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Abstract

The Wiener-Hermite Functional (WHF) method has been applied to the point reactor kinetic equation excited by Gaussian random reactivity noise under stationary conditions. Delayed neutrons and any feedback effects are disregarded. The neutron steady-state value and the power spectral density (PSD) of the neutron flux have been calculated in a second order (WHF-2) approximation. Two cases are considered : In the first case, the noise source is assumed to be white, while in the second case the source is low-pass white noise. In both cases the WHF-2 approximation of the neutron PSDs leads to relatively simple analytical expressions. The accuracy of the approach is determined by comparison with exact solutions of the problem. The investigations show that the WHF method is a powerful approximative tool for studying the nonlinear effects in the stochastic differential equation.

Keywords : Noise analysis, neutron noise, point reactor kinetics, Wiener-Hermite functional method.

I. Introduction

The modelling of random noise - in general being non-stationary and non-Gaussian - by a generalized Volterra series with a complete orthogonal set of statistical functionals (Wiener-Hermite functionals (WHFs)) is based on works by Cameron and Martin (1947) and by Wiener (1958). We will refer to this method of representing random noise as the WHF method. The WHF method is interesting for treating nonlinear effects in stochastic differential equations and for computing noise signature functions of the response for a given noise signature input. Since the expansion must ordinarily be truncated after a few terms, the results will be approximative.

Further basic considerations to the WHF method are given by Barrett (1963, 1964) and by Imamura et al. (1965). The method is also mentioned in the modern text book about random noise analysis by Priestley (1981), in the chapter dealing with poly-spectra. With regard to practical applications, there is at first a number of papers concerning various kinds of turbulence problems (Meecham and Siegel, 1964; Siegel et al., 1965; Meecham, 1965; Meecham and Jeng, 1968; Saffman, 1968 and 1969; Meecham, 1974; Meecham et al., 1975; Kerman, 1977; Hasegawa et al., 1978; Doi and Imamura, 1979; Ahmadi, 1982). Goto and Naka (1976) consider the Wiener-Hermite expansion in the Feynman path integral method appearing in particle physics. The WHF method is employed by Ahmadi (1980) to represent random ground acceleration during an earthquake. In a paper by Jahedi and Ahmadi (1983), and in an extension to this work by Orabi and Ahmadi (1987), the non-stationary random vibration of a simplified Duffing oscillator is treated by this method. Zarzycki (1986) considers nonlinear orthogonal filters which can be used for optimum prediction/modelling of stochastic sequences yielding better estimation accuracy for non-Gaussian sequences than in the linear case. Meecham and Lin (1987) apply the method to the scattering problem of waves by random rough surfaces. Surprisingly, we did not find any application in the field of reactor noise analysis.

There is the well-known problem of closure if the reactor neutron kinetics are driven by parametrically acting noise sources. The aim of this paper is to give a contribution to the power spectral density (PSD) calculation of the neutron flux under stationary conditions by applying the WHF method, up to the second order in the expansion (WHF-2 approximation). The treatment has been kept as simple as possible by restricting the neutronic model to the point reactor kinetic equation, neglecting the delayed neutrons and any feedback effects. The driving noise source is the reactivity term, which is assumed to be a stationary Gaussian random process. Two cases are considered : In the first case, the noise source is assumed to be white, while in the second case the source is low-pass white noise. In both cases the WHF-2 approximation of the neutron PSDs leads to relatively simple analytical expressions. The accuracy of the approach is determined by comparison with exact solutions. In the second case, the exact solution had to be derived.

In Section II, the model assumption and the stability criterion are summarized. Section III gives a short description of the WHF method. The application of the method to our model in the WHF-2 approximation is given in Section IV. It leads in the time domain to a system of coupled integro-differential equations which can be reduced to a solvable system of coupled integral equations in the frequency domain. In Section V, the case with white noise input is treated, while in Section VI the more tedious case of low-pass white noise input is considered.

The conclusions of the investigations are summarized in Section VII.

II. Model Assumptions and Stability Criterion

Consider the first-order stochastic differential equation :

$$\dot{X}(t) = F(t)X(t) + S_o \qquad (II.1)$$

X(t) is the neutron flux (or the reactor power). F(t) is the driving reactivity term. Its fluctuations are assumed to be stationary Gaussian random noise (which is, of course, ergodic). S_o is a time-independent constant source term. We ask for a stationary (ergodic) solution of X(t) and decompose X(t) and F(t) into the steady-state values X_o and F_o, and the fluctuating components x(t) and f(t), respectively, by

$$X(t) = X_o + x(t) \qquad (II.2)$$
$$F(t) = -F_o + f(t) \qquad (II.3)$$

with the conditions that the ensemble averages, denoted by the brackets $\langle \ldots \rangle$ are

$$\langle x(t) \rangle = 0 \qquad (II.4)$$
$$\langle f(t) \rangle = 0 \qquad (II.5)$$

For stability reasons, F_o must be a sufficiently positive quantity. The condition will be given later. Inserting equations (II.2) and (II.3) into equation (II.1) gives the stochastic differential equation for the fluctuating components alone :

$$\dot{x}(t) = -F_oX_o + S_o - F_ox(t) + X_of(t) + f(t)x(t) \qquad (II.6)$$

As long as the cross-term f(t)x(t) is negligibly small, we have the "linearized" solution for X_o and the PSD of x(t), denoted by $X_{o(lin)}$ and $S_{xx(lin)}$:

$$X_{o(lin)} = S_o/F_o \qquad (II.7)$$

$$S_{xx(lin)}(\omega) = X^2_{o(lin)} S_{ff}(\omega)/(F_o^2 + \omega^2) \qquad (II.8)$$

S_{ff} is the PSD of f(t), and ω is the angular frequency.

In power applications, the cross-term f(t)x(t) may become larger, leading to the appearance of nonlinear effects in equation (II.6). Even if f(t) is Gaussian, x(t) will no longer be Gaussian. The system can also become unstable.

If f(t) represents white noise, we assume for the physical approach that the autocovariance function (ACOF) is given by

$$C_{ff}(\tau) = \varepsilon^2 \delta(\tau) \qquad (II.9)$$

where $\delta(\tau)$ is the Dirac-Delta function and ε^2 is the PSD. Equation (II.9) characterizes continuous Gaussian white noise and is the mathematical approximation of a Gaussian process with very short, but still finite, correlation length. The difference between the continuous process and the discrete "jump-like" Wiener process is explained in the textbook of Williams (1974) and the discussion about the problem is summarized in the paper of Akcasu and Karasulu (1976). The assumption of the physical approach is certainly justified with respect to power applications. Furthermore, it is consistent with the treatment of equation (II.6) by the WHF method.

In general, if f(t) is Gaussian noise, but not necessarily white, we assume that it is obtained from passing white noise through a linear filter with the transfer function $\varphi(\omega)$, so that the PSD of f(t) is given by

$$S_{ff}(\omega) = \varepsilon^2 |\varphi(\omega)|^2 \tag{II.10}$$

φ may be assumed to be a dimensionless function of frequency, normalized to unit gain at maximum magnitude.

For white noise input, the N-th moment of the process X(t) ($N \geq 1$) can easily be obtained from the Fokker-Planck equation. It results in

$$< X^N(t)> = S_o^N / \prod_{\nu=1}^{N} (F_o - \frac{\varepsilon^2}{2}\nu) \tag{II.11}$$

Equation (II.11) indicates that the process is typically weakly ergodic up to and including the order N, if the stability condition

$$F_o - \frac{\varepsilon^2}{2} N > 0 \tag{II.12}$$

is satisfied. In particular, for N = 2 we have

$$< X(t)> = X_o = S_o/(F_o - \frac{\varepsilon^2}{2}) \tag{II.13}$$

$$< X^2(t)> = S_o^2/((F_o - \frac{\varepsilon^2}{2})(F_o - \varepsilon^2)) \tag{II.14}$$

$$Var(X(t)) = \frac{\varepsilon^2}{2} X_o^2/(F_o - \varepsilon^2) \tag{II.15}$$

It will later be shown that the stability condition (II.12) for N = 2 must also be satisfied if the input is low-pass white noise.

III. The Wiener-Hermite Functional Method

We assume that n(t) is a member function of a stationary Gaussian white-noise process with zero mean and unit PSD. The concept of the WHF method consists in representing a member function G(t) of a random process, which may be non-stationary and non-Gaussian, by n(t). Under the assumption that the mean square of G(t) exists, i.e. $< G^2(t) >$ is finite, G(t) is modelled by the integral expansion :

$$G(t) = G^{(o)}(t)H^{(o)} + \int_{-\infty}^{+\infty} dt_1 G^{(1)}(t,t_1)H^{(1)}(t_1) + \ldots$$
$$+ \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} dt_1 \ldots dt_\nu \, G^{(\nu)}(t,t_1,\ldots,t_\nu)H^{(\nu)}(t_1,\ldots,t_\nu)$$
$$+ \quad \ldots \qquad \text{(III.1)}$$

The $G^{(\nu)}$s are deterministic kernels. The $H^{(\nu)}$s are the WHFs. They have a generation law in a formal analogy to the ordinary Hermite polynomials (Barrett, 1964; Imamura et al., 1965). For the first few functionals we have :

$$H^{(o)} = 1 \qquad \text{(III.2a)}$$

$$H^{(1)}(t_1) = n(t_1) \qquad \text{(III.2b)}$$

$$H^{(2)}(t_1,t_2) = n(t_1)n(t_2) - \delta(t_1-t_2) \qquad \text{(III.2c)}$$

$$\begin{aligned} H^{(3)}(t_1,t_2,t_3) = \; & n(t_1)n(t_2)n(t_3) \\ & - n(t_1)\,\delta(t_2-t_3) \\ & - n(t_2)\,\delta(t_1-t_3) \\ & - n(t_3)\,\delta(t_1-t_2) \end{aligned} \qquad \text{(III.2d)}$$

The WHFs are statistical functionals. They contain the well-known properties of stationary Gaussian random noise (applied to the white-noise case) that all odd correlation functions vanish, while the higher order even correlation functions decay into a sum of products of ordinary correlation functions. These properties appear in the WHFs in such a way that, with the exception of the zero-th order term,

$$< H^{(\nu)}(t_1,..,t_\nu)> \; = 0 \qquad (\nu > 0) \qquad \text{(III.3)}$$

and that the WHFs represent an orthogonal set in the statistical sense by

$$< H^{(\nu)}(t_1,..,t_\nu)H^{(\mu)}(t_1',...,t_\mu')> = 0 \quad \text{for} \quad \nu \neq \mu \qquad \text{(III.4)}$$

The set is furthermore complete in the finite mean-square sense.

The deterministic kernels $G^{(\nu)}$ can be considered as the statistical projection of G(t) in the ν-th WHF.

$$G^{(\nu)}(t,t_1,...,t_\nu) \cong \; < G(t)H^{(\nu)}(t_1,..,t_\nu)> \qquad \text{(III.5)}$$

The first term, $G^{(o)}(t) = <G(t)>$, can be denoted as the trend function. The second term in the expansion by equation (III.1) is the Gaussian part, while the remaining terms are the non-Gaussian contributions to G(t).

If G(t) represents stationary noise, the fluctuating part g(t) around the mean G_o can be expressed by

$$g(t) = \int_{-\infty}^{+\infty} dt_1 G^{(1)}(t-t_1)H^{(1)}(t_1) + \iint_{-\infty}^{+\infty} dt_1 dt_2 G^{(2)}(t-t_1,t-t_2)H^{(2)}(t_1,t_2) + \ldots \qquad \text{(III.6)}$$

The kernels depend only on time lags. Without any loss of generality the higher order kernels can be considered to be symmetric functions with respect to the exchange of the time arguments. If one postulates causality, the kernels have to vanish for negative time values and equation (III.6) exhibits effectively the expansion by a Volterra series.

Using the statistical orthogonality relationship between the WHFs it is easy to calculate the ACOF of $g(t)$. Expressing the kernels in the frequency domain by the Fourier transform, and Fourier transforming the ACOF, gives the PSD of $g(t)$. The first two terms read :

$$S_{gg}(\omega) = \left|G^{(1)}(\omega)\right|^2 + \frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega' \left|G^{(2)}(\omega', \omega-\omega')\right|^2 + \dots \tag{III.7}$$

where

$$G^{(1)}(\omega) = \int_{-\infty}^{+\infty} d\sigma \; G^{(1)}(\sigma) e^{-i\omega\sigma} \tag{III.8a}$$

$$G^{(2)}(\omega_1, \omega_2) = \iint_{-\infty}^{+\infty} d\sigma_1 \, d\sigma_2 \; G^{(2)}(\sigma_1, \sigma_2) e^{-i\omega_1\sigma_1 - i\omega_2\sigma_2} \tag{III.8b}$$

The time lags have been denoted by the variable σ .

The application of the WHF method to the analysis of a stochastic single input/single output system which is nonlinear or will show nonlinear effects, consists in expanding the forcing function and the response function in terms of the same WHF set. The kernels of the forcing function must be known (either in the time domain or in the frequency domain). Under the assumption of sufficient differentiability of the noise signals, deterministic equations for determining the unknown kernels of the response function are then derived. The expansion series of the response function must generally be truncated after a few terms. To this extent the system analysis will be approximative. The number of required terms depend upon the problem. If the expansion by equation (III.6) is truncated after the ν-th term, the series represents a ν-th order approximation, denoted by WHF-ν. There is a simple case where the expansion remains finite. When we consider the square law device $g(t) = f^2(t) - G_o$, where f(t) is assumed to be band limited or low-pass stationary Gaussian white noise with zero mean, the WHF expansion of f(t) contains only the Gaussian part, while the expansion of g(t) leads to two equations, one for determining the DC value G_o and the other for determining the kernel $G^{(2)}$. All other kernels in the representation of g(t) are zero. On the other hand, the ACOF of g(t), $C_{gg}(\tau)$ can be obtained directly ($C_{gg}(\tau) = 2C_{ff}^2(\tau)$; G_o follows from $G_o = C_{ff}(0)$). The PSDs of g(t) derived directly from the ACOF or via the kernel $G^{(2)}$ are exactly the same.

IV. System Equations in the WHF-2 Approximation

Under the assumption of system stability we expand :

$$f(t) = \int_{-\infty}^{+\infty} dt_1 F^{(1)}(t-t_1) H^{(1)}(t_1) \qquad \text{(IV.1)}$$

$$x(t) = \int_{-\infty}^{+\infty} dt_1 X^{(1)}(t-t_1) H^{(1)}(t_1) + \iint_{-\infty}^{+\infty} dt_1 dt_2 X^{(2)}(t-t_1, t-t_2) H^{(2)}(t_1, t_2) \qquad \text{(IV.2)}$$

Since the forcing function f(t) is assumed to be Gaussian, there is only the Gaussian term with the given kernel $I^{(1)}$.

The expansion of the response function x(t) is truncated after the second term (the WHF-2 approximation).

Inserting equations (IV.1) and (IV.2) into equation (II.6), multiplying the resulting equation successively by $H^{(o)}$, $H^{(1)}(t_6)$ and $H^{(2)}(t_6,t_7)$, taking the ensemble averages, considering the relationships of equations (III.3) and (III.4), and observing the relationships

$$H^{(1)}(t_1)H^{(1)}(t_2) = H^{(2)}(t_1,t_2) + \delta(t_1-t_2) \qquad \text{(IV.3a)}$$

$$< H^{(1)}(t_1)H^{(1)}(t_2) > \; = \; \delta(t_1-t_2) \qquad \text{(IV.3b)}$$

$$\begin{aligned} H^{(1)}(t_1)H^{(2)}(t_2,t_3) = \; & H^{(3)}(t_1,t_2,t_3) \\ & + H^{(1)}(t_2)\,\delta\,(t_1-t_3) \\ & + H^{(1)}(t_3)\,\delta\,(t_1-t_2) \end{aligned} \qquad \text{(IV.3c)}$$

$$\begin{aligned} < H^{(2)}(t_1,t_2)H^{(2)}(t_3,t_4) > \; = \; & \delta(t_1-t_3)\,\delta\,(t_2-t_4) \\ & + \delta(t_1-t_4)\,\delta\,(t_2-t_3) \end{aligned} \qquad \text{(IV.3d)}$$

the following system of coupled integro-differential equations is obtained :

$$F_o X_o = S_o + \int_{-\infty}^{+\infty} d\sigma\, F^{(1)}(\sigma)X^{(1)}(\sigma) \qquad \text{(IV.4a)}$$

$$\frac{dX^{(1)}(\sigma)}{d\sigma} = -F_o X^{(1)}(\sigma) + X_o F^{(1)}(\sigma) + 2\int_{-\infty}^{+\infty} d\sigma' F^{(1)}(\sigma')X^{(2)}(\sigma,\sigma') \qquad \text{(IV.4b)}$$

$$\begin{aligned} \frac{\partial X^{(2)}(\sigma_1,\sigma_2)}{\partial \sigma_1} + \frac{\partial X^{(2)}(\sigma_1,\sigma_2)}{\partial \sigma_2} = \; & -F_o X^{(2)}(\sigma_1,\sigma_2) \\ & + \tfrac{1}{2}\left[F^{(1)}(\sigma_1)X^{(1)}(\sigma_2) + F^{(1)}(\sigma_2)X^{(1)}(\sigma_1)\right] \end{aligned} \qquad \text{(IV.4c)}$$

where time lags have been expressed by the variable σ .

The quantities to be determined are X_o and the kernels $X^{(1)}$ and $X^{(2)}$. For a physical solution, all kernels and the derivatives of $X^{(1)}$ and $X^{(2)}$ must vanish for infinite time lags. Due to causality the kernels must be zero for negative time lags. This implies that only the right-hand side of the derivatives may exist at zero time lag. Since we are mainly interested in the PSD of the response function, and since the equations are linear, it is straightforward to look directly for a solution in the frequency domain. Expressing all kernels by their Fourier transforms, and writing $F^{(1)}(\omega) = \varepsilon\varphi(\omega)$ (equation II.10), the following system of equations is obtained in the frequency domain :

$$F_o X_o = S_o + \frac{\varepsilon}{2\pi}\int_{-\infty}^{+\infty} d\omega\,\varphi^*(\omega) X^{(1)}(\omega) \qquad \text{(IV.5a)}$$

$$X^{(1)}(\omega) = \frac{\varepsilon}{F_o + i\omega}\left[X_o\varphi(\omega) + \frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega'\varphi^*(\omega')X^{(2)}(\omega,\omega')\right] \qquad \text{(IV.5b)}$$

$$X^{(2)}(\omega_1,\omega_2) = \frac{\varepsilon}{2(F_o + i(\omega_1+\omega_2))}\left[\varphi(\omega_1)X^{(1)}(\omega_2) + \varphi(\omega_2)X^{(1)}(\omega_1)\right] \qquad \text{(IV.5c)}$$

This system consists only of two coupled integral equations and one algebraic equation. It looks much simpler than the equations (IV.4) in the time domain. For simple cases, where the forcing kernel $\varphi(\omega)$ is represented by a broken rational function of $i\omega$, there is good chance of finding analytical expressions for the solutions X_o, $X^{(1)}$ and $X^{(2)}$ in the frequency domain.

V. WHF-2 Approximation for Gaussian White Noise Input (Case 1)

For Gaussian white noise input we have $\varphi(\omega) = 1$. Solutions of the equations (IV.5) can easily be obtained (see also Appendix A, letting $\omega_H \to \infty$). The results are :

$$X_{o(WHF-2)} = X_{o(exact)} = \frac{S_o}{F_o - \frac{\varepsilon^2}{2}} \qquad (V.1)$$

$$X^{(1)}_{(WHF-2)}(\omega) = \frac{\varepsilon S_o}{\left(F_o - \frac{\varepsilon^2}{2}\right)\left(F_o - \frac{\varepsilon^2}{2} + i\omega\right)} \qquad (V.2)$$

The kernel $X^{(2)}$ follows from equation (IV.5c). Using equation (III.7), the PSD of the response function results in

$$S_{xx(WHF-2)}(\omega) = \frac{\varepsilon^2 S_o^2}{\left(F_o - \frac{\varepsilon^2}{2}\right)^2}\left[\frac{1}{\left(F_o - \frac{\varepsilon^2}{2}\right)^2 + \omega^2} + \frac{\varepsilon^2}{2\left(F_o - \frac{\varepsilon^2}{2}\right)\left(F_o^2 + \omega^2\right)}\right] \qquad (V.3)$$

We will now compare this result with the exact form of the PSD. The exact form of the ACOF is known from the literature (Williams, 1971). It reads

$$C_{xx(exact)}(\tau) = \frac{\varepsilon^2 S_o^2 e^{-\left(F_o - \frac{\varepsilon^2}{2}\right)|\tau|}}{2\left(F_o - \frac{\varepsilon^2}{2}\right)^2\left(F_o - \varepsilon^2\right)} \qquad (V.4)$$

The Fourier transform of equation (V.4) leads to the exact form of the PSD.

$$S_{xx(exact)}(\omega) = \frac{\varepsilon^2 S_o^2}{\left(F_o - \frac{\varepsilon^2}{2}\right)\left(F_o - \varepsilon^2\right)\left[\left(F_o - \frac{\varepsilon^2}{2}\right)^2 + \omega^2\right]} \qquad (V.5)$$

If one compares equation (V.5) with equation (V.3), one can recognize that the WHF-2 approximation of the PSD gives smaller amplitude values. The first term in equation (V.3) reproduces correctly the frequency dependence, with a shift of the corner frequency down to lower values if the excitation strength given by the parameter ε^2 increases. The second term in equation (V.3) can be regarded as a second order correction to the amplitude. The accuracy of the PSD approximation can be characterized by the relative deviation

$$\Delta = \left| \frac{S_{xx(WHF-2)}(\omega) - S_{xx(exact)}(\omega)}{S_{xx(exact)}(\omega)} \right|$$

$$= \frac{r^2}{4\left(1-\frac{r}{2}\right)^2} \left(1 + \frac{2(1-r)\left(1-\frac{r}{4}\right)}{1+\left(\frac{\omega}{F_o}\right)^2} \right) \qquad \text{(V.6)}$$

where we have introduced the excitation ratio

$$r = \varepsilon^2/F_o \qquad (0 < r < 1) \qquad \text{(V.7)}$$

Fig.1 shows plots of the exact and the approximative PSDs in normalized units (ω/F_o ; $S_o = 1$, $F_o = 1$). The curves refer to three excitation ratio values, r = 0.2, 0.4 and 0.6 . For values $r \leq 0.35$ the WHF-2 approximation deviates by less than 10 % from the exact PSD.

It should be noted that the WHF-1 approximation (disregarding the kernel $X^{(2)}$ in equations (IV.5) and (III.7)) also gives the exact value of X_o. However with regard to the frequency behaviour of the PSD, it cannot lead to more than the "linearized" approach as given by equation (II.8).

VI. WHF-2 Approximation for Low-Pass Gaussian White Noise Input (Case 2)

In this section we assume that the driving noise function f(t) belongs to a Gauss-Markovian process, characterized by its ACOF

$$C_{ff}(\tau) = \frac{\varepsilon^2 \omega_H}{2} e^{-\omega_H |\tau|} \qquad \text{(VI.1)}$$

The corresponding filter function $\varphi(\omega)$ is given by

$$\varphi(\omega) = \frac{\omega_H}{\omega_H + i\omega} \qquad \text{(VI.2)}$$

ω_H is the upper cutoff frequency of the filter.

Power reactivity noise is mostly dominant in the region of very low frequencies, where delayed neutrons affect the system behaviour. Since we have disregarded delayed neutrons, our model is incomplete. Nevertheless, this omission is justified from the point of view of the mathematical effort necessary to include at least one group of delayed neutrons. The problem consists mainly in finding the exact reference solution of the PSD which we would like to have for comparison with the WHF-2 approximation.

With the filter function given by equation (VI.2), the system of equations (IV.5) can be solved. The procedure is explained in Appendix A. The formulae cover the previous case 1 if $\omega_H \to \infty$. The results are :

$$X_{o(WHF-2)} = \frac{S_o}{F_o - \frac{\varepsilon^2}{2} p} \qquad \text{(VI.3)}$$

$$\text{with } p = \frac{\omega_H (F_o + 2\omega_H)}{(a+\omega_H)(b+\omega_H)\left[1 - \frac{\varepsilon^2}{2} \frac{\omega_H}{(a+\omega_H)(b+\omega_H)}\right]} \qquad \text{(VI.4)}$$

$$a = F_o + \frac{\omega_H}{2}\left(1 + \sqrt{1 + \frac{2\varepsilon^2}{\omega_H}}\right) \qquad \text{(VI.5a)}$$

$$b = F_o + \frac{\omega_H}{2}\left(1 - \sqrt{1 + \frac{2\varepsilon^2}{\omega_H}}\right) \qquad \text{(VI.5b)}$$

$$X^{(1)}_{(WHF-2)}(\omega) = \frac{\varepsilon X_o \varphi(\omega)(c+i\omega)}{(a+i\omega)(b+i\omega)} \qquad \text{(VI.6)}$$

with $c = F_o + \omega_H + \frac{\varepsilon^2}{2} p$ (VI.7)

The kernel $X^{(2)}$ follows from equation (IV.5c), with the kernel $X^{(1)}$ given by equation (VI.6). We then obtain for the PSD of the response function :

$$S_{xx(WHF-2)}(\omega) = \varepsilon^2 X_o^2 \frac{|\varphi(\omega)|^2 (c^2+\omega^2)}{(a^2+\omega^2)(b^2+\omega^2)} + \frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega' \left| X^{(2)}(\omega', \omega-\omega')\right|^2 \qquad \text{(VI.8)}$$

The integral of the second term of the RHS of equation (VI.8) can be treated analytically by using partial fraction techniques. Thus, in principle, a closed analytical solution could be obtained. Since this integration by partial fraction techniques leads to lengthy expressions where one has to distinguish several limiting cases, we preferred to apply numerical integration using the IMSL routine QDAGI. The integral term can give a significant contribution.

The procedure for obtaining the exact DC value and the PSD is explained in Appendix B. We give here the results of these calculations :

$$X_{o(exact)} = S_o e^{-\delta} \sum_{\nu=0}^{\infty} \frac{\delta^\nu}{\nu! (A + \omega_H \nu)}$$

$$= \frac{S_o}{A} \Phi\left(1, 1 + \frac{A}{\omega_H}; -\delta\right) \qquad \text{(VI.9)}$$

A and δ are defined by equations (B.5a/b) in Appendix B. Φ denotes Kummer's confluent hypergeometrical function. The same result for X_o has also been obtained by Saito (1980).

$$S_{xx(exact)}(\omega) = 2S_o^2 e^{-2\delta} \sum_{\nu_1,\dots,\nu_6=0}^{\infty} \frac{(-1)^{\nu_5+\nu_6}\,\delta^{\nu_1+\nu_2+\nu_3+\nu_4+\nu_5+\nu_6}}{\nu_1!\cdots\nu_6!\,(A+\omega_H(\nu_1+\nu_3+\nu_5))\left[(A+\omega_H(\nu_2+\nu_3+\nu_6))^2+\omega^2\right]}$$

$$\times \left\{ \frac{\omega_H(\nu_3+\nu_4+\nu_5+\nu_6)(A+\omega_H(\nu_2+\nu_3+\nu_4))-\omega^2}{\omega_H^2(\nu_3+\nu_4+\nu_5+\nu_6)^2+\omega^2} \right.$$

$$\left. + 2\,\frac{A+\omega_H(\nu_2+\nu_3+\nu_6)}{B+\omega_H(\nu_1+\nu_2+\nu_3+\nu_4)} \right\}$$

$$\text{for } \omega \neq 0 \qquad \text{(VI.10)}$$

B is defined by equation (B.10) in Appendix B .

It should be noted that for white noise input ($\omega_H \to \infty$) the input power is infinite. The system itself acts as a low-pass filter depending on F_o and the excitation ratio r. Weak stationarity is guaranteed as long as $r < 1$. For low-pass white noise input, the input power is finite. One could conjecture that for values $\omega_H < F_o$ this weak stationarity should be preserved, even for excitation ratios exceeding 1 . When one looks at the derivation of the autocorrelation function given in Appendix B, the integrals in equation (B.6) exist only under the same condition as for white noise input, independently of the value of ω_H ($\omega_H > 0$). This is expressed equivalently in equation (VI.10) by the first term ($\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6 = 0$), which corresponds due to white noise input. The other terms behave as corrections on the band limitation of the input noise. So far, the allowed range of r is the same as for white noise input (with respect to the amplitude normalization used in equation (VI.1)).

In the numerical treatment of equation (VI.10) a summation scheme with open upper limits of the number of terms and an absolute convergence criterion was used. This scheme has previously been checked by a similar, but known, function in frequency which has arbitrarily been expanded into a 6-fold sum.

In Figs. 2-5 plots of the exact and the approximative PSDs are shown, in the same normalized units as applied to the plots in Fig.1 . The figures refer to values $\omega_H/F_0 = 0.25$, 0.5, 1.0 and 2.0 . In each figure curves are again given for the three excitation ratios, r = 0.2, 0.4 and 0.6 . One can observe that, for decreasing ratios ω_H/F_0, the WHF-2 approximation becomes closer to the exact PSD. The system is effectively less strongly excited for the same r values. Due to limitations in the CPU time, evaluations for ratios $\omega_H/F_0 < 0.25$ could not be handled since the required number of terms in equation (VI.10) increases then rapidly. For ratios $\omega_H/F_0 \gg 1$, the curves shown in Fig.1 were exactly reproduced.

The comparison of the approximative DC value with the exact one showed that equation (VI.3) is an excellent approximation within the considered range of the parameter data (r was varied up to 0.9). The strongest, but still very small, deviation appeared at $\omega_H \cong F_c$. $X_{o(WHF-2)}$ was found there to be lower against $X_{o(exact)}$ by about 1 % at r = 0.9 .

VII. Concluding Remarks

The WHF method is applied to the simplest case of a first-order stochastic differential equation with parametric random excitation. Stationary conditions are assumed. Inspite of the fact that this differential equation is linear in the usual sense, the parametric (multiplicative) excitation leads to a non-Gaussian system response, even if the input noise is Gaussian. The purpose of the investigation was to make a quantitative assessment of the DC value and the PSD of the system response in the WHF-2 approximation. For this, two cases of excitation have been selected, the first with Gaussian white noise, and the second with low-pass Gaussian white noise. In both cases the approximative results can be compared with exact solutions of the problem. For excitation ratios up to 0.3 the PSDs obtained by the WHF-2 approximation are well reproduced, if the allowance for maximum deviation from the exact values in the significant frequency range should not exceed the 10 % level. The appearance of larger excitation ratios may be unrealistic in practice due to the destabilizing effect. In case 2 (low-pass white noise input) the computational efforts required for the evaluation of the WHF-2 results are much less than for the exact solutions, which are at the upper limit of an analytical treatment. We conclude that the WHF method is a powerful approximative tool. We believe that the inclusion of delayed neutrons, as far as this can be done, should not alter this conclusion. There are other methods referred to in the literature for the approximative treatment of nonlinear stochastic differential equations. However, a comparative assessment of these other methods is outside the framework of the present paper.

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Appendix A : Solving the System Equations of the WHF-2 Approximation with Low-Pass Gaussian White Noise Input

If one inserts the kernel $X^{(2)}$ given by equation (IV.5c), with the filter function φ given by equation (VI.2) ($\varphi(\omega) = \omega_H/(\omega_H + i\omega)$), into equation (IV.5b), the following system equations result :

$$F_o X_o - S_o = \frac{\varepsilon}{2\pi} \int_{-\infty}^{+\infty} d\omega \, \varphi^*(\omega) \, X^{(1)}(\omega) \tag{A.1}$$

$$X^{(1)}(\omega) = \varepsilon \, \Phi(\omega)(F_o + \omega_H + i\omega) \Big[X_o + \frac{\varepsilon}{2\pi} \int_{-\infty}^{+\infty} d\omega' \, \frac{\varphi^*(\omega') X^{(1)}(\omega')}{F_o + i(\omega + \omega')} \Big] \tag{A.2}$$

where
$$\Phi(\omega) = \frac{\varphi(\omega)}{(a + i\omega)(b + i\omega)} \tag{A.3}$$

a and b are determined by the equation :

$$(F_o + i\omega)(F_o + \omega_H + i\omega) - \frac{\varepsilon^2}{2} \omega_H = (a + i\omega)(b + i\omega) \tag{A.4}$$

The explicit values of a and b are given by the equations (VI.5a/b). They must be real positive quantities for a stable solution of the problem.

Equations (A.1) and (A.2) represent a coupled system of two Fredholm integral equations for the unknowns X_o, which is a frequency independent constant, and the kernel $X^{(1)}(\omega)$. As soon as $X^{(1)}(\omega)$ has been determined, the kernel $X^{(2)}(\omega,\omega')$ is also known, according to equation (IV.5c).

Since we can write

$$\frac{\varphi^{*}(\omega')}{F_0 + i(\omega+\omega')} = \frac{1}{F_0+\omega_H+i\omega}\left[\varphi^{*}(\omega') + \frac{\omega_H}{F_0+i(\omega+\omega')}\right] \qquad (A.5)$$

we can bring equation (A.2) into the following form, thereby using equation (A.1) :

$$X^{(1)}(\omega) = \varepsilon\,\phi(\omega)\left[(2F_0+\omega_H+i\omega)X_0 - S_0\right] + \frac{\varepsilon^2\omega_H}{2\pi}\,\phi(\omega)\,\psi(\omega) \qquad (A.6)$$

where $$\psi(\omega) = \int_{-\infty}^{+\infty} d\omega' \,\frac{X^{(1)}(\omega')}{F_0 + i(\omega+\omega')} \qquad (A.7)$$

For the determination of ψ we perform the operations of dividing equation (A.6) by $F_0+i(\omega''+\omega)$ and integrating over ω, thereby interchanging the order of integration :

$$\psi(\omega'') = \varepsilon\, I_1(\omega'') + \frac{\varepsilon^2\omega_H}{2\pi}\int_{-\infty}^{+\infty} d\omega' X^{(1)}(\omega')\, I_2(\omega',\omega'') \qquad (A.8)$$

where $$I_1(\omega'') = \int_{-\infty}^{+\infty} d\omega\, \frac{\phi(\omega)\left[(2F_0+\omega_H+i\omega)X_0 - S_0\right]}{F_0 + i(\omega''+\omega)} = 0 \qquad (A.9)$$

$$I_2(\omega',\omega'') = \int_{-\infty}^{+\infty} d\omega\, \frac{\phi(\omega)}{\left[F_0+i(\omega'+\omega)\right]\left[F_0+i(\omega''+\omega)\right]} = 0 \qquad (A.10)$$

It follows that $\psi(\omega'') = 0$, which also implies that $\psi(\omega) = 0$. Equation (A.6) reduces to the form :

$$X^{(1)}(\omega) = \varepsilon\,\phi(\omega)\left[(2F_0+\omega_H+i\omega)X_0 - S_0\right] \qquad (A.11)$$

For the determination of X_0 we write

$$X_o = \frac{S_o}{F_o - \frac{\varepsilon^2}{2} p} \tag{A.12}$$

For determining the constant p we insert equation (A.12) into equations (A.1) and (A.11) and obtain :

$$p = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega \, \varphi^*(\omega) \phi(\omega) (F_o + \omega_H + \frac{\varepsilon^2}{2} p + i\omega) \tag{A.13}$$

The solution of equation (A.13) leads to the value of p given by equation (VI.4).

The above solution scheme seems to be generalizable for transfer functions φ which are broken rational functions of $i\omega$, in particular with isolated single poles, thereby satisfying the Routh stability criterion (e.g. Truxal, 1955). The kernel $X^{(1)}$ will then generally also be a broken rational function of $i\omega$.

Appendix B : Exact Solution of X_o and the PSD of X(t) for Low-Pass Gaussian White Noise Input

As outlined by Akcasu and Karasulu (1976), the stochastic differential equation (II.1) can be solved by using ordinary integral calculus. Assuming sufficient system stability, the stationary solution is given by

$$X(t) = S_o \int_0^\infty du\, e^{-\xi_o u + \int_0^u dv\, f(t-v)} \tag{B.1}$$

Since the forcing function f(t) is assumed to be Gaussian and its integration is a linear operation, the stochastic term in the exponent of equation (B.1) is again Gaussian. If ξ_i denotes a set of Gaussian variables with zero mean, there is a simple relationship for exponential averaging of the sum of these variables (*).

$$< e^{\sum_i \xi_i} > = e^{\frac{1}{2}\sum_{i,j} <\xi_i \xi_j>} \tag{B.2}$$

It follows that the first moment of X(t) is given by

$$< X(t)> = X_o = S_o \int_0^\infty du\, e^{-\xi_o u + \frac{u}{2}\int_{-u}^{u} dv \left(1-\frac{|v|}{u}\right) C_{ff}(v)} \tag{B.3}$$

(*) Since the sum $s = \sum_i \xi_i$ is Gaussian, the application of the Gaussian probability density function leads to $< e^s > = e^{1/2\, Var(s)}$; $Var(s) = \sum_{i,j} < \xi_i \xi_j >$.

If one inserts for C_{ff} the ACOF given by equation (VI.1), one obtains :

$$X_o = S_o \int_0^\infty du\, e^{-Au - \delta(1 - e^{-\omega_H u})} \tag{B.4}$$

$$\text{where } A = F_o - \frac{\varepsilon^2}{2} \tag{B.5a}$$

$$\delta = \frac{\varepsilon^2}{2\,\omega_H} \tag{B.5b}$$

X_o can be represented by an expansion which is given by equation (VI.9). Using the relationship of equation (B.2), the autocorrelation function (ACF) of X(t) follows from

$$< X(t)X(t+\tau) > \; = R_{xx}(\tau) = S_o^2 \iint_0^\infty du_1 du_2\, e^{P(u_1,u_2,\tau)} \tag{B.6}$$

$$\text{with } P(u_1,u_2,\tau) = -F_o(u_1 + u_2) + \frac{u_1}{2}\int_{-u_1}^{u_1} dv_1 \left(1 - \frac{|v_1|}{u_1}\right) C_{ff}(v_1) + \frac{u_2}{2}\int_{-u_2}^{u_2} dv_2 \left(1 - \frac{|v_2|}{u_2}\right) C_{ff}(v_2) + \int_0^{u_1}\int_0^{u_2} dv_1 dv_2\, C_{ff}(\tau + v_1 - v_2) \tag{B.7}$$

Since the ACF is a symmetric function in τ , we can simplify the procedure by assuming $\tau \geqq 0$ in the following formulae. With our special input ACOF, equation (B.7) takes the form :

$$P(u_1,u_2,\tau) = -2\delta - \delta e^{-\omega_H \tau} - A(u_1 + u_2) + \frac{\varepsilon^2}{2}\left[u_1 + |\tau - u_2| - |\tau + u_1 - u_2|\right] + \delta\left[e^{-\omega_H u_1} + e^{-\omega_H u_2} + e^{-\omega_H(\tau + u_1)} + e^{-\omega_H|\tau - u_2|} - e^{-\omega_H|\tau + u_1 - u_2|}\right] \tag{B.8}$$

Applying expansion techniques, the ACF of X(t) can be expressed by the 5-fold sum :

$$R_{xx}(\tau) = S_o^2 \, e^{-2\delta - \delta e^{-\omega_H \tau}} \sum_{\nu_1, \dots, \nu_5 = 0}^{\infty} \frac{(-1)^{\nu_5} \, \delta^{\nu_1+\nu_2+\nu_3+\nu_4+\nu_5}}{\nu_1! \nu_2! \nu_3! \nu_4! \nu_5! \left(A + \omega_H(\nu_1+\nu_3+\nu_5)\right)}$$

$$\times \left\{ e^{-\omega_H \tau \nu_3} \, \frac{e^{-\omega_H \tau(\nu_4+\nu_5)} - e^{-\tau(A+\omega_H \nu_2)}}{A + \omega_H(\nu_2-\nu_4-\nu_5)} + 2 \, \frac{e^{-\tau(A+\omega_H(\nu_2+\nu_3))}}{B + \omega_H(\nu_1+\nu_2+\nu_3+\nu_4)} \right\} \qquad \text{(B.9)}$$

$$\text{where } B = 2(F_o - \varepsilon^2) \qquad \text{(B.10)}$$

The first term on the RHS of equation (B.9) remains finite if the denominator approaches zero. One can show that $R_{xx}(\tau) \rightarrow X_o^2$ for $\tau \rightarrow \infty$. For $\omega_H \rightarrow \infty$ (white noise input) the sum reduces to the first term :

$$R_{xx}(\tau, \omega_H \rightarrow \infty) = X_o^2 + \frac{\varepsilon^2 S_o^2}{A^2 B} \, e^{-A\tau} \qquad \text{(B.11)}$$

The second term in equation (B.11) is the ACOF of X(t) and agrees with equation (V.4).

For the Fourier transform of equation (B.9) one needs one further expansion. The result of this calculation is given by equation (VI.10). Since the square of the DC value is included in the transform, the resulting PSD must be considered as being valid for frequencies different from zero.

Fig.1 PSD of the System Response for White Noise Input

a : Exact solutions
b : WHF-2 approximations
1,2,3 : r = 0.2, 0.4, 0.6

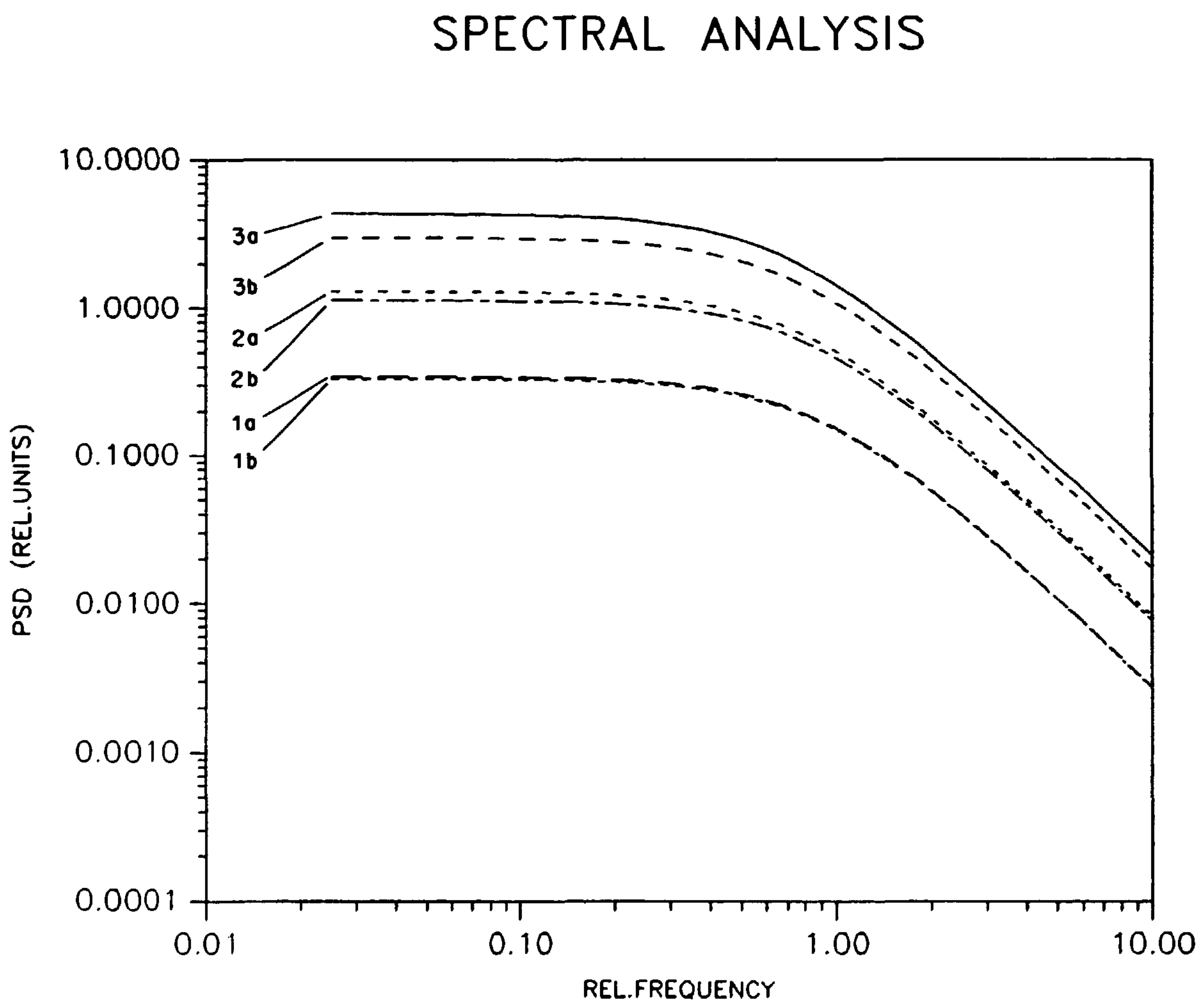

Fig.2 PSD of the System Response for Low-Pass White Noise Input
$\omega_H / \bar{r}_0 = 0.25$

a : Exact solutions
b : WHF-2 approximations
1,2,3 : r = 0.2, 0.4, 0.6

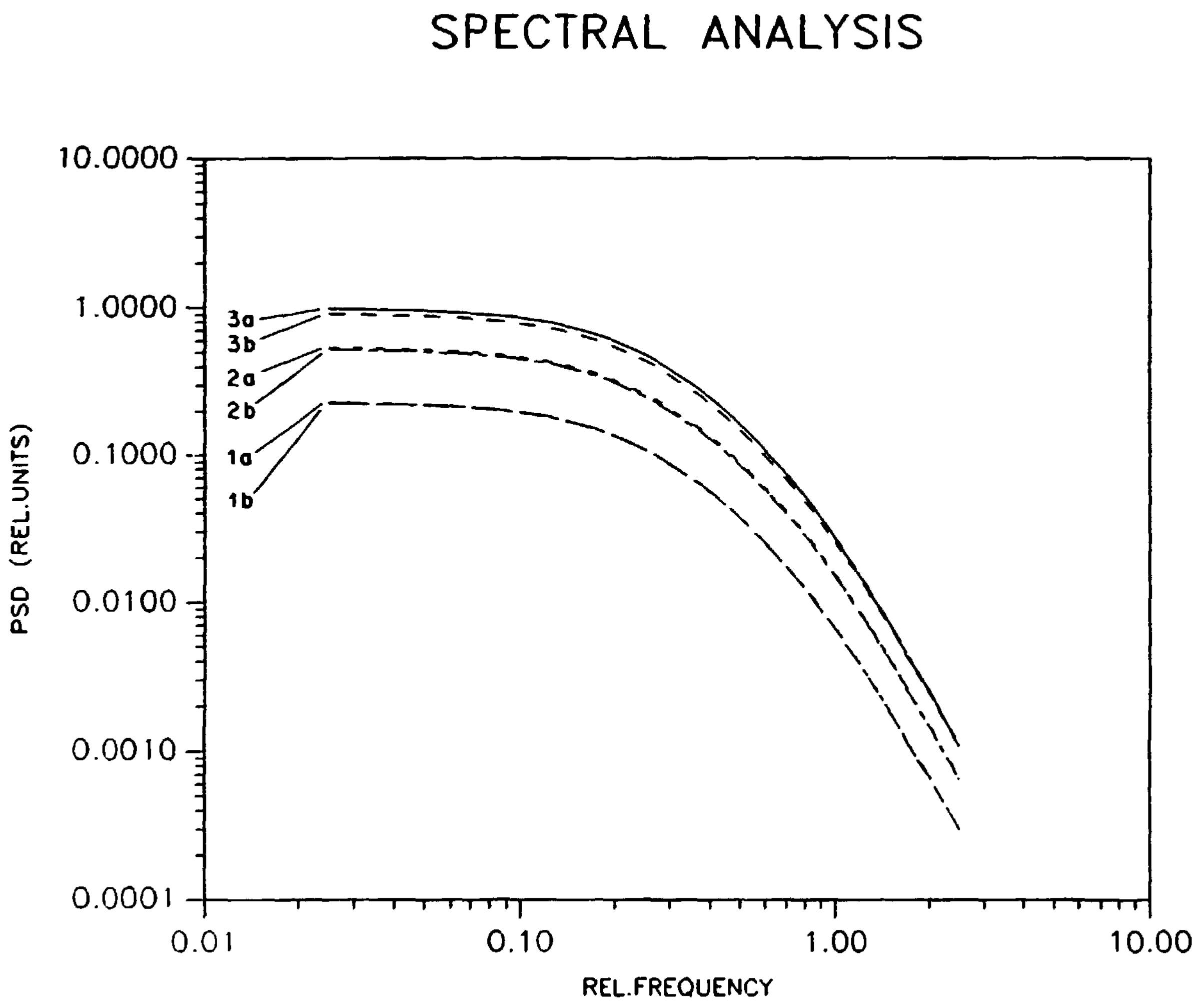

Fig.3 PSD of the System Response for Low-Pass White Noise Input
$\omega_H / F_0 = 0.5$

a : Exact solutions
b : WHF-2 approximations
1,2,3 : r = 0.2, 0.4, 0.6

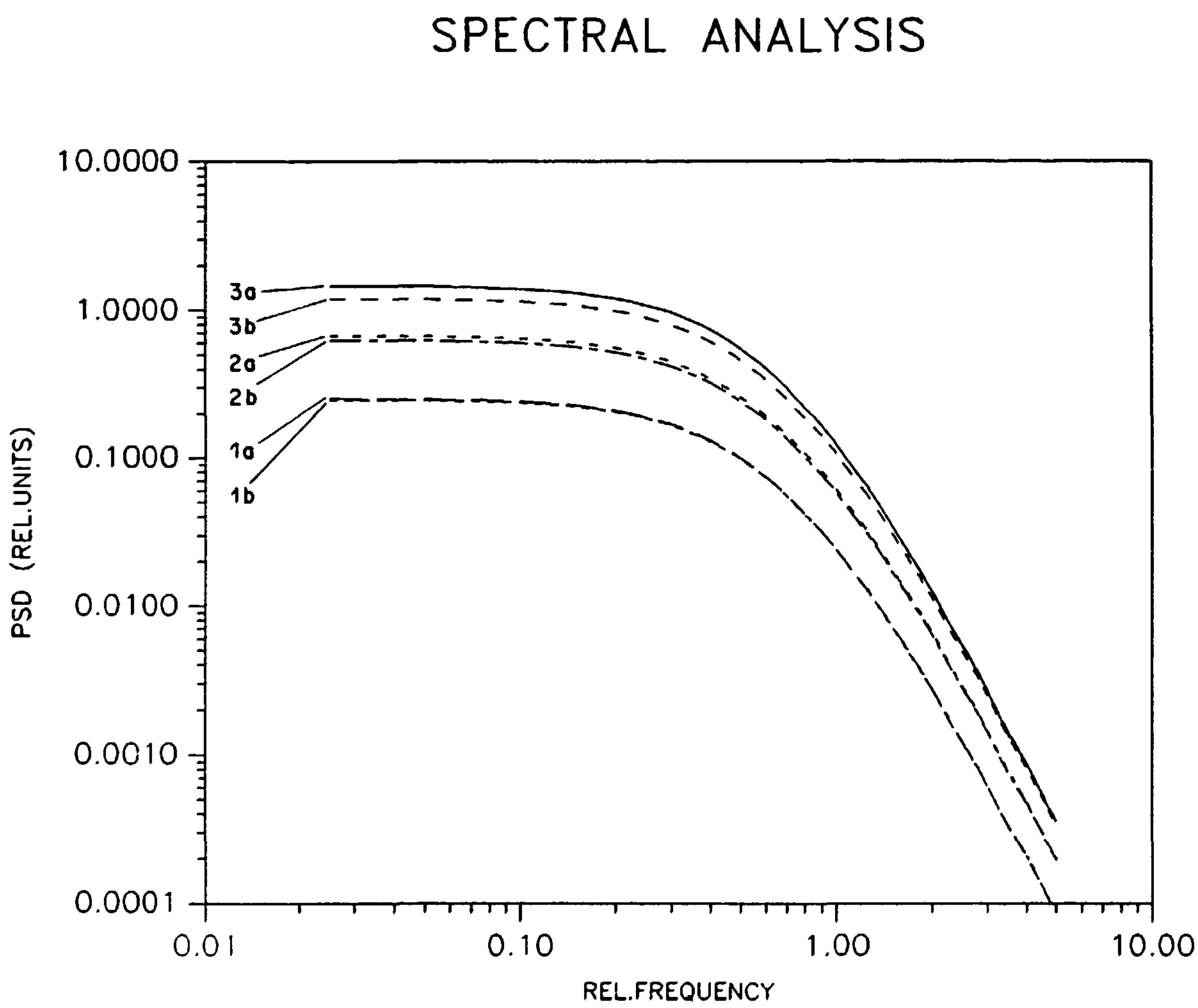

Fig.4 PSD of the System Response for Low-Pass White Noise Input
$\omega_H / F_c = 1.0$

a : Exact solutions
b : WHF-2 approximations
1,2,3 : r = 0.2, 0.4, 0.6

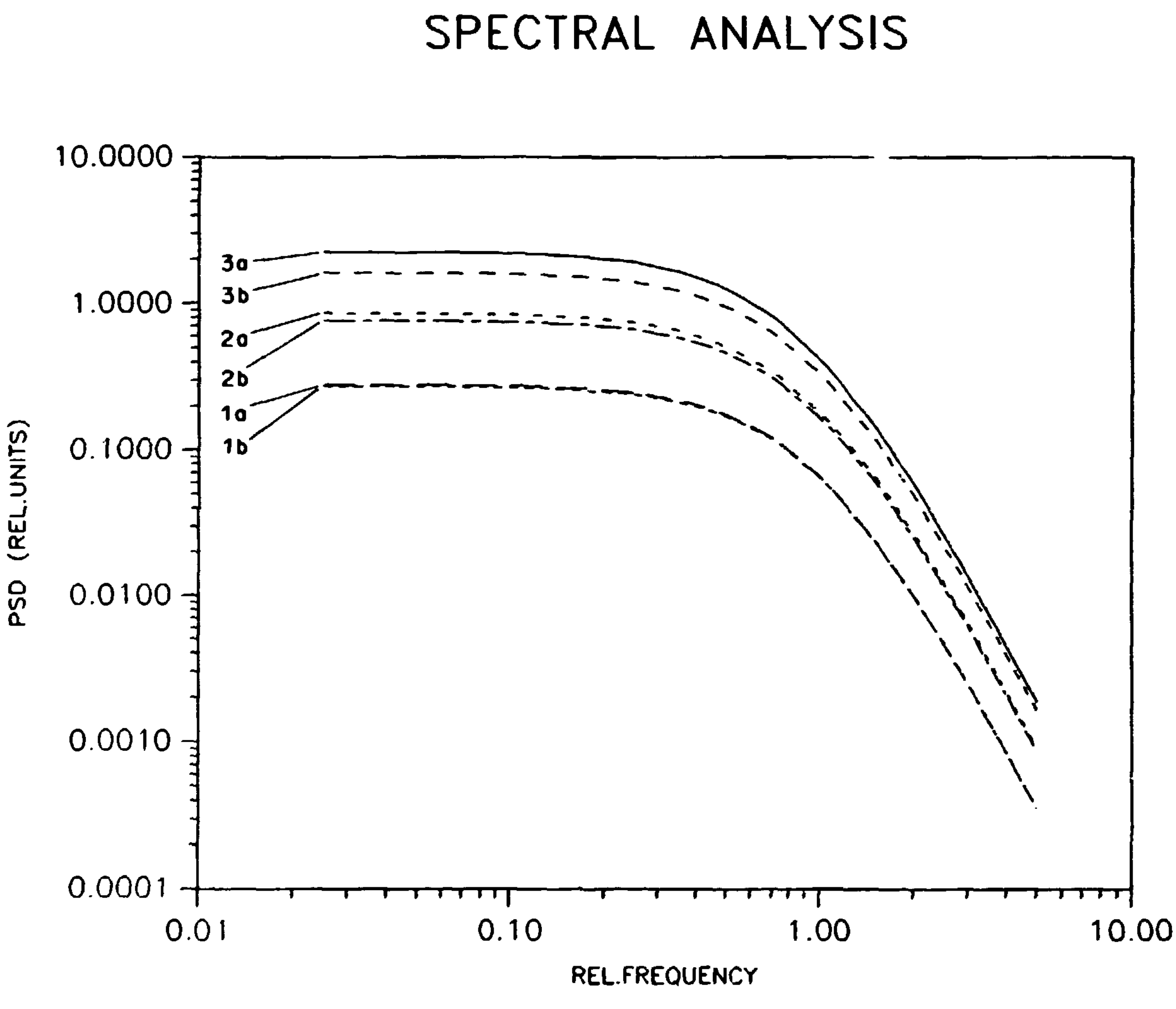

Fig.5 PSD of the System Response for Low-Pass White Noise Input

$\omega_H / F_0 = 2.0$

a : Exact solutions
b : WHF-2 approximations
1,2,3 : r = 0.2, 0.4, 0.6