,\alpha),(\eta,\beta)] = ([\xi,\eta],\omega(\xi,\eta)) \quad \xi,\eta \in Lie(LG), \quad \alpha,\beta \in R \tag{2}$$

where $\omega(\xi,\eta)$ is a Lie algebra cocycle. Such an object has to be antisymmetric (this comes from the anticommutativity of the Lie bracket) and has to satisfy the following cocycle condition (coming from the fact that the new Lie bracket has to satisfy the Jacobi identity):

$$\omega([\xi,\eta],\zeta) + \omega([\eta,\zeta],\xi) + \omega([\zeta,\xi],\eta) = 0 \quad \xi,\eta,\zeta \in Lie(LG) \tag{3}$$

One then proves [4] that such a cocycle is necessarily of the kind

$$\omega(\xi,\eta) = \frac{1}{2\pi} \int_0^{2\pi} dt < \xi(t), \frac{d}{dt}\eta(t) > \tag{4}$$

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[1] Most of the time, the functions will be taken infinitely differentiable.

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All the cocycles, and thus all central extension of $Lie(LG)$ are therefore completely determined by the choice of a symmetric invariant form $<,>$ in $Lie(G)$ (it has to be proportional to the Killing form when G is simple).

This result can be proved in (at least) two ways. The first one [4] consists in making a Fourier expansion of the periodic functions $\xi(t)$ and $\eta(t)$ and in showing that the above expression for ω is the only possible one compatible with the cocycle condition. Another proof goes along the following lines. One first notices that what really matters is the algebra $C(S^1)$ of smooth functions on the circle S^1. Indeed the algebra of $n \times n$ matrices with elements in $C(S^1)$ is the tensor product $M_n(C) \otimes C(S^1)$ ($M_n(C)$ denoting the algebra of $n \times n$ matrices on C). So, a 2-form ω on this algebra is fully determined as soon as we know its value on elements of the kind $m \otimes x$ with $m \in M_n(C)$ and $x \in C(S^1)$. Let therefore ω_c, be an antisymmetric 2-form on the commutative algebra $C(S^1)$, then $\omega = Trace \otimes \omega_c$ is an antisymmetric 2-form on the algebra $M_n(C) \otimes C(S^1)$. More precisely, $\omega(m_0 \otimes x_0, m_1 \otimes x_1) = Trace(m_0 m_1) \otimes \omega_c(x_0, x_1)$. We will explain this construction in more details in sect. 2.1.3. Moreover, if ω_c satisfy the following condition

$$\omega_c(xy, z) + \omega_c(yz, x) + \omega_c(zx, y) = 0, \quad x, y, z \in C(S^1) \tag{5}$$

then, ω will be a Lie algebra cocycle i.e. will satisfy the condition recalled previously. Such a 2-form ω_c on $C(S^1)$ is called a cyclic cocycle [5] of degree one on the algebra $C(S^1)$ and the above relation between cyclic cocycles and Lie algebra cocycles is a particular case of a general result [6]. We will give more details in what follows on cyclic cohomology but for the moment, it is enough to state the fact that the generator in degree one [2] of cyclic cohomology for the algebra $C(S^1)$ is precisely given by

$$\omega_c(x, y) = \frac{1}{2\pi} \int_0^{2\pi} dt \; x(t) \frac{d}{dt} y(t) \tag{6}$$

It is easy to check that the two properties characterizing a cyclic cocycle are indeed verified (the antisymmetry comes from the integration by part, taking into account the fact that $x(2\pi) = x(0)$. The final result about Lie algebra cocycles on the loop algebras (and their corresponding central extensions) then follows.

In order to generalize these ideas to the Z_2-graded case, we first notice that if A is a graded commutative algebra (such as the Grassmann algebra), one can define the following product in $M_n(A) \equiv M_n(C) \otimes A$:

$$[m_0 \otimes a_0 \; , \; m_1 \otimes a_1] \equiv [m_0 \; , \; m_1] \otimes a_0 a_1 = m_0 m_1 \otimes a_0 a_1 - (-1)^{\partial a_0 \partial a_1} m_1 m_0 \otimes a_1 a_0$$
$$= (m_0 \otimes a_0)(m_1 \otimes a_1) - (-1)^{\partial a_0 \partial a_1}(m_1 \otimes a_1)(m_0 \otimes a_0)$$

where ∂a denotes the intrinsic Z_2-grading of the element $a \in A$. It is easy to check that $M_n(A)$ satisfies the graded Jacobi identity and is therefore a graded Lie algebra. The rest of

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[2] Notice that a cyclic cocycle of degree p is a function of p+1 arguments.

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the discussion is similar to the previous case: in order to characterize the central extension of super-loop algebras (i.e. super-Kač-Moody algebras), we have to classify the solutions of the equation determining the cocycles. This, in turn, boils down to determine the cyclic cocycles of degree one on the super-circle. In the non supersymmetric case, there exists only one solution that we mentionned previously. However, in the supersymmetric case, there are many independent solutions (this comes from the fact that the cyclic cohomology of Grassmann algebras is in a sense "richer" than the cyclic cohomology of the algebra $C(S^1)$). In the supersymmetric case, the super Lie bracket is no longer anticommutative but satisfies the equation:

$$[\xi, \eta] = -(-1)^{\partial \xi \partial \eta}[\eta, \xi] \tag{7}$$

Also the super-Jacobi identity reads:

$$(-1)^{\partial \xi \partial \eta}[\xi, [\zeta, \eta]] + (-1)^{\partial \xi \partial \zeta}[\zeta, [\eta, \xi]] + (-1)^{\partial \eta \partial \zeta}[\eta, [\xi, \zeta]] = 0 \tag{8}$$

Consequently, the cocycle ω determining the extension is no longer antisymmetric but graded antisymmetric and has to satisfy a Z_2-graded version of the cocycle condition. These two equations read:

$$\omega(\xi, \eta) = -(-1)^{\partial \xi \partial \eta}\omega(\eta, \xi) \tag{9}$$

and

$$(-1)^{\partial \xi \partial \eta}\omega(\xi, [\zeta, \eta]) + (-1)^{\partial \zeta \partial \xi}\omega(\zeta, [\eta, \xi]) + (-1)^{\partial \eta \partial \zeta}\omega(\eta, [\xi, \zeta]) = 0 \tag{10}$$

In order to discuss these equations, one has to use the relation between the cohomology of (super)Lie algebras and the (Z_2-graded) cyclic cohomology of the non-commutative (but associative) algebra $A_N = C(S^1) \otimes G_N$, i.e. the super-circle. A cyclic cocycle of degree one Φ on the super-circle has to satisfy the following conditions:

$$\Phi(Y, X) = -(-1)^{\partial X \partial Y}\Phi(X, Y) \tag{11}$$

$$-\Phi(X, YZ) + \Phi(XY, Z) + (-1)^{\partial Z(\partial X + \partial Y)}\Phi(ZX, Y) = 0 \tag{12}$$

where $X, Y, Z \in A_N$. We will discuss the solution of these equations in section 2.1.6.

2.1.3. Cyclic cohomology of associative algebras and cohomology of Lie algebras

We already gave the definition of cyclic cocycles of degree one in the previous paragraph and we will not need more for our purpose. However, it may be interesting to recast our present problem in the more general framework of "Non Commutative Geometry". We will first define Hochschild cohomology and cyclic cohomology for associative algebras and then make the link with the cohomology of algebras of matrices.

Let A be an associative algebra (not necessarily commutative) which we will take Z_2 graded. Then, complex (or real) valued n-linear forms on A are particularly interesting

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objects to consider. A Hochschild cochain of degree n is defined as a multilinear form $\Phi(a_0, a_1, \ldots, a_n)$ of degree $n + 1$ [3] . We define the following operator b from cochains of degree n to cochains of degree $n + 1$:

$$[b\Phi](a_0, \ldots, a_n, a_{n+1}) =$$

$$\sum_{j=0}^{n} (-1)^j \Phi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) \; - \; (-1)^{\epsilon_{n+1}} \Phi(a_{n+1} a_0, a_1, \ldots, a_n) \qquad (13)$$

where $\epsilon_{n+1} = \partial a_{n+1} \sum_{i=0}^{n} \partial a_i$. It is not too difficult to show that $b^2 = 0$. This operator b is called the Hochschild coboundary operator and the corresponding cohomology, Hochschild cohomology: Hochschild cocycles Φ are such that $b\Phi = 0$, Hochschild coboundaries Ψ are such that $\Psi = b\Phi$ for some Φ. The space of Hochschild cocycles of degree n modulo Hochschild coboundaries is the Hochschild cohomology group of degree n.

The cyclicity operator λ is defined as follows [4]

$$[\lambda\Phi](a_0, a_1, \ldots, a_n) = (-1)^n (-1)^{\epsilon_n} \Phi(a_n, a_0, \ldots, a_{n-1}) \qquad (14)$$

The multilinear form Φ is called cyclic whenever $\lambda\Phi = \Phi$. It can be shown [5] that if Φ is cyclic, so is $b\Phi$. Therefore it is tempting to consider the subcomplex of Hochschild cochains built out of cyclic cochains. One defines therefore the space of cyclic cocycles (i.e. Hochschild cocycles which are cyclic) and the space of cyclic coboundaries (they are of the kind $b\Phi$ for some cyclic cochain Φ). The quotient of cyclic cocycles of degree n modulo cyclic coboundaries defines the cyclic cohomology group of degree n. The theory of cyclic cohomology has been introduced and developed in [5]; the Z_2-graded case has been particularly studied in [12]; for a general review, see [8].

Given an algebra A, it becomes possible to consider matrices with elements in A (such an algebra $M_n(A)$ is obtained by taking the tensor product of A with the algebra of $n \times n$ matrices over the complex numbers). As we saw in sect. 2.1.2., in the case where A is (graded)commutative, it is possible to define a (graded)Lie bracket on $M_n(A)$. Cohomology for Lie algebras of matrices is well known and we will not recall the definitions here (see for instance [9]). The main idea is that it is possible to relate the cohomology of Lie algebras to the cyclic cohomology of A itself. Notice first that, by linearity, a Lie algebra cocycle will be defined as soon as we know its definition on elementary matrices (which are of the kind $M_j^i \otimes a$ where $a \in A$ and where the only non zero entry of M_j^i is 1 in position (i, j)). The result (due to [6]) is the following. Let Φ be a cyclic cocycle on A, that we suppose commutative; in order to get a multilinear form on $M_n(C) \otimes A$, one composes it with the trace operator. The object that we obtain is then "cyclically antisymmetric"

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[3] Actually, there are several kinds of Hochschild complexes depending upon the choice of a module for A. Here, the module is the dual of the algebra itself.

[4] The notation λ differs by a sign from the one introduced in [5].

but not fully antisymmetric. Therefore, in order to get a multilinear antisymmetric form on the Lie algebra, we have to fully antisymmetrize it:

$$\omega(m_0 \otimes a_0, \ldots, m_n \otimes a_n) = Ant(Trace(m_0 \ldots m_n) \; \Phi(a_0, \ldots, a_n)) \qquad (15)$$

The result ω is a Lie algebra cocycle [5] .

In the case where we start from a graded commutative algebra rather than a commutative one, and from a super Lie algebra rather than a Lie algebra, the results are analogous.

In the present situation, we are only interested in Lie algebra cocycles of degree two – since they determine central extensions – for a Lie algebra of matrices whose elements are functions on the super-circle. We have therefore to determine cyclic cocycles of degree one on the algebra $A_N = C(S^1) \otimes G_N$. Notice that for cyclic cocycles of degree one, we do not have to antisymmetrize, since, at this order, cyclic antisymmetry coincides with full antisymmetry.

2.1.4. Hochschild and cyclic cohomology of the algebra of functions on the circle

As already stated in sect. 2.1.1, it is rather straightforward to check the following results.

1) The cyclic cohomology group of degree zero coincides with the Hochschild cohomology group in the same dimension and with the space of distributions on the circle $C(S^1)$.

2) Cyclic and Hochschild cohomology groups also coincide in degree one (cf eq. (11) and (12)) and all the cocycles are of the kind $a\Phi$ where a is an arbitrary constant and $\omega_c(x, y)$ is given by the formula (6) [6] .

2.1.5. Hochschild and cyclic cohomology of Grassmann algebras

Before studying the cocycles on the super-circle, it is necessary to study those on Grassmann algebras. To our knowledge, the cyclic cocycles have not been explicitly computed in the mathematical litterature, only [7] determines the dimension of the cohomology

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[5] More precisely, cyclic cohomology coincides with the (primitive part of the) cohomology of the Lie algebra of matrices.

[6] The general result for the algebra $C(\mathcal{M})$ of smooth functions on a smooth manifold $\mathcal{M}$ is the following [5]: The Hochschild cohomology group of degree k is canonically isomorphic with the space of de Rham currents (i.e. distributional forms) of dimension k on $\mathcal{M}$ and the cyclic cohomology group of degree k is canonically isomorphic to the direct sum $Ker\partial \oplus H_{k-2} \oplus H_{k-4} \oplus \ldots$ where $Ker\partial$ is the space of closed k-dimensional currents and where H_k denotes the usual de Rham homology of $\mathcal{M}$.

groups of degree k (but he does not give their explicit expression). We will therefore discuss quite explicitly the cases $N = 1, 2$ and 3 before considering the general case.

- Cyclic cocycles of degree 0 on G_N

Cyclic cocycles of degree 0 are one-forms on G_N that have to satisfy the conditions $b\vartheta = 0$ and $\lambda\vartheta = \vartheta$. At this order the second condition gives nothing and the first is trivially satisfied because $b\vartheta(X, Y) = XY - (-1)^{\partial X \partial Y} YX = 0$ since the algebra is graded commutative. One can therefore always identify the space of cyclic cocycles of degree zero (and the group of cyclic cohomology of degree 0) with the dual of G_N. It is therefore of dimension 2^N.

- Cyclic cocycles of degree 0 and 1 on G_1

G_1 is generated by 1 and θ with $\theta^2 = 0$. According to the previous discussion, cyclic cocycles of degree zero are generated by the two elements ϑ_1 and ϑ_θ with

$$\vartheta_1(1) = 1, \ \vartheta_1(\theta) = 0, \ \vartheta_\theta(1) = 0, \ \vartheta_\theta(\theta) = 1 \tag{16}$$

Cyclic cocycles of degree one on G_1 have to satisfy the two following conditions (already given previously)

$$\varphi(Y, X) = -(-1)^{\partial X \partial Y} \varphi(X, Y)$$

and

$$-\varphi(X, YZ) + \varphi(XY, Z) + (-1)^{\partial Z(\partial X + \partial Y)}\varphi(ZX, Y) = 0$$

where $X, Y, Z \in G_1$. The last equation gives in fact 2^3 equations since X, Y, Z have to be chosen in the set $\{1, \theta\}$. The solution of this system leads to the following result. There is only one cyclic cocycle of degree one in G_1. We will denote it by φ. It is characterized by the equations

$$\varphi(\theta, \theta) = 1, \ \varphi(\theta, 1) = \varphi(1, \theta) = \varphi(1, 1) = 0 \tag{17}$$

It is possible to write the previous cocycle by using a Berezin integral. Let $X, Y \in G_1$, then $\varphi(X, Y) = \int X \frac{d}{d\theta} Y d\theta$. Indeed, if we set $X = X_0 1 + X_1\theta$ and $Y = Y_0 1 + Y_1\theta$, then $\varphi(X, Y) = X_1 Y_1$.

- Cyclic cocycles of degree 0 and 1 on G_2

G_2 is generated by 1, θ^1 and θ^2 with $(\theta^1)^2 = 0, (\theta^2)^2 = 0$ and $\theta^1\theta^2 = -\theta^2\theta^1$. According to the previous discussion, cyclic cocycles of degree zero are generated by the four elements $\vartheta_1, \ \vartheta_{\theta^1}, \ \vartheta_{\theta^2}, \ \vartheta_{\theta^1\theta^2}$. These one-forms are dual to the basis $\{1, \theta^1, \theta^2, \theta^1\theta^2\}$. Notice that $\vartheta_1, \ \vartheta_{\theta^1\theta^2}$ are even cocycles whereas $\vartheta_{\theta^1}, \ \vartheta_{\theta^2}$ are odd.

Cyclic cocycles of degree one are still determined by the two equations given previously, however, the second equation gives in fact $4^3 = 64$ equations since X, Y, Z have to be chosen in the set $\{1, \theta^1, \theta^2, \theta^1\theta^2\}$. This time, one finds five independent cyclic cocycles (which can be taken as generators). The generators will be denoted by $\varphi_1, \ \varphi_2, \ \varphi_3, \ \varphi_4, \ \varphi_5$

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and are explicitly given as follows (the first three are even and the last two are odd): $\varphi_1(\theta^1, \theta^1) = 1$, $\varphi_2(\theta^2, \theta^2) = 1$, $\varphi_3(\theta^1, \theta^2) = \varphi_3(\theta^2, \theta^1) = 1$, $\varphi_4(\theta^1, \theta^1\theta^2) = -\varphi_4(\theta^1\theta^2, \theta^1) = 1$, $\varphi_5(\theta^2, \theta^1\theta^2) = -\varphi_5(\theta^1\theta^2, \theta^2) = 1$. In the above, we only gave those non trivial relations which are not a direct consequence of linearity or of elementary rules of calculation in G_2, like for instance $\varphi_4(\theta^1, \theta^1\theta^2) = -\varphi_4(\theta^1, \theta^2\theta^1)$. All other $\varphi_i(X, Y)$ are zero (for instance $\varphi_2(\theta^1, \theta^2) = 0$).

It is possible to write the previous cocycles of degree one by using a Berezin integral. Let $X, Y \in G_2$, and write $X = X_0 + X_1\theta^1 + X_2\theta^2 + X_{12}\theta^1\theta^2$ (same thing for Y). One gets

$$\varphi_1(X, Y) = X_1Y_1 = \int X[\theta^2\partial_1]Y d\theta^1 d\theta^2$$

$$\varphi_2(X, Y) = X_2Y_2 = -\int X[\theta^1\partial_2]Y d\theta^1 d\theta^2$$

$$\varphi_3(X, Y) = X_1Y_2 + X_2Y_1 = \int X[-\theta^1\partial_1 + \theta^2\partial_2]Y d\theta^1 d\theta^2 \qquad (18)$$

$$\varphi_4(X, Y) = X_1Y_{12} - X_{12}Y_1 = \int X[-\partial_1 + 2\theta^2\partial_2\partial_1]Y d\theta^1 d\theta^2$$

$$\varphi_5(X, Y) = X_2Y_{12} - X_{12}Y_2 = \int X[-\partial_2 - 2\theta^2\partial_2\partial_1]Y d\theta^1 d\theta^2$$

The reader can easily check that the previous one-cocycles are independent and non trivial, i.e. they are not cyclic coboundaries, by computing explicitly the coboundaries $b\vartheta_I$ with $I \in \{1, \theta^1, \theta^2, \theta^1\theta^2\}$, i.e. writing $[b\vartheta_I](X, Y) = \vartheta_I(XY)$. Notice that the last two cocycles involve differential operators of degree two in the Grassmann algebra; however, by introducing graded derivations, we will see that they can be written in term of differential operators of degree one only. Let $\mathcal{D}$ be a generic differential operator of degree one in the Grassmann algebra; it can be written as a linear combination of $\mathcal{D}_i$ where $\mathcal{D}_i = c_i(\theta^1, \theta^2)\partial_i$, c_i being a monomial in the Grassmann variables θ^i. If X and Y are two monomials in the Grassmann algebra, let us define $X\mathcal{D}_i^g Y = (-1)^{\partial X(\partial \mathcal{D}_i)}X\mathcal{D}_iY$ where $\partial\mathcal{D}_i$ denotes the intrinsic Z_2-grading of $\mathcal{D}_i$. We now extend the definition to generic $\mathcal{D}$ and X by linearity. Then the previous cocycles can be written as follows:

$$\varphi_1(X, Y) = \int X[\theta^2\partial_1]^g Y d\theta^1 d\theta^2$$

$$\varphi_2(X, Y) = -\int X[\theta^1\partial_2]^g Y d\theta^1 d\theta^2$$

$$\varphi_3(X, Y) = \int X[-\theta^1\partial_1 + \theta^2\partial_2]^g Y d\theta^1 d\theta^2 \qquad (19)$$

$$\varphi_4(X, Y) = -\int X[\partial_1]^g Y d\theta^1 d\theta^2$$

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$$\varphi_5(X,Y) = -\int X[\partial_2]^g Y d\theta^1 d\theta^2$$

One can remark that the Z_2-grading of φ is given by the Z_2-grading of the differential operator which defines it.

- Cyclic cocycles of degree 0 and 1 on G_3

For G_3, we can choose the basis $\{1,\theta^1,\theta^2,\theta^3,\theta^1\theta^2,\theta^2\theta^3,\theta^3\theta^1,\theta^1\theta^2\theta^3\}$. Again, cyclic cocycles of degree zero are generated by the dual basis (this space has now dimension $2^3 = 8$). These elements will be denoted as follows:

$$\vartheta_1, \quad \vartheta_{\theta^1}, \quad \vartheta_{\theta^2}, \quad \vartheta_{\theta^3}, \quad \vartheta_{\theta^1\theta^2}, \quad \vartheta_{\theta^2\theta^3}, \quad \vartheta_{\theta^3\theta^1}, \quad \vartheta_{\theta^1\theta^2\theta^3}$$

Cyclic cocycles of degree one have to satisfy the cyclicity condition ($8^2 = 64$ equations) and the Hochschild condition ($8^3 = 512$ equations). One finds that this vector space has dimension 17 and is generated by the following cocycles (the first nine – that we will denote by φ_i – are even and the last eight – that we will denote by ψ_i – are odd). They are explicitly given as follows (i.e. they are non vanishing only on the following arguments).

$$\varphi_1(\theta^1,\theta^1) = \varphi_2(\theta^2,\theta^2) = \varphi_3(\theta^3,\theta^3) = 1$$

$$\varphi_4(\theta^1,\theta^2) = \varphi_4(\theta^2,\theta^1) = \varphi_5(\theta^2,\theta^3) = \varphi_5(\theta^3,\theta^2) = \varphi_6(\theta^3,\theta^1) = \varphi_6(\theta^1,\theta^3) = 1$$

$$\varphi_7(\theta^1,\theta^1\theta^2\theta^3) = \varphi_7(\theta^1\theta^2\theta^3,\theta^1) = \varphi_7(\theta^1\theta^2,\theta^3\theta^1) = -\varphi_7(\theta^3\theta^1,\theta^1\theta^2) = 1$$

$$\varphi_8(\theta^2,\theta^1\theta^2\theta^3) = \varphi_8(\theta^1\theta^2\theta^3,\theta^2) = -\varphi_8(\theta^1\theta^2,\theta^2\theta^3) = \varphi_8(\theta^2\theta^3,\theta^1\theta^2) = 1$$

$$\varphi_9(\theta^3,\theta^1\theta^2\theta^3) = \varphi_9(\theta^1\theta^2\theta^3,\theta^3) = -\varphi_9(\theta^3\theta^1,\theta^2\theta^3) = \varphi_9(\theta^2\theta^3,\theta^3\theta^1) = 1$$

$$\psi_1(\theta^1,\theta^1\theta^2) = -\psi_1(\theta^1\theta^2,\theta^1) = 1 \tag{20}$$

$$\psi_2(\theta^1,\theta^3\theta^1) = -\psi_2(\theta^3\theta^1,\theta^1) = 1$$

$$\psi_3(\theta^2,\theta^2\theta^3) = -\psi_3(\theta^2\theta^3,\theta^2) = 1$$

$$\psi_4(\theta^2,\theta^1\theta^2) = -\psi_4(\theta^1\theta^2,\theta^2) = 1$$

$$\psi_5(\theta^3,\theta^3\theta^1) = -\psi_5(\theta^3\theta^1,\theta^3) = 1$$

$$\psi_6(\theta^3,\theta^2\theta^3) = -\psi_6(\theta^2\theta^3,\theta^3) = 1$$

The last two, ψ_7 and ψ_8 have to be independent and should vanish as follows

$$-\psi(\theta^1,\theta^2\theta^3) + \psi(\theta^1\theta^2,\theta^3) + \psi(\theta^3\theta^1,\theta^2) = 0$$

As in the previous case, it is easy to rewrite the cocycles of degree one by using a Berezin integral. Let $X,Y \in G_3$ and write the following decomposition on the Grassmann variables

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$$X = X_0 + X_1\theta^1 + X_2\theta^2 + X_3\theta^3 + X_{12}\theta^1\theta^2 + X_{23}\theta^2\theta^3 + X_{31}\theta^3\theta^1 + X_{123}\theta^1\theta^2\theta^3 \text{ (and the}$$

same thing for Y). One obtains therefore

$$\varphi_1(X,Y) = X_1Y_1 = -\int X[\theta^2\theta^3\partial_1]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\varphi_2(X,Y) = X_2Y_2 = -\int X[\theta^3\theta^1\partial_2]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\varphi_3(X,Y) = X_3Y_3 = -\int X[\theta^1\theta^2\partial_3]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\varphi_4(X,Y) = X_1Y_2 + X_2Y_1 = -\int X[\theta^3\theta^1\partial_1 - \theta^3\theta^2\partial_2]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\varphi_5(X,Y) = X_2Y_3 + X_3Y_2 = -\int X[\theta^1\theta^2\partial_2 - \theta^1\theta^3\partial_3]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\varphi_6(X,Y) = X_3Y_1 + X_1Y_3 = -\int X[\theta^2\theta^3\partial_3 - \theta^2\theta^1\partial_1]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\varphi_7(X,Y) = X_1Y_{123} + X_{123}Y_1 + X_{12}Y_{31} - X_{31}Y_{12} = -\int X[\partial_1]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\varphi_8(X,Y) = X_2Y_{123} + X_{123}Y_2 + X_{23}Y_{12} - X_{12}Y_{23} = -\int X[\partial_2]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\varphi_9(X,Y) = X_3Y_{123} + X_{123}Y_3 + X_{23}Y_{31} - X_{31}Y_{23} = -\int X[\partial_3]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\psi_1(X,Y) = X_1Y_{12} - X_{12}Y_1 = -\int X[\theta^3\partial_1]^g Y d\theta^1 d\theta^2 d\theta^3 \qquad (21)$$

$$\psi_2(X,Y) = X_1Y_{31} - X_{31}Y_1 = -\int X[\theta^2\partial_1]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\psi_3(X,Y) = X_2Y_{23} - X_{23}Y_2 = -\int X[\theta^1\partial_2]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\psi_4(X,Y) = X_2Y_{12} - X_{12}Y_2 = -\int X[\theta^3\partial_2]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\psi_5(X,Y) = X_3Y_{31} - X_{31}Y_3 = -\int X[\theta^2\partial_3]^g Y d\theta^1 d\theta^2 d\theta^3$$

$$\psi_6(X,Y) = X_3Y_{23} - X_{23}Y_3 = -\int X[\theta^1\partial_3]^g Y d\theta^1 d\theta^2 d\theta^3$$

For the last two cocycles, one can write them as two independent combinations of the form

$$-\int X[\eta_1\theta^1\partial_1 + \eta_2\theta^2\partial_2 + \eta_3\theta^3\partial_3]^g Y d\theta^1 d\theta^2 d\theta^3$$

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with $\eta_1 + \eta_2 + \eta_3 = 0$. This example shows that, as soon as $N > 2$, it is not possible to canonically associate a basis of one-cocycles to a given basis in G_N.

- Cyclic cocycles on G_N, general results.

Although the cyclic cohomology of Grassmann algebras has not been determined explicitly, the dimensionality of the groups of cyclic cohomology $H_\lambda^p(G_N)$ has been calculated [7] (thanks to a Künneth-like formula that we will discuss later). The results are the following. For $p = 0$, the dimension is 2^N – this, we already know since the space of zero cocycles coincides with the vector space of one forms on G_N. For any p, the "Z_2-graded dimension" d is the coefficient of t^p in the expansion of $P(t) = [2^{N-1}(1+\epsilon)-(1-t)^N]/[(1+t)(1-t)^N]$ where ϵ is a Z_2 generator. Actually, one can write d as $d = d_e + d_o\epsilon$ where d_e (resp. d_o) is the number of even (resp. odd) cyclic cocycles of degree p. The total number of cyclic cocycles of degree p is therefore $d_t = d_e + d_o$. For $N = 2$ and $p = 1$ the reader can check that $d = 3 + 2\epsilon$ (i.e. three even generators and two odd for the one-cocycles), in accordance with the previous explicit results. An analogous situation holds for $N = 3$ where one gets $d = 9 + 8\epsilon$ one-cocycles. The number of independent one-cocycles grows rapidly with N; it is easy to show that one gets $(N - 1)2^{N-1} + 1$ even one-cocycles and $(N - 1)2^{N-1}$ odd one-cocycles. Altogether, $d_t = (N - 1)2^N + 1$, ($d = 25 + 24\epsilon \approx 49$ for $N = 4$).

We have seen that for $N = 1$, 2 and 3, the cyclic cocycles of degree one may be expressed as Berezin integrals $\omega_{\mathcal{D}}(X, Y) = \int X\mathcal{D}^g Y d^N\theta$, where $\mathcal{D}$ is a differential operator of degree one in the Grassmann algebra. This result can be extended for general N as follows. We first take a bilinear form $\omega_{\mathcal{D}}$ from $G_N \times G_N$ in R, $\omega_{\mathcal{D}}(X, Y) = \int X\mathcal{D}^g Y d^N\theta$. We may assume that $\mathcal{D} = \mathcal{D}^+ + \mathcal{D}^-$ where $\mathcal{D}^\pm = \sum_{i=1}^N P_i^\mp \partial_i$ with $P_i^+(\theta)$ (resp. $P_i^-(\theta)$) is an even (resp. odd) polynomial in the Grassmann variables θ. In this case, one can write $\omega_{\mathcal{D}}$ as

$$\omega_{\mathcal{D}}(X, Y) = \int X\mathcal{D}^+ Y d^N\theta + \int(-1)^{\partial X} X\mathcal{D}^- Y d^N\theta \qquad (22)$$

Integrating by parts (taking care of graded commutation $XY = (-1)^{\partial X \partial Y} YX$) leads to the following expression:

$$\omega_{\mathcal{D}}(X, Y) = (-1)^{1+\partial X \partial Y}\omega_{\mathcal{D}}(X, Y) - \int X(div^g\mathcal{D})Y d^N\theta \qquad (23)$$

where the graded divergence is defined by [7]

$$div^g\mathcal{D} \equiv \sum_{i=1}^N \partial_i P_i^* \equiv \sum_{i=1}^N \partial_i P_i^+ - \partial_i P_i^- \qquad (24)$$

The cocycle condition is then trivially satisfied thanks to the divergence equation:

$$\omega_{\mathcal{D}}(XY, Z) + \omega_{\mathcal{D}}(YZ, X) + \omega_{\mathcal{D}}(ZX, Y) = \sum_{i=1}^N \int \partial_i(P_i^* XYZ)d^N\theta = 0 \qquad (25)$$

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[7] For any super-function P given by $P^+ + P^-$ where $\partial P^+ = 0$ and $\partial P^- = 1$, one defines the star super-function by $P^* = P^+ - P^-$

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Therefore, $\omega_{\mathcal{D}}(X, Y)$ is a cyclic cocycle provided the differential operator $\mathcal{D}$ is graded divergenceless: $div^g \mathcal{D} = 0$.

One has to verify that ω_D is not a trivial one-cocycle, i.e. a coboundary $b\Omega$ where Ω is a zero-cocycle. But the space of cyclic cocycle of degree zero (of dimension 2^N) is identified with the dual of G_N. Therefore, it can be written as $\Omega(X) = \int P(\theta).X d^N \theta$ where $P(\theta)$ is a polynomial in the θ variables. More precisely, if ϑ_I denotes the dual basis of θ^I, and if $X \in G_N$, one gets $\vartheta_I(X) = \int [\partial/\partial \theta^I (\theta^1 \theta^2 ... \theta^N) X] d^N \theta$. One deduces that the coboundaries are of the form $[b\Omega](X, Y) = \Omega(XY) = \int P(\theta) XY d^N \theta$ and are not associated with (divergenceless) differential operators. Thus ω_D is not a trivial cyclic cocycle of degree one. On the other hand, the differential operator $\mathcal{D}$ depends on $N2^N$ constants. The cocycle condition $div^g \mathcal{D} = 0$, projected on each product of θ variables leads to $2^N - 1$ independent equations of constraints. One obtains in this way $N2^N - (2^N - 1) = (N - 1)2^N + 1$ independent cocycles as Berezin integrals. As we know, this is just the number of independent cocycles for G_N, and so we conclude that every non trivial cyclic cocycle of degree one on G_N has the form

$$\omega_{\mathcal{D}}(X, Y) = \sum_{i=1}^{N} \int \left(X P_i^+ \partial_i Y + (-1)^{\partial X} X P_i^- \partial_i Y \right) d^N \theta = \int X \mathcal{D}^g Y d^N \theta \qquad (26)$$

with $div^g \mathcal{D} = 0$.

Remark that if $\mathcal{D}$ and $\mathcal{D}'$ are differential operators defining cyclic cocycles, $[\mathcal{D}, \mathcal{D}']$ is such that $div^g[\mathcal{D}, \mathcal{D}'] = \mathcal{D}(div^g \mathcal{D}') - (-1)^{\partial \mathcal{D} \partial \mathcal{D}'} \mathcal{D}'(div^g \mathcal{D}) = 0$, so $[\mathcal{D}, \mathcal{D}']$ also defines a cyclic cocycle. The set of derivations $\mathcal{D}$ such that $div^g \mathcal{D} = 0$ is a Lie subalgebra of the algebra of differential operators of degree one in the Grassmann variables.

In general, the cocycle $\omega_{\mathcal{D}}$ has non zero values on any couples (X, Y) whatever the Z_2-grading of X and Y. It might be interesting to know under which condition the cocycle $\omega_{\mathcal{D}}$ vanishes. The cases where $\omega_{\mathcal{D}} = 0$ for $\partial X \neq \partial Y$ is of particular interest. Actually, since the integration over the θ's selects only the $\theta^1 ... \theta^N$ component in $X\mathcal{D}^g Y$, one obtains the following result (which is easy to check by choosing specific values for X and Y): the cocycle $\omega_{\mathcal{D}}$ vanishes on couples (X, Y) such that $\partial X \neq \partial Y$ (respectively $\partial X = \partial Y$) if and only if the differential operator $\mathcal{D}$ has the same grading $\partial \mathcal{D}$ than the parity of N (respectively of $N + 1$).

2.1.6. The cyclic cohomology of the super-circle

We now want to determine the cyclic cocycles of degree one on the super-circle, i.e. on the algebra $A_N = C(S^1) \otimes G_N$. Rather than trying to solve directly the equations of cyclicity and of vanishing of $b\Phi$ in this case, it is easier to use the knowledge that we have already on the pieces G_N and $C(S^1)$ along with a theorem which allows us to compute the cyclic cohomology of the algebra $A \otimes B$ as soon as we know those of A and of B. Such

a theorem is proven in [7][10] and we will describe it below. As far as one-cocycles are concerned, the result is rather intuitive. One finds [8] :

$$\left[\text{Space of cyclic cocycles of degree one on the supercircle}\right] =$$

$$\left[\text{Space of distributions on } S^1 \text{ (i.e. zero cocycles)}\right] \otimes \left[\text{Cyclic cocycles of degree one on } G_N\right]$$

$$\oplus$$

$$\left[\text{Cyclic cocycles of degree one on } S^1\right] \otimes \left[\text{Linear forms on } G_N \text{ (i.e. zero cocycles)}\right]$$

The reader should not however believe that this formula can be easily generalized to other spaces $\mathcal{M} \neq S^1$ and to cyclic cocycles of degree $p > 1$. There is a simple "Künneth formula" for the Hochschild cohomology. However, the "Künneth formula" for cyclic cohomology is not simple, but it simplifies here because $\mathcal{M} = S^1$ and $p = 1$. We will indicate below how to study such more general situations. The reader only interested in the one-cocycles on A_N should just read the following results and skip the discussion that ends the paragraph.

One gets
- $(N-1)2^N + 1$ cocycles of the kind $\alpha_i(t) \otimes \varphi_i$, where α_i denote arbitrary distributions on the circle and φ_i are the cyclic cocycles of degree one on G_N determined previously.
- 2^N cocycles of the kind $a_k \omega_c \otimes \vartheta_k$ where a_k are arbitrary constants, $\omega_c(X,Y)$ is given by the formula (6) and ϑ_k are elements of the dual basis of G_N.

It is of course possible to write these cocycles by using integration on the super-circle. For instance, in the case of G_2, the cocycle $\Phi = \alpha_2(t) \otimes \varphi_2$ reads

$$\Phi(X,Y) = \int \alpha_2(t) X_2(t) Y_2(t) dt = \int \alpha_2(t) X[\theta^1 \partial_2]^g Y \, dt d\theta^1 d\theta^2$$

the cocycle $\Phi = a_1 \omega_c \otimes \vartheta_1$ reads

$$\Phi(X,Y) = a_1 \int X_0 \partial_t Y_0 dt = a_1 \int X[\theta^1 \theta^2 \partial_t]^g Y \, dt d\theta^1 d\theta^2$$

and the cocycle $\Phi = a_2 \omega_c \otimes \vartheta_{\theta^1}$ reads

$$\Phi(X,Y) = a_2 \int (X_0 \partial_t Y_1 + X_1 \partial_t Y_0) dt = -a_2 \int X[\theta^2 \partial_t]^g Y \, dt d\theta^1 d\theta^2$$

with $X(t, \theta^1, \theta^2) = X_0(t) + X_1(t)\theta^1 + X_2(t)\theta^2 + X_{12}(t)\theta^1\theta^2$ and similarly for Y [9] .

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[8] What else could it be !

[9] We extend the definition of $\mathcal{D}^g$ from G_N to A_N by putting the intrinsic Z_2-grading of ∂_t to zero.

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According to [7], the Hochschild cohomology of a product is given by the following relation: $H^*(A \otimes B) = H^*(A) \otimes H^*(B)$ [10]. The formula giving the cyclic cohomology groups $H_\lambda^p(A \otimes B)$ is slightly more involved. The first result that we need (and which is rather easy to prove, cf [5][8]) is that the cyclic cohomology of the algebra of complex numbers $H_\lambda^p(C)$ is 0 if p is odd and equal to C if p is even. One can indeed prove that it is generated in degree two by the following form $\sigma(1,1,1) = 2i\pi$. Moreover, it is possible to show [5] that for any algebra A, $H_\lambda^*(A)$ is a module over $H_\lambda^*(C)$; indeed there is an operation (called "the operation S of Connes") sending $H_\lambda^p(A)$ to $H_\lambda^{p+2}(A)$ and this action is fully characterized by the action of the generator σ. In order to prove the relevant formula, one has to assume that the algebra B is such that $H_\lambda^*(B)$ can be written as

$$H_\lambda^*(B) = H_\lambda^*(C) \otimes U^* \oplus V^* \tag{27}$$

where $U^* = \bigoplus_{p>0} U^p$ is a finite dimensional vector space and where V^* is a trivial module for the action of the operation S, i.e. $S = 0$ on V^*. The only thing that we need here is that this property is indeed satisfied for the algebra $B = G_N$. Indeed, one can show that $H_\lambda^*(G_N) = H_\lambda^*(C) + W^*$. The (graded) Poincare polynomial of the vector space W^* is actually the quantity $P(t)$ that we introduced at the end of sect. 2.1.5. For instance, in the case of G_1, we had $W^0 = C$ and $W^1 = C$ so that $H_\lambda^0 = C \oplus C$ and $H_\lambda^1 = C$. In the case of G_2, we had $W_0 = C \oplus C \oplus C$ and $W_1 = (C \oplus C \oplus C) \oplus (C \oplus C)$. More generally, G_N has the required property, with $U^0 = C, U^i = 0$ for $i > 0$ and $V^* = W^*$. For such an algebra B one proves that

$$H_\lambda^n(A \otimes B) = \bigoplus_{p+q=n} [H_\lambda^p(A) \otimes U^q \oplus H^p(A) \otimes V^q] \tag{28}$$

Notice that the second term involves Hochschild cohomology. In the present case, there are two substantial simplifications. The first comes from the fact that for a Grassmann algebra $U^p = 0$ as soon as $p \geq 1$. Let us assume that $A = C^\infty(\mathcal{M})$ where $\mathcal{M}$ is some finite dimensional smooth manifold. For one-cocycles, the formula reads

$$H_\lambda^1(C^\infty(\mathcal{M}) \otimes G_N) = \Lambda_0(\mathcal{M}) \otimes W^1 + H_\lambda^1(C^\infty(\mathcal{M})) \otimes C + \Lambda_1(\mathcal{M}) \otimes W^0 \tag{29}$$

where Λ_0 denotes the space of all currents of degree zero on $\mathcal{M}$ (distributions), Λ_1 denotes the space of currents of degree one on $\mathcal{M}$ and $H_\lambda^1(C^\infty(\mathcal{M}))$ coincides with the space of closed currents of degree one on $\mathcal{M}$ [11]. The next simplification comes from taking $\mathcal{M} = S^1$; then all the currents of degree one are closed since the circle is one-dimensional.

From our separate study of the cyclic cohomology of the circle and of Grassmann algebras, we conclude that, in the case of the super-circle, $H_\lambda^1(A_N)$ is spanned by the following three kinds of cocycles:

$d = (N-1)2^N + 1$ cocycles of the kind $\alpha_i(t) \otimes \varphi_i$, where the α_i's denote arbitrary functions on the circle (actually distributions) and the φ_i's are cyclic cocycles of degree

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[10] The notation H^* denotes the collection of H^p.

[11] Here we make use of the result given in the footnote 6 of sect. 2.1.4.

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one on G_N – which have the same Z_2-grading as the cocycles of G_N. This first class of cocycles corresponds to the first term of the general "Künneth-like" formula.

One cocycle of the kind $a_1\omega_c \otimes \vartheta_1$ where a_1 is an arbitrary constant, ω_c is the usual cyclic cocycle on the circle and ϑ_1 is the cocycle of degree zero on G_N defined as the one form vanishing on all the generators of G_N but the unit. Notice that this cocycle is even. This second class of cocycles (containing only one element) corresponds to the second term of the general formula [12] .

Finally, we obtain $2^N - 1$ cocycles of the kind $a_k\omega_c \otimes \vartheta_k$ where a_k are arbitrary constants, ω_c is the usual cocycle on $C(S^1)$ and ϑ_k are cyclic cocycles of degree zero on G_N – the special cocycle $k = 1$ has already been singled out. It is clear that 2^{N-1} among those cocycles are odd and the others $(2^{N-1} - 1)$ are even. This last class of cocycles corresponds to the last term of the general formula.

Altogether, we find that the cyclic cocycles of degree one on the super-circle A_N span a vector space of dimension $N2^N + 1$. They depend upon $(N - 1)2^N + 1$ arbitrary distributions and 2^N constants. In the following, we will concentrate our attention to the distributions which are associated to functions.

2.2. Explicit results

For the convenience of the reader, and because we will need them later, we gather the results for $N = 1, 2$ and 3.

- The case $N = 1$

According to the results of section 2.1.6., the most general one-cocycles are linear combinations of $a_1\omega_c \otimes \vartheta_1$, $a_0\omega_c \otimes \vartheta_\theta$ and $\alpha \otimes \varphi$, where a_0, a_1 are constants, α is a function of t and $\omega_c(X, Y)$ is given by (6). We find it more convenient to express the results in terms of Berezin integrals and use the characterization of cyclic cocycles by the graded divergenceless differential operators described previously (we shall continue to adopt this attitude in the following). An arbitrary differential operator of order one on A_1 can be written as:

$$\mathcal{D} = a_0\partial_t \ + \ a_1\theta\partial_t \ + \ \epsilon\partial_\theta \ + \ \sigma\theta\partial_\theta \tag{30}$$

Operators characterizing cyclic cocycles have to satisfy a divergence equation which implies that $\sigma = 0$. This is of course in accordance with the results of sect. 2.1.6.
The $N = 1$ cocycle takes the form

$$\Phi_{N=1}(X, Y) = \int X\big[a_0\partial_t \ + \ a_1\theta\partial_t \ + \ \epsilon\partial_\theta\big]^\vartheta Y \, d\theta dt \tag{31}$$

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[12] The particular cyclic cocycle of degree zero on G_N called ϑ_1 in sect. 2.1.5. is actually rather special. Indeed, by repeated action of the operator S, it leads to a hierarchy of non trivial cyclic cocycles of degree 2n. Repeated action of S (stabilisation) on the other zero cocycles lead to cocycles that are cohomologically trivial. This is discussed in [11] along with several elementary properties of Grassman algebras.

which depends on 2 constants a_0, a_1 and one function ϵ of t.

- The case $N = 2$

The most general differential operator of order one of A_2 is:

$$\mathcal{D} = a_0 \partial_t + a_{12}\theta^1\theta^2\partial_t + \sum_i (a_i\theta^i\partial_t + \epsilon_i\partial_i + \lambda_i\theta^1\theta^2\partial_i) + \sum_{ij}\sigma_{ij}\theta^i\partial_j \qquad (32)$$

The divergence equation implies

$$\sigma_{11} + \sigma_{22} = 0 \qquad \text{and} \qquad \lambda_i = 0 \qquad (i = 1,2) \qquad (33)$$

Therefore the $N = 2$ one-cocycle has the following form:

$$\Phi_{N=2}(X,Y) = \int X \left[a_0\partial_t + a_{12}\theta^1\theta^2\partial_t + \sum_i (a_i\theta^i\partial_t + \epsilon_i\partial_i) + \sum_{ij}\sigma_{ij}\theta^i\partial_j \right]^g Y d\theta^1 d\theta^2 dt$$

$$(34)$$

where a_0, a_1, a_2, a_{12} are constants and ϵ_i, σ_{ij} are functions of t.

- The case $N = 3$

The most general differential operator of order one of A_3 is:

$$\mathcal{D} = a_0\partial_t + a_{123}\theta^1\theta^2\theta^3\partial_t + \sum_i (a_i\theta^i\partial_t + \epsilon_i\partial_i + \lambda_i\theta^1\theta^2\theta^3\partial_i) +$$

$$\sum_{ij}\sigma_{ij}\theta^i\partial_j + \sum_{cyclic(i,j,k)} \left(b_k\theta^i\theta^j\partial_t + \sum_m \omega_{mk}\theta^i\theta^j\partial_m \right) \qquad (35)$$

The notation $cyclic(i,j,k)$ means that we sum over cyclic permutations of $1,2,3$.

The divergenceless condition leads to the following constraints on the coefficients:

$$\sigma_{11} + \sigma_{22} + \sigma_{33} = 0, \qquad \lambda_i = 0, \qquad \omega_{ij} = \omega_{ji}, \qquad i,j = 1,2,3 \qquad (36)$$

One obtains therefore for the $N = 3$ one-cocycle:

$$\Phi_{N=3}(X,Y) = \int X \left[a_0\partial_t + a_{123}\theta^1\theta^2\theta^3\partial_t + \sum_i (a_i\theta^i\partial_t + \epsilon_i\partial_i) + \sum_{ij}\sigma_{ij}\theta^i\partial_j + \right.$$

$$\left. + \sum_{cyclic(i,j,k)} \left(b_k\theta^i\theta^j\partial_t + \omega_{ij}(\theta^j\theta^k\partial_j - \theta^i\theta^k\partial_i) + \omega_{kk}\theta^i\theta^j\partial_k \right) \right]^g Y d\theta^1 d\theta^2 d\theta^3 dt \qquad (37)$$

where a_0, a_i, a_{123}, b_i are constants and $\epsilon_i, \sigma_{ij}, \omega_{ij}$ are functions of t.

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3. Algebras of Super-Derivations (the general framework)

3.1. General discussion

3.1.1. Derivations and Super-derivations

Let $Der(A_0)$ be the algebra of vector fields on the circle, it is also the algebra of derivations of $A_0 = C(S^1)$. In the case $N \neq 0$, we will call $Der(A_N)$ the algebra of graded derivations of $A_N = C(S^1) \otimes G_N$. It is clear that $Der(A_N)$ is generated by derivations of the type $\mathcal{L} = V^t \partial_t + V^\alpha \partial_\alpha$ where $\partial_t = \partial/\partial t$ and $\partial_\alpha = \partial/\partial\theta^\alpha$. This notation can be condensed as $\mathcal{L} = V^I \partial_I$ where $I \in \{0, 1, 2, \ldots, N\}$ and where we set $V^t = V^0, \partial_t = \partial_0$. The quantities V^I are, of course, functions on the super-circle, $V^I = V^I(t, \theta) = V_A^I(t)\theta^A$ where $A \in \{1, \ldots, 2^N\}$ and θ^A is a monomial element of G_N. Therefore $V_A^I(t)$ is a rectangular matrix of dimension $2^N(N+1)$ whose elements are functions of $t \in S^1$. Notice that for an arbitrary algebra A, the algebra of its derivations $Der A$ is usually not a module over A (i.e. if $a \in A$ and $\delta \in Der(A)$, then $a \times \delta$ is usually not in $Der(A)$). However, when A is commutative or graded-commutative (as here), it is so. Indeed, $\mathcal{D}_N$ is a module of dimension $N + 1$ over A_N (and of dimension $2^N(N+1)$ over $C(S^1)$).

We will now build central extensions of super-loop algebras and examine what are the derivations compatible with a given extension.

3.1.2. The compatibility equation

If ω is a Lie algebra cocycle on the super-loop algebra K_N, one can build a central extension (cf. eq. (2)) $K_{N,\omega} = K_N \oplus R \times e$ where e is a new (central) generator and with a new Lie product, $[X, Y]_{new} = [X, Y]_{old} + \omega(X, Y) \times e$. It is clear that not any (graded) derivation of K_N will be a graded derivation of $K_{N,\omega}$. Such a graded derivation $\mathcal{L}$ should satisfy the property

$$\mathcal{L}([X, Y]) = [\mathcal{L}X, Y] + (-1)^{\partial X \partial \mathcal{L}}[X, \mathcal{L}Y] \tag{38}$$

where $\partial \mathcal{L}$ is zero if it is a true derivation and one if it is an antiderivation. From the above definition of $[\ ,\ ]_{new}$, it is clear that $\mathcal{L}$ will be a derivation if

$$\omega(\mathcal{L}X, Y) + (-1)^{\partial X \partial \mathcal{L}}\omega(X, \mathcal{L}Y) = 0 \quad \text{for any} \quad X, Y \in K_N \tag{39}$$

As described in section 1., K_N is obtained from A_N by tensorialization by matrices; in the same way, the Lie algebra cocycle ω can be gotten from a cyclic cocycle Φ on the super-circle A_N. In order to remove unnecessary complications, we can work with A_N rather than K_N: determining the Lie algebra of super-derivations of $K_{N,\omega}$ compatible with a given Lie algebra cocycle ω amounts to determining the super-derivations $\mathcal{L}$ of A_N compatible with

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a cyclic cocycle Φ of A_N, i.e. it amounts to selecting a subalgebra $Der_\Phi(A_N)$ of $Der(A_N)$ characterized by the $\mathcal{L}$ such that

$$\Phi(\mathcal{L}X, Y) + (-1)^{\partial X \partial \mathcal{L}} \Phi(X, \mathcal{L}Y) = 0 \quad \text{for any} \quad X, Y \in A_N \tag{40}$$

If we write $\Phi(X, Y)$ as $\int X\mathcal{D}^g Y$ where $\mathcal{D}$ is a differential operator of first order (we saw that it is always possible), the previous equation simply reads (using $\Phi(X, \mathcal{L}Y) = (-)^{1+\partial X(\partial \mathcal{L}+\partial Y)}\Phi(\mathcal{L}Y, X)$ and integrating by part)

$$[\mathcal{L}, \mathcal{D}] + (div^g \mathcal{L})\mathcal{D} = 0 \tag{41}$$

the definition of div^g given in eq. (24) being now extended to the $N+1$ variables $t, \theta^1, ..., \theta^N$ of A_N.

Notice that if Φ is associated to a distribution, this equation has to be understood in the sense of the distribution, i.e. applied to any test functions X and Y:

$$\int X([\mathcal{L}, \mathcal{D}] + (div^g \mathcal{L})\mathcal{D})^g Y\, dt d^N \theta = 0 \quad \text{for any} \quad X, Y \in A_N$$

A derivation $\mathcal{L}$ which satisfies such an equation will be called a derivation compatible with the cyclic cocycle Φ (given by the differential operator $\mathcal{D}$) or $\mathcal{D}$-compatible.

This formula suggests a few comments. First, using the relation

$$div^g[\mathcal{L}, \mathcal{L}'] = \mathcal{L}(div^g \mathcal{L}') - (-1)^{\partial \mathcal{L} \partial \mathcal{L}'} \mathcal{L}'(div^g \mathcal{L})$$

the compatibility equation for some $\mathcal{L}$ and some $\mathcal{L}'$ implies the compatibility equation for $[\mathcal{L}, \mathcal{L}']$:

$$(div^g[\mathcal{L}, \mathcal{L}'])\mathcal{D} + [[\mathcal{L}, \mathcal{L}'], \mathcal{D}] = 0$$

In other words, the subset of $\mathcal{D}$-compatible super-derivations $\mathcal{L}$ is a graded Lie subalgebra of the graded Lie algebra of super-derivations. Secondly, since $div^g \mathcal{D} = 0$, the differential operator $\mathcal{D}$ defining the cocycle $\omega_\mathcal{D}$ is itself a $\mathcal{D}$-compatible super-derivation if $\mathcal{D}$ is even (whilst if $\mathcal{D}$ is odd $[\mathcal{D}, \mathcal{D}]$ represents an anticommutator which does not vanish necessarily). Finally, we notice that the compatibility equation represents actually two sets of equation since we demand the super-derivation V to be of definite Z_2-grading.

Thus, for a given cocycle Φ – a given differential operator $\mathcal{D}$ –, the previous compatibility equation selects a sub-algebra $Der_\Phi(A_N)$ of the algebra of all derivations. This sub-algebra is the algebra of derivations of the extension $A_{N,\Phi}$. Notice that these algebras $Der_\Phi(A_N)$ are also graded Lie algebra and could themselves possess non trivial extensions: we do not study here these possible extensions of these algebras of derivations.

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3.1.3. A problem of stratification

An arbitrary cocycle of A_N is a priori a linear combination of $(N-1)2^N + 1 + 2^N$ generators, with coefficients which are either constants or functions on the circle (cf. the previous section). Therefore, the solution of the compatibility equations (for given N) will a priori depends upon the constants (or the functions) entering the expression of the cocycle. It is already clear that the structure of the algebra $Der_\Phi(A_N)$ will depend on the vanishing or not of these coefficients. One is naturally led to consider the vector space of one-dimensional cyclic cocycles as a representation space for the super-derivation algebra. Then the subalgebra $Der_\Phi(A_N)$ appears as the stabilizer of the cocycle Φ. A stratum being the set of orbits associated, up to a conjugation, to the same little group, the problem consists of determining strata in the cocycle vector space. To each stratum, one can associate a particular sub-algebra $Der_\Phi(A_N)$ and the situation is completly understood when one is able to write the partial ordered set of subalgebras corresponding to the different strata. We will examine below the cases corresponding to $N = 1, 2$ and 3.

3.2. Study of the cases N=1, N=2, N=3

3.2.1. The case with one Grassmann generator

The algebra K_1 built from a simple Lie algebra $\mathcal{K}$ with generators T^a where $a = 1, ..., \dim \mathcal{K}$ will be described by T_m^a, $T_m^a\theta$ ($m \in \mathbf{Z}$) satisfying the commutation relations:

$$
\begin{aligned}
[T_m^a, T_n^b] &= if^{abc}T_{m+n}^c \\
[T_m^a, T_n^b\theta] &= if^{abc}T_{m+n}^c\theta \\
\{T_m^a\theta, T_n^b\theta\} &= 0
\end{aligned}
\tag{42}
$$

The super-Kač-Moody algebra obtained from the $N = 1$ cocycle given by (31) has the following non trivial commutation relations:

$$
\begin{aligned}
[T_m^a, T_n^b] &= if^{abc}T_{m+n}^c + a_1 m\delta_{m+n}\delta^{ab} \\
[T_m^a, T_n^b\theta] &= if^{abc}T_{m+n}^c\theta + a_0 m\delta_{m+n}\delta^{ab} \\
\{T_m^a\theta, T_n^b\theta\} &= -\epsilon_{m+n}\delta^{ab}
\end{aligned}
\tag{43}
$$

where we used the mode expansion of $\epsilon(t) = \sum_{m\in\mathbf{Z}} \epsilon_m t^{-m}$.

The most general even super-derivation is

$$
\mathcal{L}_{even} = A\partial_t + D\theta\partial_\theta
\tag{44}
$$

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and the most general odd super-derivation is

$$\mathcal{L}_{odd} = C\theta\partial_t + B\partial_\theta \tag{45}$$

The compatibility equations have the form

$$
\begin{aligned}
a_0 D &= 0 \\
\partial_t(\epsilon A) - 2\epsilon D &= 0 \\
a_0 \partial_t B &= 0 \\
\epsilon C + a_1 B &= 0
\end{aligned}
\tag{46}
$$

which lead to different super-derivation algebras according to the vanishing or not of the one-cocycle coefficients. One obtains explicitly: [13]

• $Der(A_1, a_0)$

The solutions of the compatibility equations are $D = \partial_t B = 0$ and therefore the super-derivation algebra is given by

$$\mathcal{L} = A\partial_t + C\theta\partial_t + k\partial_\theta \tag{47}$$

where k is a complex constant.
Taking for the super-derivations a basis of functions $L_n = -t^{n+1}\partial_t$, $C_n = t^{n+1}\theta\partial_t$ and $c = \partial_2$, we find the following commutation relations:

$$
\begin{array}{lll}
[L_n, L_m] = (n-m)L_{n+m} & \{C_n, C_m\} = 0 & [L_n, c] = 0 \\
[L_n, C_m] = (n-m)C_{n+m} & \{C_n, c\} = -L_n &
\end{array}
\tag{48}
$$

• $Der(A_1, a_1)$

The solution is now $B = 0$ and one has

$$\mathcal{L} = A\partial_t + D\theta\partial_\theta + C\theta\partial_t \tag{49}$$

Taking for the super-derivations a basis of functions $L_n = -t^{n+1}\partial_t$, $C_n = t^{n+1}\theta\partial_t$ and $D_n = t^n\theta\partial_\theta$, we find the following commutation relations:

$$
\begin{array}{ll}
[L_n, L_m] = (n-m)L_{n+m} & [L_n, C_m] = (n-m)C_{n+m} \\
[L_n, D_m] = -mD_{n+m} & [D_n, D_m] = 0 \\
[D_n, C_m] = C_{n+m} & \{C_n, C_m\} = 0
\end{array}
\tag{50}
$$

---

[13] The notation $Der(A_1, c)$ means that we choose a cocycle Φ of the kind (31) with all coefficients set to zero but c. The corresponding super-Kač-Moody algebra is given by eq. (43), the non-vanishing central extension terms being those where c appears.

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• $Der(A_1, \epsilon)$

The solutions are given by $D = \frac{1}{2\epsilon}\partial_t(\epsilon A)$ and $C = 0$. One obtains

$$\mathcal{L} = A\partial_t + \frac{1}{2\epsilon}\partial_t(\epsilon A)\theta\partial_\theta + B\partial_\theta \qquad (51)$$

Taking for the super-derivations a basis of functions $L_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n\theta\partial_\theta$ and $B_n = t^n\partial_\theta$, and for ϵ a constant function, we find the following commutation relations:

$$[L_n, L_m] = (n-m)L_{n+m} \qquad [L_n, B_m] = (\frac{n+1}{2} - m)B_{n+m} \qquad \{B_n, B_m\} = 0 \qquad (52)$$

•. $Der(A_1, a_0, a_1)$

One has for the compatibility equations $D = B = 0$. One obtains the super-derivation algebra

$$\mathcal{L} = A\partial_t + C\theta\partial_t \qquad (53)$$

The commutations relations are those of the super-derivation algebra $Der(A_1, a_0)$ restricted to the generators L_n and C_n.

• $Der(A_1, a_1, \epsilon)$

The solutions are $D = \frac{1}{2\epsilon}\partial_t(\epsilon A)$ and $B = -\frac{\epsilon}{a_1}C$. Thus the super-derivation algebra is

$$\mathcal{L} = A\partial_t + \frac{1}{2\epsilon}\partial_t(\epsilon A)\theta\partial_\theta + C(\theta\partial_t - \frac{\epsilon}{a_1}\partial_\theta) \qquad $$

However, a redefinition of the θ parameter leads to the following expression:

$$\mathcal{L} = A\partial_t + \frac{1}{2\epsilon}\partial_t(\epsilon A)\theta\partial_\theta + C(\theta\partial_t - \epsilon\partial_\theta) \qquad (54)$$

This super-derivation algebra is actually the $N = 1$ super-Virasoro conformal algebra (also called Neveu-Schwarz-Ramond algebra) [1]. The commutation relations between the generators $L(u_0) = u_0\partial_t + \frac{1}{2\epsilon}(\partial_t(\epsilon u_0))\theta\partial_\theta$ and $G(u_\theta) = u_\theta(\theta\partial_t - \epsilon\partial_\theta)$ are the following:

$$[L(u_0), L(v_0)] = L(u_0\partial_t v_0 - v_0\partial_t u_0)$$

$$[L(u_0), G(v_\theta)] = G(u_0\partial_t v_\theta - \frac{1}{2}v_\theta\partial_t u_0) \qquad (55)$$

$$\{G(u_\theta), G(v_\theta)\} = L(-2u_\theta v_\theta)$$

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Taking $\epsilon(t) \doteq t^{2\kappa-1}$ with $\kappa = 0$ (Ramond sector) or $1/2$ (Neveu-Schwarz sector), the Laurent expansions of L and G, $L_n = -t^{n+1}\partial_t - \frac{n}{2}t^n\theta\partial_\theta$ and $G_n = t^{n+1}(\theta\partial_t - t^{2\kappa-1}\partial_\theta)$, lead to the relations:

$$[L_m, L_n] = (m-n)L_{m+n}$$
$$[L_m, G_r] = (\frac{m+2\kappa}{2} - r)G_{m+r} \tag{56}$$
$$\{G_r, G_s\} = 2L_{r+s}$$

with $m, n \in Z$ and $r, s \in Z + \kappa$.

- $Der(A_1, a_0, \epsilon)$

The solutions are given by $D = C = 0$ and $\epsilon A = $ constant, $B = $ constant. The super-derivation algebra is generated by

$$\mathcal{L} = \frac{c_1}{\epsilon}\partial_t + c_2\partial_\theta \tag{57}$$
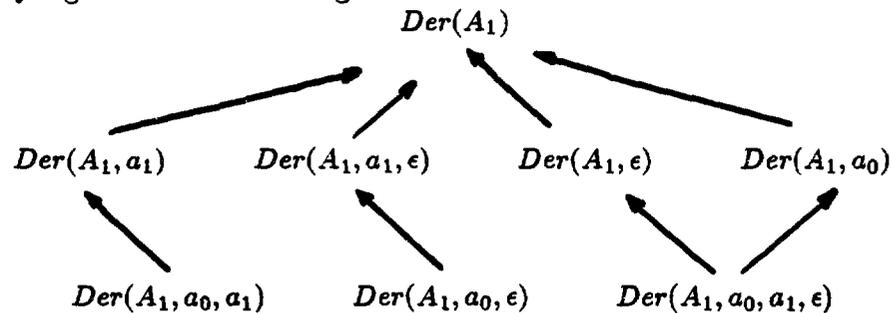
where c_1 and c_2 are complex constants.

- $Der(A_1, a_0, a_1, \epsilon)$

The solutions are given by D and $\epsilon A = $ constant, $B = $ constant, $C = -\frac{a_1}{\epsilon}B$. The super-derivation algebra is generated by

$$\mathcal{L} = \frac{c_1}{\epsilon}\partial_t - a_1\frac{c_2}{\epsilon}\theta\partial_t + c_2\partial_\theta \tag{58}$$

where c_1 and c_2 are complex constants.

Finally the partially ordered set of the super-derivation algebras of the $N = 1$ super-Kač-Moody algebras is the following:

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3.2.2. The case with two Grassmann generators

The most general even super-derivation is

$$\mathcal{L}_{even} = A\partial_t + B\theta^1\theta^2\partial_t + \sum_{ij} W_{ij}\theta^i\partial_j \tag{59}$$

and the most general odd super-derivation is

$$\mathcal{L}_{odd} = \sum_i L_i\theta^i\partial_t + C_i\partial_i + D_i\theta^1\theta^2\partial_i \tag{60}$$

The compatibility equations $N = 2$ read (the indices i,j run from 1 to 2):

$$a_0(W_{11} + W_{22}) = 0$$

$$\epsilon_i B - a_1 W_{2i} + a_2 W_{1i} = 0$$

$$\partial_t(\sigma_{ij}A) - a_0\partial_t W_{ij} + \sum_m (\sigma_{mj}W_{im} - \sigma_{im}W_{mj} - \sigma_{ij}W_{mm}) = 0$$

$$\partial_t(\epsilon_i A) - \sum_m (\epsilon_m W_{mi} + \epsilon_i W_{mm}) = 0 \tag{61}$$

$$\sum_m (\epsilon_m L_m + a_m C_m) = 0$$

$$a_{12}C_i + a_0 D_i - \sigma_{2i}L_1 + \sigma_{1i}L_2 = 0$$

$$a_0\partial_t C_i - \sum_m \sigma_{im}C_m = 0$$

$$\partial_t(\epsilon_i L_j) + a_j\partial_t C_i + (-1)^j(\epsilon_i D_j + \epsilon_j D_i) = 0$$

Since the cocycle $N = 2$ contains 9 independent coefficients, these equations lead to $2^9 = 512$ different algebras, many of them being isomorphic. This gives an estimate of the large number of inequivalent strata. Although we have not performed a detailed analysis of all these cases, however, it is not too difficult to convince oneself that none of these algebras of super-derivations contains the $N = 2$ super-conformal algebra that will be defined in (80). Indeed, when putting the explicit expression of the generators of the $N = 2$ super-conformal algebra in terms of super-derivations (cf. eq. (80)) into the compatibility equations (61), one finds that the only solution is given by the zero cocycle: $\mathcal{D} = 0$.

3.2.3. The case with three Grassmann generators

The most general even super-derivation is

$$\mathcal{L}_{even} = A\partial_t + \sum_i K_i \theta^1\theta^2\theta^3 \partial_i + \sum_{ij} H_{ij}\theta^i\partial_j + \sum_{cyclic(i,j,k)} D_k \theta^i\theta^j \partial_t \qquad (62)$$

and the most general odd super-derivation is

$$\mathcal{L}_{odd} = B\theta^1\theta^2\theta^3 \partial_t + \sum_i (C_i \theta^i\partial_t + E_i\partial_i) + \sum_{cyclic(i,j,k)}\sum_m W_{mk}\theta^i\theta^j \partial_m \qquad (63)$$

where $\sum_{cyclic(i,j,k)}$ denotes a sum over the cyclic permutations of (1,2,3).

The compatibility equations $N = 3$ read (the indices i,j runs from 1 to 3 in the equations (64.3) to (64.7), and the indices i,j,k are circular permutations of (1,2,3) in the equations (64.8) to (64.12), whilst $l = 1,2,3$ in the three last equations:

$$a_0 \sum_m H_{mm} = 0$$

$$\sum_m (a_m E_m + \epsilon_m C_m) = 0$$

$$\partial_t(\epsilon_i A) - \sum_m (\epsilon_i H_{mm} + \epsilon_m H_{mi}) = 0$$

$$a_0 K_i + \sum_m b_m H_{mi} - \sum_m \sigma_{mi} D_m = 0$$

$$\sum_m (a_m W_{im} + \omega_{im} C_m) + a_{123} E_i + \epsilon_i B = 0$$

$$a_0 \partial_t E_i - \sum_m \sigma_{mi} E_m = 0 \qquad (64)$$

$$\partial_t(\sigma_{ij} A) - a_0 \partial_t H_{ij} + \sum_m (\sigma_{mj} H_{im} - \sigma_{im} H_{mj} - \sigma_{ij} H_{mm}) = 0$$

$$\epsilon_i D_j - \epsilon_j D_i - a_k(H_{ii} + H_{jj}) + a_i H_{ki} + a_j H_{kj} = 0$$

$$a_0(W_{ij} - W_{ji}) + b_j E_i - b_i E_j - \sum_m \sigma_{km} C_m = 0$$

$$\partial_t(\omega_{il} A) + \partial_t(\epsilon_l D_i) - \epsilon_i K_i - \epsilon_i K_l - a_j \partial_t H_{kl} + a_k \partial_t H_{jl} - \sum_m (\omega_{jm} H_{mi} + \omega_{im} H_{mj}) = 0$$

$$\partial_t(\epsilon_l C_i + a_i E_l) - \epsilon_l W_{jk} - \epsilon_j W_{lk} + \epsilon_l W_{kj} + \epsilon_k W_{lj} + \omega_{lj} E_k - \omega_{kl} E_j = 0$$

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$$\partial_t(\sigma_{kl}C_j - \sigma_{jl}C_k - a_0 W_{li} - b_i E_l) + \sum_m (\sigma_{ml}W_{im} + \sigma_{mi}W_{lm}) = 0$$

4. Superconformal algebras and algebras of super-derivations. Discussion and open problems

In string theory, the supersymmetric extension ($N = 1$) of the Virasoro algebra is known as the Ramond-Neveu-Schwarz (or RNS) algebra. A natural generalization to extended supersymmetries ($N > 1$) has been proposed by Ademollo et al. [2] a long time ago. The Virasoro algebra can be seen as the derivation algebra of the circle compatible with a Kač-Moody algebra. In the same way, the RNS algebra can be seen as a derivation algebra of the super-circle A_1 compatible with a non trivial cocycle associated with a super-Kač-Moody algebra constructed from A_1 [1]. The situation changes drastically for $N \geq 2$: as we have seen by direct computation, there is no non trivial cocycle compatible with the $N = 2$ super-conformal algebra. We will show below that this result holds for any N. However, it might seem reasonable, as for $N = 0$ and $N = 1$, and in the spirit of considering Kač-Moody algebras as ancestors of the super-Virasoro algebras, to associate for $N > 1$ superderivation algebras on extended super-circles with super-Kač-Moody algebras with non trivial cocycles.

We will start this section by a rapid survey on the Ademollo algebras, recalling the definitions and pointing out an interesting property which will be used in the study of the compatibility equation. Then in the light of this property, we present a construction of the $N = 2$ Ademollo algebra. Finally we suggest two approaches leading to the construction of super-derivation algebras based on super-Kač-Moody algebras and generalizing the $N = 1$ super-Virasoro algebra.

4.1. The Ademollo et al. algebras

4.1.1. Generalities

In the light of the RNS algebra, Ademollo et al. were able to combine in a consistent way conformal transformations with extended supersymmetric ones. These transformations can be reformulated as follows. In the N-extended superspace parametrized by the complex coordinates $(t, \theta^1, ..., \theta^N, \bar{t}, \bar{\theta}^1, ..., \bar{\theta}^N)$ with the metric δ_{ij} on the Grassmannian coordinates, we define as usual the covariant derivatives according to:

$$D_i = \theta^i \partial_t + \partial_i \quad \text{with} \quad \{D_i, D_j\} = 2\delta_{ij}\partial_t \quad \text{and} \quad \partial_i \equiv \frac{\partial}{\partial\theta^i} \tag{65}$$

and consider the super-analytic transformations

$$(t, \theta^1, ..., \theta^N) \rightarrow (\tilde{t}, \tilde{\theta}^1, ..., \tilde{\theta}^N) \tag{66}$$

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which satisfy for all $i, j = 1, ..., N$

$$\overline{D}_j \tilde{t} = \overline{D}_j \tilde{\theta}^i = 0 \quad \text{with} \quad \overline{D}_i = \vec{\theta}^i \partial_{\tilde{t}} + \partial_{\tilde{t}} \quad \text{and} \quad \partial_{\tilde{t}} \equiv \frac{\partial}{\partial \vec{\theta}^i} \tag{67}$$

(i.e. we restrict our attention to the $t, \theta^1, ..., \theta^N$ sector). Under such a tranformation, the (super)covariant derivatives transform as:

$$D_i \to \tilde{D}_i = (\tilde{D}_i \theta^j) D_j + (\tilde{D}_i z - \theta^j \tilde{D}_i \theta^j) \partial_t \tag{68}$$

Then the Ademollo et al. transformations can be seen as a super-analytic transformation satisfying

$$\tilde{D}_i z = \theta^j \tilde{D}_i \theta^j \tag{69}$$

i.e. the D_i's transform homogeneously as

$$D_i \to \tilde{D}_i = (\tilde{L}_i \partial^j) D_j \tag{70}$$

At fixed N, the Ademollo et al. algebra contains 2^N sets of generators, and among them can be recognized an $SO(N)$ Kač-Moody algebra. As examples, for $N = 0$, we recover the Virasoro algebra generated by the L_m $(m \in Z)$ and for $N = 1$, the RNS algebra generated by the L_m $(m \in Z)$ and the fermionic generators G_m, $(m \in Z$ or $m \in Z + \frac{1}{2})$, while for $N = 2$ an SO(2) Kač-Moody algebra shows up. We also note the existence at the order N of the finite simple superalgebra $OSp(N|2)$ generated in the Neveu-Schwarz sector by $L_{\pm 1,0}$ $G^i_{\pm 1/2}$ $(i = 1, ..., N)$ and the $T_0^{ij} = -T_0^{ji} = \{G^i_{1/2}, G^j_{-1/2}\}$. Denoting by $\mathcal{L} = u \partial_t + \xi^i D_i$ the generators associated to eq. (66), where $u = u(t, \theta^1, ..., \theta^N)$ and $\xi^i = \xi^i(t, \theta^1, ..., \theta^N)$ are super-functions, we deduce

$$[\mathcal{L}, D_i] = \delta D_i = \tilde{D}_i - D_i \tag{71}$$

Now equation (68) can be written as infinitesimal level as

$$[\mathcal{L}, D_i] = A_i^j D_j \tag{72}$$

and we obtain

$$\xi_i = \frac{1}{2} D_i u^* \quad \text{and} \quad [\mathcal{L}, D_i] = (D_i \xi^{j*}) D_j = -\frac{1}{2}(D_i D_j u) D_j \quad (i = 1, ..., N) \tag{73}$$

It is worthwhile to emphasize the relevance of the above equations, which appear as a set of *necessary and sufficient conditions for determining the Ademollo algebras.*

4.1.2. The Ademollo algebras as super-derivation algebras

We will determine in this section whether the N-Ademollo algebra may be viewed as a super-derivation algebra $Der_\Phi(A_N)$ compatible with some cocycle Φ. When looking

at the defining relations (73) of the Ademollo algebras and (43) of the super-derivation algebras, it is tempting to connect these two approaches. Let us write these two relations:

$$[\mathcal{L}, D_i] = -\frac{1}{2}(D_i D_j u) D_j \qquad (i = 1, ..., N) \qquad \text{Ademollo algebras}$$

with $\mathcal{L} = u\partial_t + \frac{1}{2}(D_i u^*)D_i$

$$[\mathcal{L}, \mathcal{D}] = -(div^g \mathcal{L})\mathcal{D} \qquad \text{superderivation algebras}$$

One could think a priori that the N-Ademollo algebra is compatible with N cocycles, each being a covariant derivative D_i. However, we will see that it is not the case. Actually, any cocycle $\mathcal{D}$ can be decomposed on the basis $\{\partial_t, D_1, ..., D_N\}$ as

$$\mathcal{D} = \alpha_0(t, \theta)\partial_t + \sum_{i=1}^{N} \alpha_i(t, \theta)D_i$$

with $\partial_t(\alpha_0 + \sum_{i=1}^{N} \alpha_i \theta^i) = 0$ and $\sum_{i=1}^{N} \partial_i \alpha_i^* = 0$ (cf. sect. 2.1.6).

The relation (43) leads then to two sets of equations, corresponding to the coefficient of ∂_t and D_i respectively:

$$u\partial_t \alpha_0 - (\partial_t u)\alpha_0 - \frac{1}{2}\sum_{k=1}^{N+1}(D_k u^*)(D_k \alpha_0) = -(1 - N/2)(\partial_t u)\alpha_0$$

$$u\partial_t \alpha_i - \frac{1}{2}(\partial_t D_i u^*)\alpha_0^* - \frac{1}{2}\sum_{k=1}^{N+1}\left((D_k u^*)(D_k \alpha_i) + (D_k D_i u)\alpha_k\right) = -(1 - N/2)(\partial_t u)\alpha_i$$

$$(74)$$

In order to have the N-Ademollo algebra as a $\mathcal{D}$-compatible algebra, these equations have to be satisfied for any super-function u. One is led to the following equations:

$$\alpha_0 = 0 \qquad \partial_t \alpha_i = D_k \alpha_i = 0 \qquad \alpha_k = (2 - N)\alpha_k \delta_{ik} \quad \forall i, k \qquad (75)$$

Thus, for the cocycle $\mathcal{D}$ to be non-zero, one must have $(2-N)\delta_{ik} = 1$ for all $i, k = 1, ..., N$. This clearly imposes $N = 1$.

Therefore, the N-Ademollo algebras are super-derivation algebras compatible with some cocycle $\mathcal{D}$ if and only if $N = 1$. In this case, the cocycle $\mathcal{D}$ is given by the covariant derivative D, which is just the case found in sect. 3.2.1. and in ref. [1].

4.1.3. The N=2 case

Let us illustrate the above property (73) in the $N = 2$ case. We set

$$\mathcal{L} = u\partial_t + \xi^1 D_1 + \xi^2 D_2 \qquad (76)$$

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and impose the conditions (72) for $i = 1, 2$. Then we deduce the following relations

$$\xi_i = \frac{1}{2}D_i u^* \qquad A_i^j = -\frac{1}{2}D_i D_j u \qquad (77)$$

Using the decomposition $u = u_0 + 2u_i\theta^i + 2u_{12}\theta^1\theta^2$, we find the generators of the $N = 2$ Ademollo algebra:

$$
\begin{aligned}
L(u_0) &= u_0\partial_t + \frac{1}{2}(\partial_t u_0)(\theta^1\partial_1 + \theta^2\partial_2) \\
G_1(u_1) &= u_1(\theta^1\partial_t - \partial_1) + (\partial_t u_1)\theta^1\theta^2\partial_2 \\
G_2(u_2) &= u_2(\theta^2\partial_t - \partial_2) - (\partial_t u_2)\theta^1\theta^2\partial_1 \\
T(u_{12}) &= u_{12}(\theta^2\partial_1 - \theta^1\partial_2)
\end{aligned} \qquad (78)
$$

The commutation relations between these generators can be written as

$$
\begin{aligned}
[L(u_0), L(v_0)] &= L(u_0\partial_t v_0 - v_0\partial_t u_0) & \qquad [L(u_0), T(v_{12})] &= T(u_0\partial_t v_{12}) \\
[L(u_0), G^i(v_i)] &= G^i(u_0\partial_t v_i - \frac{1}{2}v_i\partial_t u_0) & \qquad [T(u_{12}), G^i(v_i)] &= \epsilon^{ij}G^j(u_{12}v_i) \\
\{G^i(u_i), G^j(v_j)\} &= \delta^{ij}L(-2u_iv_j) + \epsilon^{ij}T(u_i\partial_t v_j - v_j\partial_t u_i)
\end{aligned}
$$

$$(79)$$

the Laurent expansion of which lead to

$$
\begin{aligned}
[L_m, L_n] &= (m - n)L_{m+n} & \qquad [T_m, G_r^i] &= \epsilon^{ij}G_{m+r}^j \\
[L_m, T_n] &= -nT_{m+n} & \qquad \{G_r^i, G_s^j\} &= 2\delta^{ij}L_{r+s} + (r - s)\epsilon^{ij}T_{r+s} \\
[L_m, G_r^i] &= (m/2 - r)G_{m+r}^i
\end{aligned} \qquad (80)
$$

ϵ^{ij} being the antisymmetric tensor of order 2. Here, one has $m, n \in Z$ and $r, s \in Z$ (Ramond sector) or $r, s \in Z + \frac{1}{2}$ (Neveu-Schwarz sector).

The $N = 2$ two-dimensional superspace is a reducible representation of supersymmetry, which can be split into two parts, using the notion of U(1)-chirality. Defining D and $\overline{D}$ as $D = \overline{\theta}\partial_t + 2\partial_\theta$ and $\overline{D} = \theta\partial_t + 2\partial_{\overline{\theta}}$, with $\theta = \theta^1 + i\theta^2$ and $\overline{\theta} = \theta^1 - i\theta^2$ (note that $\overline{\theta}$ has nothing to do with the $\overline{\theta}^i$'s defined in sect. 4.1.), a chiral (or antichiral) superfield $\Phi(t, \theta^1, \theta^2)$ satisfies $\overline{D}\Phi = 0$ (or $D\Phi = 0$).

Owing to equation (72) we will see that the $N = 2$ Ademollo algebra is naturally related to this chirality. The A matrix defined in equation (77) can be decomposed for any N as $A = -\frac{1}{2}(\partial_t u)1_N + \hat{A}$, with 1_N being the $N \times N$ identity matrix and $\hat{A}$ an antisymmetric matrix. For $N = 2$, we have

$$A = -\frac{1}{2}(\partial_t u)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}D_1 D_2 u \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (81)$$

which, after diagonalisation, leads to

$$
\begin{aligned}
[\mathcal{L}, D] &= -\frac{1}{2}(\partial_t u - iD_1 D_2 u)D = -\frac{1}{4}(D\overline{D}u)D \\
[\mathcal{L}, \overline{D}] &= -\frac{1}{2}(\partial_t u + iD_1 D_2 u)\overline{D} = -\frac{1}{4}(\overline{D}Du)\overline{D}
\end{aligned} \qquad (82)
$$

One easily deduces that (anti)chirality is preserved under the action of an $\mathcal{L}$ generator:

$$D\Phi = 0 \implies \mathcal{L}D\Phi = 0 \quad\text{and}\quad \overline{D}\Phi = 0 \implies \mathcal{L}\overline{D}\Phi = 0 \tag{83}$$

Note that this last property can be viewed as a *definition* of the $N = 2$ Ademollo algebra.

We already mentioned in sect. 3.2. that there is no cocycle of a super-Kač-Moody algebra associated with a derivation algebra containing the $N = 2$ Ademollo algebra. Even more, one can check that there is no chiral (antichiral) super-Kač-Moody algebra with a non trivial cocycle whose derivation algebra contains the above algebra.

4.2. New kinds of super-conformal algebras ?

As already mentionned in this paper, the Virasoro and the RNS algebras which are of particular physical interest are naturally related to (super)Kač-Moody algebras. At this point, it is tempting to try new kinds of $N > 1$ super-conformal algebras which would be related to super-Kač-Moody ones. Hereafter, we will adopt two possible attitudes. First we will look for N super-derivation algebras containing as a subalgebra the $(N - 1)$ Ademollo algebra. Then in a completely different framework, we will propose a recurrent class of cocycles $\mathcal{D}_N$, with $\mathcal{D}_1$ being associated with the RNS algebra and $\mathcal{D}_{N+1}$ easily deduced from $\mathcal{D}_N$, and such that the corresponding derivation algebras are included into one another: $Der_{\mathcal{D}_N}(A_N) \subset Der_{\mathcal{D}_{N+1}}(A_{N+1})$.

4.2.1. Embedding of the N-Ademollo algebra in $Der(A_{N+1})$

Considering a super-loop algebra on A_{N+1}, we determine the possible cocycles $\omega_{\mathcal{D}}$ such that the $\mathcal{D}$-compatible super-derivation algebra contains the N-Ademollo algebra. The calculation is quite similar (although a little more tedious) to the previous one done in sect 4.1.2. Up to a rotation in the space of the θ variables, one can assume that the N-Ademollo algebra is described in terms of the N first variables $\theta^1, ..., \theta^N$. One can write therefore $\mathcal{L}$ as $\mathcal{L} = w\partial_t + \sum_{i=1}^{N+1} \Psi_i D_i$ with

$$w = u + \theta^{N+1}v \quad\text{and}\quad \Psi_a = \frac{1}{2}D_a u^* + \theta^{N+1}\eta_a$$

where $u, v \in A_N$, $\eta_a \in C(A_N)$ with $a = 1, ...N$, and $\Psi_{N+1} \in C(A_{N+1})$.
Computing $[\mathcal{L}, \mathcal{D}] = -(div^s\mathcal{L})\mathcal{D}$, one obtains

$$w\partial_t\alpha_0 - (\partial_t w)\alpha_0 + \sum_{k=1}^{N+1}\left(\Psi_k D_k\alpha_0 - (D_k w^*)\alpha_k^* + 2\Psi_k\alpha_k^*\right) = -\left(\partial_t w + \sum_{k=1}^{N+1}D_k\Psi_k^*\right)\alpha_0$$

$$w\partial_t\alpha_i - (\partial_t\Psi_i)\alpha_0^* + \sum_{k=1}^{N+1}\left(\Psi_k D_k\alpha_i + (D_k\Psi_i^*)\alpha_k\right) = -\left(\partial_t w + \sum_{k=1}^{N+1}D_k\Psi_k^*\right)\alpha_i$$

$$\tag{84}$$

Demanding these equations to be satisfied for any super-function u (in order to have the N-Ademollo algebra), we obtain the following relations (with $a = 1, ..., N$ and $j = 1, ..., N+1$):

$$\partial_t \alpha_0 = 0 \qquad D_a \alpha_0 = 0 \qquad \theta^{N+1} \alpha_{N+1} = -\frac{N}{2} \alpha_0$$
$$\partial_t \alpha_j = 0 \qquad D_a \alpha_j = 0 \qquad \alpha_{N+1}^* = \alpha_{N+1} \tag{85}$$
$$(N-2)\alpha_{N+1} = 0 \qquad \alpha_a = 0$$

(the equation $\alpha_a = 0$ does not exist for $N = 1$).

These equations lead to the following results. *The N-Ademollo algebra is embedded in a super-derivation algebra $Der(A_{N+1})$ compatible with some non trivial cocycle $\mathcal{D}$ if and only if $N = 1$ or $N = 2$.*

In the case of $N = 1$, the cocycle is given by $\mathcal{D} = (c_0 + c_2 \theta^2)(\theta^1 \partial_t + \partial_1) = (c_0 + c_2 \theta^2)D_1$ where c_0 and c_2 are complex constants. The (non)vanishing of the coefficients c_0 and c_2 leads to three different super-derivation algebras.

In the case of $N = 2$, there is only one solution given by the cocycle $\mathcal{D} = c_3 \partial_3$ where c_3 is a complex constant.

These algebras are shown in the appendix.

4.2.2. A very particular chain of cocycles and their corresponding algebras

In this paragraph, we look for a chain of super-derivation algebras, each of them being compatible with a non trivial cocycle, the Nth order algebra being included into the $(N + 1)$th order one, and such that for $N = 1$, one recovers the RNS algebra.

When one looks at the super-derivation algebras $Der_\Phi(A_2)$ containing the RNS algebra, two one-cocycles are pointed out: $\mathcal{D}_2 = D_1$ and $\mathcal{D}_2' = \theta^2 D_1$. The generalization of these cocycles is very easy: $\mathcal{D}_N = D_1$ and $\mathcal{D}_N' = \theta^N \dots \theta^2 D_1$. We will study the two chains of super-derivation algebras, compatible with these two chains of cocycles.

• Super-derivation algebra $Der_{D_1}(A_N)$

The super-derivation $\mathcal{L} = w\partial_t + \sum_{i=1}^{N} \Psi_i D_i$ satisfies the compatiblity equation $[\mathcal{L}, D_1] = -(div^g \mathcal{L})D_1$. One has

$$[\mathcal{L}, D_1] = (2\Psi_1 - D_1 w^*)\partial_t - \sum_{i=1}^{N}(D_1 \Psi_i^*)D_i \quad \text{and} \quad div^g \mathcal{L} = \partial_t w + \sum_{i=1}^{N} D_i \Psi_i^*$$

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Therefore one obtains

$$2\Psi_1 - D_1 w^* = 0 \quad \text{and} \quad D_1 \Psi_k^* = \delta_{1k}(\partial_t w - \sum_{i=1}^{N} D_i \Psi_i^*) \quad (k = 1, ..., N) \qquad (86)$$

Thus the super-derivation $\mathcal{L}$ has to satisfy

$$
\begin{aligned}
&2\Psi_1 - D_1 w^* = 0 \\
&D_1 \Psi_n^* = 0 \qquad (n > 1) \\
&\sum_{n=2}^{N} D_n \Psi_n^* = 0
\end{aligned}
\qquad (87)
$$

Notice that the second equation implies that $\partial_t \Psi_n = 0$ (for $n \neq 1$) i.e. the functions Ψ_n are constant in t for $n \neq 1$. We will call these generators "discrete generators". They commute with the Virasoro generators $L_n = -t^{n+1} - \dfrac{n+1}{2} t^n \theta^1 \partial_1$ obtained for $w = -t^{n+1}$. Altogether this algebra contains $N 2^{N-1} + 1$ generators among which $(N-2)2^{N-1} + 1$ "discrete" ones.

- Super-derivation algebra $Der_{\mathcal{D}'_N}(A_N)$

A similar computation for the compatibility equation leads to the equations

$$
\begin{aligned}
&(2\Psi_1 - D_1 w^*)\theta^N \ldots \theta^2 = 0 \\
&D_1(\Psi_n^* \theta^N \ldots \theta^2) = 0 \qquad (n > 1) \\
&\sum_{n=2}^{N} D_n(\Psi_n^* \theta^N \ldots \theta^2) = 0
\end{aligned}
\qquad (88)
$$

The last two equations are equivalent to the relations $\Psi_n^0 = \Psi_n^1 = 0$ (with $n > 1$) where Ψ_n is decomposed as $\Psi_n(t,\theta) = \Psi_n^0(t) + \sum_{k=1}^{N} \Psi_n^k(t)\theta^k + \ldots$.

This algebra contains $2^N + 2N(2^{N-1} - 1)$ generators. If we don't worry about the discrete generators of $Der_{D_1}(A_N)$, we see that the defining relation $2\Psi_1 - D_1 w = 0$ for $Der_{D_1}(A_N)$ is stronger than the one of $Der_{\mathcal{D}'_N}(A_N)$. So we conclude that, up to the discrete generators, one has the inclusion

$$Der_{D_1}(A_N) \subset Der_{\mathcal{D}'_N}(A_N)$$

From the relations (73) and (74), one also has the embeddings (including the discrete generators)

$$Der_{D_1}(A_N) \subset Der_{D_1}(A_{N+1})$$

$$Der_{\mathcal{D}'_N}(A_N) \subset Der_{\mathcal{D}'_{N+1}}(A_{N+1})$$

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Notice that the algebras $Der_{D_1}(A_2)$ and $Der_{\mathcal{D}'_2}(A_2)$ have been given in sect. 4.2.1. (for the commutation relations of these algebras, see the appendix).

5. Conclusion

In this paper, a classification of the possible central extensions for super-loop algebras defined on the super-circle A_N, where N is the number of generators of the corresponding Grassmann algebra, has been achieved. A general method for determining the super-derivation algebra compatible with each super-Kač-Moody algebra has also been obtained, and explicited in the cases $N = 1, 2, 3$.

Suprisingly enough, the usual super-conformal algebras for $N > 1$ cannot be associated to super-Kač-Moody companions, contrarily to the case $N = 0$ (Virasoro algebra) and $N = 1$ (RNS algebra). Such a result led us to investigate new types of extended conformal algebras for $N > 1$.

Finally, let us note that a semi-direct sum of a super affine algebra by its corresponding super-derivation algebra might be exploited in the construction of two dimensional rational conformal field theories. In particular new kinds of Wess-Zumino-Witten models [13] might be considered, generalizing the $N = 1$ case treated in ref. [14].

Acknowledgment

Appendix

In this appendix, we will explicit the commutation relations of some remarkable super-derivation algebras used in sect. 4.2.

We recall that if K is a loop-algebra with generators T_m^a ($m \in \mathbf{Z}$), constructed from a simple Lie algebra $\mathcal{K}$ with generators T^a ($a = 1, ..., \dim \mathcal{K}$), one can associate a super-loop algebra K_N with the generators T_m^a, $T_m^a \theta^i$, $T_m^a \theta^i \theta^j$, ..., $T_m^a \theta^1 ... \theta^N$. In the case $N = 2$, the non trivial commutation relations of the super-loop algebra take the explicit following form:

$$[T_m^a, T_n^b] = if^{abc}T_{m+n}^c \qquad\qquad [T_m^a, T_n^b\theta^i] = if^{abc}T_{m+n}^c\theta^i$$
$$[T_m^a, T_n^b\theta^1\theta^2] = if^{abc}T_{m+n}^c\theta^1\theta^2 \qquad \{T_m^a\theta^1, T_n^b\theta^2\} = if^{abc}T_{m+n}^c\theta^1\theta^2 \qquad (A.1)$$

For a super-derivation algebra, we will take for each independent generator $\phi(t)\partial_I$ where ∂_I is a linear combination of ∂_t and ∂_{θ_i}, a basis of function $\phi_m = t^{m+d_\phi-1}\partial_I$ where the conformal weight d_ϕ of the generator $\phi(t)\partial_I$ is defined through the action of the Virasoro generator L_m on it, i.e. $[L_n, \phi_m] = (n(d_\phi - 1) - m)\phi_{n+m}$.

- Super-derivation algebra $Der_{\mathcal{D}}(A_2)$ with $\mathcal{D} = a_2\theta^2(\theta^1\partial_t + \partial_1)$

The super-Kač-Moody algebra obtained from this one-cocycle has the following non trivial commutation relations:

$$[T_m^a, T_n^b] = if^{abc}T_{m+n}^c - a_2m\delta_{m+n}\delta^{ab} \qquad \{T_m^a\theta^1, T_n^b\theta^1\} = a_2\delta_{m+n}\delta^{ab}$$
$$[T_m^a, T_n^b\theta^1\theta^2] = if^{abc}T_{m+n}^c\theta^1\theta^2 \qquad\qquad \{T_m^a\theta^1, T_n^b\theta^2\} = if^{abc}T_{m+n}^c\theta^1\theta^2 \qquad (A.2)$$
$$[T_m^a, T_n^b\theta^i] = if^{abc}T_{m+n}^c\theta^i$$

The super-derivations compatible with $\mathcal{D}$ are (after a redefinition of some generators)

$$\mathcal{L}_{even} = A\partial_t + \frac{1}{2}(\partial A)(\theta^1\partial_1 + \theta^2\partial_2) + T\theta^2\partial_1 + W\theta^2\partial_2 + B\theta^1\theta^2\partial_t$$
$$\mathcal{L}_{odd} = G^1(\theta^1\partial_t - \partial_1) + (\partial G^1)\theta^1\theta^2\partial_2 + G^2\theta^2\partial_t - (\partial G^2)\theta^1\theta^2\partial_1 + D^1\theta^1\theta^2\partial_1 + D^2\theta^1\theta^2\partial_2$$
$$(A.3)$$

and a basis of generators is given by

$$L_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n(\theta^1\partial_1 + \theta^2\partial_2)$$
$$T_n = t^n\theta^2\partial_1 \quad \text{and} \quad W_n = t^n\theta^2\partial_2$$
$$B_n = t^n\theta^1\theta^2\partial_t$$
$$G_n^1 = t^{n+1/2}(\theta^1\partial_t - \partial_1) + (n + \frac{1}{2})t^{n-1/2}\theta^1\theta^2\partial_2 \qquad (A.4)$$
$$G_n^2 = t^{n+1/2}\theta^2\partial_t - (n + \frac{1}{2})t^{n-1/2}\theta^1\theta^2\partial_1$$
$$D_n^1 = t^{n-1/2}\theta^1\theta^2\partial_1 \quad \text{and} \quad D_n^2 = t^{n-1/2}\theta^1\theta^2\partial_2$$

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The conformal weights are given by $d_\phi = \frac{1}{2}$ for D_m^1 and D_m^2, $d_\phi = 1$ for T_m, W_m, B_m, $d_\phi = \frac{3}{2}$ for G_m^1 and G_m^2, and of course $d_\phi = 2$ for L_m.

The non trivial commutation relations between the generators are

$$
\begin{aligned}
&[W_n, T_m] = T_{n+m} && [W_n, D_m^1] = D_{n+m}^1 \\
&[W_n, G_m^1] = -nD_{n+m}^2 && [W_n, G_m^2] = G_{n+m}^2 + nD_{n+m}^1 \\
&[W_n, B_m] = B_{n+m} && [B_n, G_m^1] = G_{n+m}^2 + nD_{n+m}^1 \\
&[T_n, D_m^2] = -D_{n+m}^1 && [T_n, G_m^1] = G_{n+m}^2 \\
&\{G_n^1, G_m^1\} = 2L_{n+m} && \{G_n^1, G_m^2\} = (m-n)T_{n+m} \\
&\{G_n^1, D_m^1\} = -T_{n+m} + B_{n+m} && \{G_n^1, D_m^2\} = -W_{n+m} \\
&\{G_n^2, D_m^2\} = B_{n+m}
\end{aligned}
\tag{A.5}
$$

Note that the generators L_n, G_n^i and T_n form a subalgebra of $Der_\mathcal{D}(A_2)$ which is a contraction of the $N = 2$ Ademollo algebra. If we start from the algebra (80) and define the generators $\hat{L}_n = L_n$, $\hat{G}_n^1 = G_n^1$, $\hat{G}_n^2 = \epsilon G_n^2$, $\hat{T}_n = \epsilon T_n$, and compute the commutation relations between these generators at the limit $\epsilon \to 0$, one finds

$$
\begin{aligned}
&[\hat{L}_n, \hat{T}_m] = -m\hat{T}_{n+m} \quad \text{and} \quad [\hat{L}_n, \hat{G}_m^i] = (n/2 - m)\hat{G}_{n+m}^i \\
&[\hat{T}_n, \hat{G}_m^1] = \hat{G}_{n+m}^2 \quad \text{and} \quad [\hat{T}_n, \hat{G}_m^2] = 0 \\
&\{\hat{G}_n^1, \hat{G}_m^1\} = 2L_{n+m} \quad \text{and} \quad \{\hat{G}_n^2, \hat{G}_m^2\} = 0 \\
&\{\hat{G}_n^1, \hat{G}_m^2\} = (n - m)\hat{T}_{n+m}
\end{aligned}
\tag{A.6}
$$

which has to be compared with the equations $(A.5)$.

• Super-derivation algebra $Der_\mathcal{D}(A_2)$ with $\mathcal{D} = a_0(\theta^1 \partial_t + \partial_1)$

The super-Kač-Moody algebra obtained from this one-cocycle has the following non trivial commutation relations:

$$
\begin{aligned}
&[T_m^a, T_n^b] = if^{abc}T_{m+n}^c && [T_m^a \theta^1 \theta^2, T_n^b \theta^1] = a_0 \delta_{m+n} \delta^{ab} \\
&[T_m^a, T_n^b \theta^1 \theta^2] = if^{abc}T_{m+n}^c \theta^1 \theta^2 && \{T_m^a \theta^1, T_n^b \theta^2\} = if^{abc}T_{m+n}^c \theta^1 \theta^2 \\
&[T_m^a, T_n^b \theta^1] = if^{abc}T_{m+n}^c \theta^1 && \\
&[T_m^a, T_n^b \partial^2] = if^{abc}T_{m+n}^c \theta^2 + a_0 m \delta_{m+n} \delta^{ab} &&
\end{aligned}
\tag{A.7}
$$

The super-derivations compatible with $\mathcal{D}$ are

$$
\begin{aligned}
\mathcal{L}_{even} &= A\partial_t + \frac{1}{2}(\partial A)\theta^1 \partial_1 - T\theta^2(\theta^1 \partial_t - \partial_1) \\
\mathcal{L}_{odd} &= G^1(\theta^1 \partial_t - \partial_1) + G^2 \theta^2 \partial_t - \frac{1}{2}(\partial G^2)\theta^1 \theta^2 \partial_1 + k\partial_2
\end{aligned}
\tag{A.8}
$$

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A basis of generators is given by

$$L_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n\theta^1\partial_1$$

$$T_n = -t^{n+1/2}\theta^2(\theta^1\partial_t - \partial_1)$$

$$G_n^1 = t^{n+1/2}(\theta^1\partial_t - \partial_1) \quad \text{and} \quad G_n^2 = t^{n+1}\theta^2\partial_t - \frac{n+1}{2}t^n\theta^1\theta^2\partial_1$$

$$c = \partial_2$$

$$(A.9)$$

The conformal weights are $d_\phi = \frac{3}{2}$ for G_m^1 and T_m, $d_\phi = 2$ for G_m^2.
The non trivial commutation relations are given by

$$[T_n, G_m^1] = 2G_{n+m}^2$$

$$\{G_n^1, G_m^1\} = 2L_{n+m} \qquad \{G_n^1, G_m^2\} = (\frac{m}{2} - n)T_{n+m} \qquad (A.10)$$

$$[T_n, c] = -G_n^1 \qquad \{G_n^2, c\} = -L_n$$

Notice that the generators T_n and G_n^2 do not obey to a "conformal spin-statistic theorem", i.e. T_n (resp. G_n^2) is even (bosonic) with a half-integer conformal spin (resp. odd (fermionic) with an integer conformal spin). In this respect, they behave like conformal ghost fields, and the c generator plays the role of the BRST charge.

- Super-derivation algebra $Der_{\mathcal{D}}(A_2)$ with $\mathcal{D} = a_2\theta^2(\theta^1\partial_t + \partial_1) + a_0(\theta^1\partial_t + \partial_1)$

The super-Kač-Moody algebra obtained from this one-cocycle has the following non trivial commutation relations:

$$[T_m^a, T_n^b] = if^{abc}T_{m+n}^c - a_2m\delta_{m+n}\delta^{ab} \qquad [T_m^a\theta^1\theta^2, T_n^b\theta^1] = a_0\delta_{m+n}\delta^{ab}$$

$$[T_m^a, T_n^b\theta^1\theta^2] = if^{abc}T_{m+n}^c\theta^1\theta^2 \qquad \{T_m^a\theta^1, T_n^b\theta^1\} = a_2\delta_{m+n}\delta^{ab}$$

$$[T_m^a, T_n^b\theta^1] = if^{abc}T_{m+n}^c\theta^1 \qquad \{T_m^a\theta^1, T_n^b\theta^2\} = if^{abc}T_{m+n}^c\theta^1\theta^2$$

$$[T_m^a, T_n^b\theta^2] = if^{abc}T_{m+n}^c\theta^2 + a_0m\delta_{m+n}\delta^{ab}$$

$$(A.11)$$

The super-derivations are

$$\mathcal{L}_{even} = A\partial_t + \frac{1}{2}(\partial A)\theta^1\partial_1 - T\theta^2(\theta^1\partial_t - \partial_1)$$

$$\mathcal{L}_{odd} = G^1(\theta^1\partial_t - \partial_1) + G^2\theta^2\partial_t - \frac{1}{2}(\partial G^2)\theta^1\theta^2\partial_1$$

$$(A.12)$$

This algebra is a subalgebra of the previous one, where the discrete generator c is set to zero.

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- Super-derivation algebra $Der_{\mathcal{D}}(A_3)$ with $\mathcal{D} = a_3 \partial_3$

The super-Kač-Moody algebra obtained from this one-cocycle has the following non trivial commutation relations:

$$[T^a_m, T^b_n] = i f^{abc} T^c_{m+n}$$
$$[T^a_m, T^b_n \theta^i \theta^j] = i f^{abc} T^c_{m+n} \theta^i \theta^j$$
$$[T^a_m, T^b_n \theta^i] = i f^{abc} T^c_{m+n} \theta^i$$
$$[T^a_m, T^b_n \theta^1 \theta^2 \theta^3] = i f^{abc} T^c_{m+n} \theta^1 \theta^2 \theta^3$$
$$\{T^a_m \theta^i, T^b_n \theta^j\} = i f^{abc} T^c_{m+n} \theta^i \theta^j$$
$$\{T^a_m \theta^3, T^b_n \theta^1 \theta^2 \theta^3\} = -a_3 \delta_{m+n} \delta^{ab}$$

$$[T^a_m \theta^1 \theta^2, T^b_n \theta^3] = i f^{abc} T^c_{m+n} \theta^1 \theta^2 \theta^3$$
$$[T^a_m \theta^2 \theta^3, T^b_n \theta^1] = i f^{abc} T^c_{m+n} \theta^1 \theta^2 \theta^3$$
$$[T^a_m \theta^3 \theta^1, T^b_n \theta^2] = i f^{abc} T^c_{m+n} \theta^1 \theta^2 \theta^3$$
$$[T^a_m \theta^3 \theta^1, T^b_n \theta^2 \theta^3] = a_3 \delta_{m+n} \delta^{ab}$$

$$(A.13)$$

The super-derivations are

$$
\begin{aligned}
\mathcal{L}_{even} &= A \partial_t + \frac{1}{2}(\partial A)(\theta^1 \partial_1 + \theta^2 \partial_2) + B(\theta^3 \partial_3 - \theta^1 \partial_1 - \theta^2 \partial_2) \\
&\quad + H^1 \theta^1 \partial_3 + H^2 \theta^2 \partial_3 + U(\theta^1 \partial_1 - \theta^2 \partial_2) + T(\theta^1 \partial_2 - \theta^2 \partial_1) \\
&\quad + S(\theta^1 \partial_2 + \theta^2 \partial_1) + K \theta^1 \theta^2 \partial_t + \frac{1}{2} \partial K \theta^1 \theta^2 \theta^3 \partial_3 \\
\mathcal{L}_{odd} &= G^1(\theta^1 \partial_t - \partial_1) + (\partial G^1) \theta^1 \theta^2 \partial_2 + G^2(\theta^2 \partial_t - \partial_2) - (\partial G^2) \theta^1 \theta^2 \partial_1 \\
&\quad + W^1(\theta^2 \theta^3 \partial_3 + 2\theta^1 \theta^2 \partial_1) + W^2(\theta^1 \theta^3 \partial_3 - 2\theta^1 \theta^2 \partial_2) + D \theta^1 \theta^2 \partial_3 \\
&\quad + E^1 \partial_1 + E^2 \partial_2 + E^3 \partial_3
\end{aligned}
$$

$$(A.14)$$

A basis of generators is given by

$$
\begin{aligned}
L_n &= -t^{n+1} \partial_t - \frac{n+1}{2}(\theta_1 \partial_1 + \theta_2 \partial_2) \\
B_n &= t^n(\theta^3 \partial_3 - \theta^1 \partial_1 - \theta^2 \partial_2) \\
H^1_n &= t^{n-1/2} \theta^1 \partial_3 \quad \text{and} \quad H^2_n = t^{n-1/2} \theta^2 \partial_3 \\
U_n &= t^n(\theta^1 \partial_1 - \theta^2 \partial_2) \qquad T_n = t^n(\theta^1 \partial_2 - \theta^2 \partial_1) \qquad S_n = t^n(\theta^1 \partial_2 + \theta^2 \partial_1) \\
K_n &= t^n \theta^1 \theta^2 \partial_t + \frac{n}{2} t^{n-1} \theta^1 \theta^2 \theta^3 \partial_3 \\
G^1_n &= t^{n+1/2}(\theta^1 \partial_t - \partial_1) + (n + \frac{1}{2}) t^{n-1/2} \theta^1 \theta^2 \partial_2 \\
G^2_n &= t^{n+1/2}(\theta^2 \partial_t - \partial_2) - (n + \frac{1}{2}) t^{n-1/2} \theta^1 \theta^2 \partial_1 \\
W^1_n &= t^{n-1/2}(\theta^2 \theta^3 \partial_3 + 2\theta^1 \theta^2 \partial_1) \quad \text{and} \quad W^2_n = t^{n-1/2}(\theta^1 \theta^3 \partial_3 - 2\theta^1 \theta^2 \partial_2) \\
D_n &= t^{n-1} \theta^1 \theta^2 \partial_3 \\
E^1_n &= t^{n+1/2} \partial_1 \qquad E^2_n = t^{n+1/2} \partial_2 \qquad E^3_n = t^n \partial_3
\end{aligned}
$$

$$(A.15)$$

The conformal weights are $d_\phi = 0$ for D_m, $d_\phi = \frac{1}{2}$ for W^1_m, W^2_m, H^1_m and H^2_m, $d_\phi = 1$ for T_m, U_m, S_m, B_m, K_m, E_3, $d_\phi = \frac{3}{2}$ for G^1_m, G^2_m, E^1_m, E^2_m.

The non trivial commutation relations between the generators are

$[T_n, S_m] = 2U_{n+m}$

$[U_n, S_m] = 2T_{n+m}$

$[T_n, H_m^1] = -H_{n+m}^2$

$[S_n, H_m^1] = H_{n+m}^2$

$[U_n, H_m^1] = H_{n+m}^1$

$[B_n, H_m^1] = -2H_{n+m}^1$

$[T_n, U_m] = -2S_{n+m}$

$[B_n, K_m] = -2K_{n+m}$

$[T_n, H_m^2] = H_{n+m}^1$

$[S_n, H_m^2] = H_{n+m}^1$

$[U_n, H_m^2] = -H_{n+m}^2$

$[B_n, H_m^2] = -2H_{n+m}^2$

$[T_n, G_m^1] = G_{n+m}^2$

$[T_n, W_m^1] = W_{n+m}^2$

$[T_n, E_m^1] = -E_{n+m}^2$

$[S_n, G_m^1] = G_{n+m}^2$

$[S_n, W_m^1] = W_{n+m}^2$

$[S_n, E_m^1] = -E_{n+m}^2$

$[U_n, G_m^1] = G_{n+m}^1 + 2E_{n+m}^1$

$[U_n, W_m^1] = -W_{n+m}^1$

$[U_n, E_m^1] = -E_{n+m}^1$

$[B_n, G_m^1] = -G_{n+m}^1 - nW_{n+m}^2$

$[B_n, W_m^1] = -W_{n+m}^1$

$[B_n, E_m^1] = E_{n+m}^1$

$[B_n, E_m^3] = -E_{n+m}^3$

$[K_n, G_m^1] = -[K_n, E_m^1] = G_{n+m}^2 + E_{n+m}^2 + \frac{n}{2}W_{n+m}^1$

$[K_n, G_m^2] = -[K_n, E_m^2] = -G_{n+m}^1 - E_{n+m}^1 - \frac{n}{2}W_{n+m}^2$

$[H_n^i, G_m^j] = \epsilon_{ij}(n+m)D_{n+m} + \delta_{ij}E_{n+m}^3$

$[H_n^1, W_m^1] = -3D_{n+m}$

$[H_n^1, E_m^1] = [H_n^2, E_m^2] = -E_{n+m}^3$

$[T_n, G_m^2] = -G_{n+m}^1$

$[T_n, W_m^2] = -W_{n+m}^1$

$[T_n, E_m^2] = E_{n+m}^1$

$[S_n, G_m^2] = G_{n+m}^1$

$[S_n, W_m^2] = W_{n+m}^1$

$[S_n, E_m^2] = -E_{n+m}^1$

$[U_n, G_m^2] = -G_{n+m}^2 - 2E_{n+m}^2$

$[U_n, W_m^2] = W_{n+m}^2$

$[U_n, E_m^2] = E_{n+m}^2$

$[B_n, G_m^2] = -G_{n+m}^2 - nW_{n+m}^1$

$[B_n, W_m^2] = -W_{n+m}^2$

$[B_n, E_m^2] = E_{n+m}^2$

$[B_n, D_m] = -3D_{n+m}$

$[H_n^2, W_m^2] = 3D_{n+m}$

$[K_n, E_m^3] = (m - \frac{n}{2})D_{n+m}$

$\{G_n^1, G_m^1\} = \{G_n^2, G_m^2\} = 2L_{n+m}$

$\{G_n^1, E_m^1\} = -L_{n+m} - \frac{n-m}{2}U_{n+m}$

$\{G_n^1, E_m^2\} = -\frac{n+m}{2}T_{n+m} - \frac{n-m}{2}S_{n+m}$

$\{G_n^1, E_m^3\} = mH_{n+m}^1$

$\{G_n^1, D_m\} = H_{n+m}^2$

$\{G_n^1, G_m^2\} = (n-m)T_{n+m}$

$\{G_n^2, E_m^1\} = \frac{n+m}{2}T_{n+m} - \frac{n-m}{2}S_{n+m}$

$\{G_n^2, E_m^2\} = -L_{n+m} + \frac{n-m}{2}U_{n+m}$

$\{G_n^2, E_m^3\} = mH_{n+m}^2$

$\{G_n^2, D_m\} = -H_{n+m}^1$

$$\{G_n^1, W_m^1\} = T_{n+m} - S_{n+m} + 2K_{n+m} \quad \{G_n^2, W_m^1\} = U_{n+m} - B_{n+m}$$

$$\{G_n^1, W_m^2\} = -U_{n+m} - B_{n+m} \qquad \{G_n^2, W_m^2\} = -T_{n+m} - S_{n+m} - 2K_{n+m}$$

$$\{E_n^1, W_m^1\} = -T_{n+m} + S_{n+m} \qquad \{E_n^2, W_m^1\} = U_{n+m} + B_{n+m}$$

$$\{E_n^1, W_m^2\} = -U_{n+m} + B_{n+m} \qquad \{E_n^2, W_m^2\} = T_{n+m} + S_{n+m} \qquad (A.16)$$

$$\{E_n^3, W_m^1\} = -H_{n+m}^2 \qquad \{E_n^3, W_m^2\} = -H_{n+m}^1$$

$$\{E_n^1, D_m\} = H_{n+m}^2 \qquad \{E_n^2, D_m\} = -H_{n+m}^1$$

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