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EQUIVALENCE BETWEEN DEEP ENERGY-DEPENDENT AND SHALLOW ANGULAR MOMENTUM DEPENDENT POTENTIALS

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Abstract:

Recently Baye showed that supersymmetry can be applied to determine a shallow ℓ-dependent potential phase equivalent to a deep potential, assumed to be energy-independent and having Pauli forbidden states (PFS), for α-α scattering. The PFS are eliminated by this procedure. The shallow potential has a r^{-2} singularity to allow the fulfillment of a generalised Levinson theorem.

Such deep potentials are generated as equivalent local potentials (ELP) to the Resonating Group Model (RGM) by the WKB procedure of Horiuchi and are generally energy-dependent. To eliminate this E-dependence as required for the application of Baye's method, we generate ℓ-dependent, but E-independent, deep local potentials equivalent to the deep RGM + WKB local potential, by the exact inversion method of Marchenko. In this way we fully preserve the spectrum including the PFS. Subsequently we use the supersymmetric method, to eliminate the PFS, ensuring that the generalized Levinson theorem is satisfied.

As an example we apply this method to the well known simple model of two dineutrons scattering in the RGM, where the deep ELP of Horiuchi has a substantial energy-dependence and one PFS only for ℓ=0.

1. Introduction

Baye[1] recently investigated the controversy between deep or shallow nucleus-nucleus potentials, a question also discussed in detail by Horiuchi[2]. The Resonating Group Model (RGM) approach to two cluster scattering provides phase shifts which obey a modified Levinson theorem[3]. The Pauli forbidden states (PFS) which occur in the nonlocal RGM interactions have to be taken into account in addition to the physical bound states (PBS) in this modified Levinson theorem. It has been shown by Horiuchi[2,4] that in the WKB approximation deep equivalent local potentials (ELP) are obtained for the nonlocal RGM interactions, which in addition to the PBS also have the PFS as bound states and therefore necessarily satisfy the generalized Levinson theorem. Their real parts also show in general a substantial energy-dependence and at most a weak angular momentum dependence. These potentials generally reproduce the RGM phase shifts to a good approximation.

On the other hand good fits to the same data have been obtained employing shallow phenomenological potentials, which are strongly ℓ-dependent. As an illustration we refer to $\alpha+\alpha$ scattering and the phenomenological Ali-Bodmer potentials[5]. However they have the disadvantage of not reproducing the high energy behaviour of the RGM phase shifts (since only the physical bound states occur in the Levinson relation in this case). Recently Michel and Reidemeister[6] showed that phenomenological shallow potentials having a $\frac{1}{r^2}$ singularity can reproduce the microscopic phase shifts in agreement with another extension of the Levinson theorem proposed by Swan[7]. Subsequently Baye[1] showed that supersymmetry applied to the Schrödinger equation establishes an exact relation between deep and shallow but singular local potentials, both of which are energy-independent. The potentials related in this way are exactly phase equivalent, while their wave functions are related, by a differential operator. However, they produce a different number of bound states, while preserving the Levinson relation. The shallow potential has the required r^{-2} singularity to make this possible.

Baye succesfully applied this procedure to the case of $\alpha+\alpha$ scattering. Using the energy-independent local potential of Buck et al[3], which approximates the ELP of the RGM interaction in the low energy region, he applied the supersymmetric relations to eliminate the two unphysical PFS for $\ell=0$ and the one PFS for $\ell=1$ and obtained shallow but strongly ℓ-dependent potentials. These potentials compared well with the well-known shallow potentials of Ali and Bodmer[5]. Therefore they offer a more rigorous explanation of the phenomenological shallow potentials[6] by relating them to microscopic RGM interactions.

However the exact ELP to a RGM interaction is intrinsically energy-dependent, since the nonlocality of the RGM interaction is represented by the energy-dependence of the equivalent local potential. Horiuchi[3,9] explicitly derived the ELP for the α-α nonlocal RGM interaction and even in this case found a substantial energy-dependence over a more extended energy region. This energy-dependence becomes even more pronounced for the scattering of heavier clusters than $\alpha+\alpha$, e.g. α-O^{16} [10].

Unfortunately the supersymmetric relations between deep and shallow potentials employed by Baye are only applicable to energy-independent local potentials, since they are intrinsically related to inverse scattering theory as discussed by Sukumar[11]. The application of Baye's procedure therefore depends crucially on the approximation of the energy-dependent ELP of Horiuchi by an energy-independent local potential at low energies. Such an approximation neglects theenergy-dependence, which simulates the nonlocality of the RGM interaction. It is this nonlocality of the RGM which is responsible for the Pauli forbidden states in the first place however. Consequently the ELP for the RGM interaction as demonstrated in Horiuchi's theory is inherently energy-dependent.

Fortunately there is a way out of this dilemma as we will demonstrate in detail in the next sections. From the phase shifts and bound states (including the PFS) of the energy-dependent ELP of Horiuchi we construct by means of the exact inverse scattering formalism of Marchenko[12], E-independent but ℓ-dependent equivalent local potentials. These potentials are deep and non-singular since they have the generalized PFS as the ELP of Horiuchi. Consequently they obey the generalized Levinson theorem.

$$\delta_\ell(0) - \delta_\ell(\infty) = n_P \pi + n_{PS} \pi , \qquad (1)$$

where n_P = number of physical bound states and n_{PS} = number of Pauli forbidden states.

In the second step we eliminate the PFS by means of supersymmetry according to Baye's prescription. The potentials obtained in this way are shallow and even more strongly ℓ-dependent. They are also singular and satisfy Eq. (1) and therefore represent a rigorous explanation of the phenomenological shallow potentials based on extensions of the Levinson theorem in the context of the RGM theory and Horiuchi's "RGM + WKB" approach.

2. Scattering of Two Dineutrons

To illustrate our method, we consider a particularly simple model example of the scattering of two dineutrons in the RGM, which was originally studied by Kukulin et al[13]. The internal spatial function of the dineutron is chosen as simple as possible, i.e. as a translationally-invariant shell-model function of $(0s)^2$ configuration in its ground state in a harmonic oscillator well of width parameter α:

$$\varphi(r) = (\alpha/\pi)^{\frac{1}{4}} \exp\left(-\tfrac{1}{2}\alpha r^2\right) \qquad (2)$$

$$\alpha = m\omega/2h$$

The (2+2)-system corresponds to the shell-model configuration s^2p^2. In this system one S-state of the relative motion is forbidden by the Pauli principle and has the wave function

$$\phi_0(R) = (2\alpha/\pi)^{\frac{1}{2}} \exp\left(-\alpha R^2\right) \qquad (3)$$

The effective interaction is chosen to be of the form

$$V(r) = -V_0(w+mP_x) \exp\left(-\gamma r^2\right) \qquad (4)$$

where w and m are the usual Wigner and Majorana admixtures respectively and P_x is the Majorana operator. The direct potential is then given by

$$V_D(R) = -V_0 \left[\frac{2\alpha}{2\alpha+\gamma}\right]^{3/2} (4w-2m) \exp\left[-\frac{2\alpha\gamma}{2\alpha+\gamma} R\right] \qquad (5)$$

The exchange kernel of the RGM consists of several terms given in [13]. The solution $\phi(R)$ of the integro-differential equation of the RGM is not necessarily orthogonal to the function $\phi_0(R)$. To obtain a physically meaningful solution $\tilde{\phi}_0(R)$ however it should be orthogonal to $\phi_0(R)$.

We therefore have $\tilde{\phi}(R) \equiv P_0 \phi(R)$

where

$$P_0 \equiv 1 - |\phi_0\rangle\langle\phi_0|$$

Kukulin et al[13] showed that the RGM equation can be written as

$$(T_R + V_D + \hat{V}_N - E)\, \phi(r) = 0 \tag{6}$$

where

$$\hat{V}_N \phi(R) \equiv \int dR'\ U_N \left[\frac{R+R'}{2}\right] \exp\left[-\left(\tfrac{1}{4}\alpha+\gamma\right)(R-R')^2\right] \phi(R') \tag{7}$$

and

$$U_N(R) = -\ V_0\ (4m - 2\omega) \left[\frac{4\alpha}{\alpha+2\gamma}\right]^{3/2} \exp(-2\alpha R^2) \tag{8}$$

with the additional orthogonality condition

$$\langle\phi|\phi_0\rangle = 0 \tag{9}$$

The values of the parameters of the NN force are taken from Kukulin et al[13] and are given by

$$V_0 = 73 \text{ MeV}, \quad \gamma = 0{,}46 \text{ fm}^{-2}, \quad \omega = m = 0{,}5 \text{ while } \alpha = 0{,}22 \text{ fm}^{-2}.$$

The ELP of the nonlocal interaction of Eq. (7) is obtained by using Horiuchi's "WKB + RGM" method, which in this case reduces to the old Pery-Buck procedure[14]. The ELP, V_D+V_L phase equivalent to the potential V_D+V_N is determined from

$$V_L(E,R) \approx U_N(R) \exp\{(m/\hbar^2(\alpha+2\gamma))\ [V_D(R) + V_L(E,R) - E]\} \tag{10}$$

Note that this potential is independent of ℓ.

This energy-dependent potential has a forbidden 0S state with a binding energy of about 8,3 MeV, which according to Kukulin et al[13] has a wave function very closely approximating $\phi_0(R)$. Therefore the orthogonality condition Eq. (9) is automatically satisfied in this case. They also found that in the low energy region for $E \approx 0$, $V_L(R)$ can be approximately represented by

$$V_L(R) \approx - V_{OL} \exp(-\kappa R^2) \tag{11}$$

where $V_{OL} = 20{,}5$ MEV and $\kappa = 0{,}30\ fm^{-2}$ respectively.

The ELP however has a substantial energy-dependence which at low energies can be represented by

$$V_{OL}(E) \approx V_{OL}(E{=}0) \exp(-E/E_0)$$

where $E_0 = 84$ MeV. For $E \leq E_0$ the well depth of $V(R) = V_D(R) + V_L(E,R)$ has the approximate form $V_0(E) \approx V_0 - \frac{1}{4} E$ which clearly indicates that it is not negligeable at all. This also shows up in the fact that $V_L(R)$ given by Eq. (11) has a binding energy of only 7.67 MeV instead of 8.3 MeV for $V_L(E,R)$ given by Eq. (10).

3. Energy-independent Deep Potentials by Marchenko's Inversion Method

To be able to apply the supersymmetric scheme of Baye we first have to generate E-independent but ℓ-dependent potentials which have the same spectra as the ELP of Eq. (10), including the PFS, by means of the Marchenko theory.

The Marchenko spectral inversion equation for each angular momentum ℓ is given by:

$$K_\ell(r,s) + F_\ell(r,s) + \int_r^\infty K_\ell(r,t)\, F_\ell(t,s)\, dt = 0\ , \tag{12}$$

where the input kernel $F_\ell(r,s)$ is obtained from the S-matrix via

$$F_\ell(r,s) = (2\pi)^{-1} \int_{-\infty}^{\infty} h_\ell^{(+)}(kr)\ [1-S_\ell(k)]\ h_\ell^{(+)}(ks)dk$$

$$+ \sum_{i=1}^{n_p+n_{FS}} M_i\ h_0^{(+)}(k_i,r)\ h_0^{(+)}(k_i,s) \tag{13}$$

and where the $h_\ell^{(+)}$ are Riccati-Hankel functions for outgoing wave conditions, while the M_i are the bound state normalization constants (where both the physical and forbidden bound states are included). The local potential producing $S_\ell(k)$ is obtained from the diagonal part of the output kernel by

$$V_\ell(r) = -2 \frac{d}{dr} K_\ell(r,r) \tag{14}$$

We represent $S_\ell(k)$ in rational form by

$$S_\ell(k) = \prod_{n=1}^{N} \frac{k+a_n}{k-a_n} \tag{15}$$

which allows us to perform the integral in Eq. (13) analytically and solve Eq. (12) for $K_\ell(r,s)$[14,15,16] assuming arbitrary values for the normalization constants M_i. It can be shown[14] that only for a unique set of values of the M_i a short range potential is obtained. Otherwise the potentials $V_\ell^M(R)$ determined from Eq. (14) have long range tails. In our simple example only one Pauli forbidden state occurs for ℓ=0 at -8.3 MeV, while no bound states are present for ℓ=1, guaranteeing uniqueness of the potential in that case[14]. The bound state normalization parameter M_1 is determined according to the short range criterium as described in Refs. 14 and 15 and is given accurately by M_1 = 9.99. For $\ell \geq 1$ the potential $V_\ell(r)$ obtained by inversion already are the shallow ℓ-dependent potentials we were looking for. However for $\ell=0$ we still have a deep potential (although now independent of the energy) with a PFS. In Fig. 1 we show $V_D(r)$, $V_0(r) + V_L(0,r)$, $V_D(r) + V_L(E_b,r)$ (where E_b = -8.3 MeV) and $V_0^M(r)$, the potential obtained by Marchenko inversion of the whole spectrum of $V_D(r) + V_L(E,r)$ for $\ell=0$.

4. Removal of Pauli Forbidden State

We now proceed to remove the PFS at E_b = -8.3 MeV from the spectrum of $V_0^M(r)$ using Baye's supersymmetric scheme[1)].

In this method the Hamiltonian in units where $\hbar^2/2m = 1$ is given by:

$$H_0^\ell = \left[- \frac{d^2}{dr^2} + V_0{}^\ell\right]$$

where $V_0{}^\ell$ includes the centrifugal, Coulomb and nuclear potentials. The Hamiltonian H_0 and its supersymmetric partner H_1 are factorized as

$$H_0 = A_0^+ A_0^- + \epsilon_0, \qquad H_1 = A_0^- A_0^+ + \epsilon_0 \tag{16}$$

where

$$A_0^- = (A_0^+)^\dagger = - \frac{d}{dr} + \frac{d}{dr} \ln \Psi_0(\epsilon_0)$$

The nodeless wave function $\Psi_0(\epsilon_0)$ is the solution of the Schrödinger equation at the factorization energy ϵ_0. We must therefore have $\epsilon_0 \leq E_0^{(0)}$, where $E_0^{(0)}$ is the groundstate energy of H_0, while the eigenfunctions $\Psi_0(E)$ and $\Psi_1(E)$ of H_0 and H_1 are related by

$$\Psi_1(E) = (E - \epsilon_0)^{-\frac{1}{2}} A_0^- \Psi_0(E) \tag{17}$$

The choice of $\epsilon_0 = E_0^{(0)}$ makes the spectrum of H_1 identical to the spectrum of H_0, except for the absence of the ground state of H_0. In this case the potential corresponding to H_1 is given by

$$V_1 = V_0 - 2 \frac{d^2}{dr^2} \ell n \ \Psi_0(E_0^{(0)}) \tag{18}$$

For small values of r we have $V_1 \sim V_0 + 2(\ell+1)\ r^{-2} \sim (\ell+1)(\ell+2)\ r^{-2}$, which behaves as a nonsingular potential with angular momentum ℓ. However V_1 and V_0 are not phase equivalent. To achieve this the supersymmetric partner H_2 of H_1 with $\epsilon_1 = E_0^{(0)}$ as factorization energy is required. Since $\epsilon_1 < E_1^{(0)}$ the ground state energy of H_1, the function $\Psi_1(\epsilon_1)$ is not square integrable and behaves asymptotically as $\exp\ (+\ \gamma_0 r)$, where $\gamma_0 = (-E_0^{(0)})^{\frac{1}{2}}$, but can be chosen to be regular at r=0. The bound state spectra of H_1 and H_2 are identical. To any physical state of H_1 with energy E corresponds a state of H_2 with the wave function

$$\Psi_2(E) = (E_0^{(0)} - E)^{-1}\ A_1^-\ A_0^-\ \Psi_0(E) \tag{19}$$

where $\Psi_2(E)$ and $\Psi_0(E)$ have the same phase shift, and $\delta_2^{\ell}(E) = \delta_0^{\ell}(E)$ and where $\Psi_2(E)$ is normalizable for the bound states. The corresponding potential is given by

$$V_2 = V_0 - 2 \frac{d^2}{dr^2} \ell n\ [\Psi_0(E_0^{(0)})\ \Psi_1(E_0^{(0)})] \tag{20}$$

which behaves for small r like

$$V_2 \sim V_0 + 2(2\ell+3)\ r^{-2} \sim (\ell+2)(\ell+3)\ r^{-2} \tag{21}$$

The singular potential V_2 is shallow and has one bound state less than V_0. From the equality of the phase shifts it follows that $\delta_\ell(0) - \delta_\ell(\infty)$ is the same for V_0 and V_2, in spite of V_2 having one bound state less than V_0 as proved by Swan[7]. This two-step process can be iterated to remove all the nonphysical bound states.

5. Results

Applying the procedure described in the previous section to eliminate the PFS at -8.3 MeV from the deep potential $V_0^M(r)$, obtained by Marchenko inversion and shown in Fig. 1, we find the corresponding phase equivalent shallow potential $V_0^{S_2}(r)$ having no bound state. It is shown in Fig. 2, together with the unphysical potential $V_0^{S_1}(r)$ obtained in the intermediate step which corresponds to Eq. (18). It is seen that the potential $V_0^{S_2}(r)$ is purely repulsive in spite of having a phase shift with the property $\delta_0(0) = \pi$, like the original $V_G(E,r)$ and $V_0^M(r)$. This is possible as discussed in Section 4 because of its r^{-2} singularity. The potentials $V_0^M(r)$, $V_0^{S_2}(r)$ and $V_1^M(r)$ are compared in Fig. 3, which demonstrates that the ℓ-dependence of the shallow potentials represented by $V_0^{S_2}(r)$ and $V_1^M(r)$, is very pronounced. Even the first step, consisting of just removing the E-dependence by inversion, already results in a strong ℓ-dependence as is seen by comparing $V_0^M(r)$ and $V_1^M(r)$.

Finally like Baye for α-α scattering, we take the potential $V_D(r) + V_L(0,r)$ which approximately represents the ELP at low energies and is E-independent as the starting point of the supersymmetric construction of the shallow potential. This potential has a PFS at $E_b = -6.74$ MeV. In this case the Marchenko inversion as an intermediate step can be omitted even for $\ell=0$ due to the absence of an E-dependence. This shallow potential is compared in Fig. 4, to the one derived previously for $\ell=0$, using $V_D(r) + V_L(E,r)$ by employing Marchenko inversion, while their phase shifts are compared in Fig. 5. It is seen that neglecting the energy dependence of $V(E,r)$ of Eq. (10)

and replacing it by $V_L(0,r)$, does have a considerable effect on the phase shift at energies 50 MeV. The potentials themselves also differ substantially in the region from 1 to 3 fm.

6. Conclusions

In conclusion we have presented a mathematically rigorous scheme to obtain shallow singular potentials which are strictly phase equivalent to a given RGM nonlocal interaction excluding the Pauli forbidden states while still satisfying the generalized Levinson theorem for the RGM phase shifts.

The scheme does not depend on the neglect of the energy-dependence of the deep local potential of Horiuchi equivalent to the RGM. Its aim is achieved by converting the E-dependence of the ELP into an ℓ-dependence by means of Marchenko inversion method in a first step. In the second step these deep ℓ-dependent potentials are transformed into shallow ℓ-dependent singular potentials by using Baye's supersymmetric scheme for removing all the Pauli forbidden states. This two-step procedure in general increases the ℓ-dependence as compared to the one obtained when an approximate E-independent ELP valid for low energies only is employed as in Baye's example for α-α scattering.

In the first step (Marchenko inversion) the phase shift and bound state spectrum of the RGM interaction can be used directly as input. The construction of the deep ELP is not even necessary for this purpose. In this way exact phase equivalence between the RGM interaction and the shallow potentials can be achieved. This could be of some importance if the energy-dependent ELP of Horiuchi is not quite phase equivalent to the RGM interaction within the accuracy required. Even in the case the construction of the deep local and energy-dependent ELP to the RGM interaction is of course still very useful for comparison purposes.

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FIGURE CAPTIONS

Fig. 1 The direct potential $V_D(r)$, the ELP at zero energy $V_D(r) + V_L(0,r)$, and at $E_b = -8.3$ MeV i.e. $V_D(r) + V_L(E_b,r)$, and the potential $V_0^M(r)$, obtained by Marchenko inversion of the whole spectrum of $V_D(r) + V_L(E,r)$ for $\ell=0$, are plotted.

Fig.2 Tje potentials $V_0^M(r)$ - - - -, for S-waves and $V_1^M(r)$ -·-·-·-, for p-waves obtained by Marchenko inversion of the corresponding spectra produced by the full equivalent local potential $V_D(r) + V_L(E,r)$ are shown. We also plot the potential ${V_0}^{S_1}(r)$ ...... obtained by the first step described by Eq. (18) in the supersymmetric procedure to remove the PFS and the fianl potential ${V_0}^{S_2}(r)$ ——— , which is phase equivalent to $V_0^M(r)$, but has no bound state. The potential ${V_0}^{S_2}(r)$ has a r^{-2} singularity described by Eq. (21).

Fig.3 We compare the potentials $V_0^M(r)$ -·-·-·-· , $V_1^M(r)$ - - - - - , and ${V_0}^{S_2}(r)$ ——— described in the caption of Fig.2, to illustrate their ℓ-dependence.

Fig.4 The potntial $V_0^M(r)$ -·-·-·-, is compared to the potential ${\tilde{V}_0}^{S_2}(r)$ ——— obtained by applying Baye's supersymmetric removal of the PFS an -7.76 MeV of the energy-indepdendent potential $V_D(r) + V_L(0,r)$, i.e. ${\tilde{V}_0}^{S_2}(r)$ have the same phase shift.

Fig.5 The phase shifts of ${\tilde{V}_0}^{S_2}(r)$——— ${\tilde{V}_0}^{S_2}(r)$ - - - - - (which is phase equivalent to the potential $V_D(r) + V_L(E,r)$) are compared. For the definition of the potentials see previous figure captions.

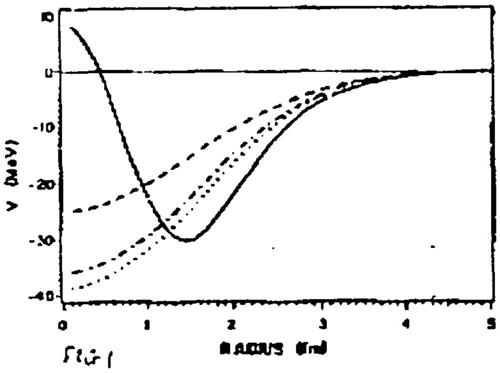

FIG 1

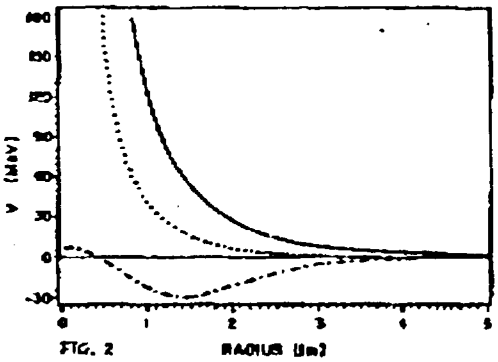

FIG. 2

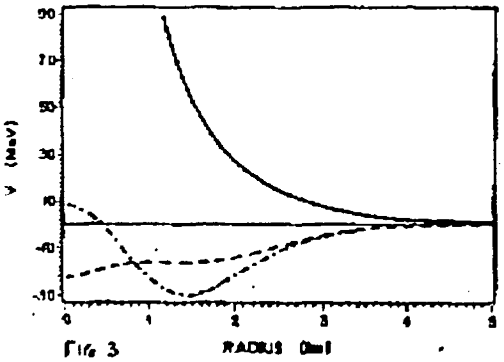

FIG 3

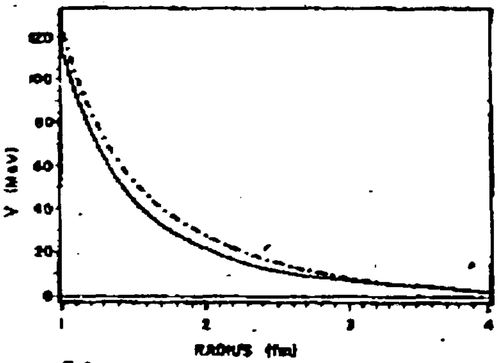

FIG 1

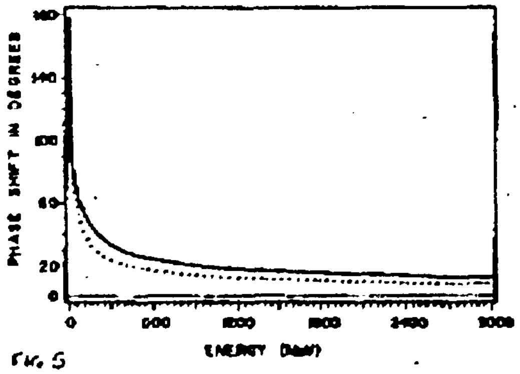

FIG. 5