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THE LOCAL STRUCTURE OF A LIOUVILLE VECTOR FIELD *

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ABSTRACT

In this work we investigate the local structure of a Liouville vector field ξ of a Kähler manifold (P, Ω) which vanishes on an isotropic submanifold Q of P. Some of the eigenvalues of its linear part at the singular points are zero and the remaining ones are in resonance. We show that there is a C^1-smooth linearizing conjugation between the Liouville vector field ξ and its linear part. To do this we construct Darboux coordinates adapted to the unstable foliation which is provided by the Centre Manifold Theorem. We then apply recent linearization results due to G. Sell.

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1 Introduction

In this work we study the local structure of a Liouville vector field ξ of a $2n$-dimensional Kähler manifold (P,Ω) which vanishes on a k-dimensional isotropic submanifold Q of P. We assume that at every point q of Q $\xi(q) = 0$ and the eigenvalues of its 1-jet are 1, 1/2 and 0. This means that in some coordinates

$$J^1_q(\xi) = \sum_{i=1}^{k}(0x_i\frac{\partial}{\partial x_i} + 1y_i\frac{\partial}{\partial y_i}) + \sum_{r=k+1}^{n}(\frac{1}{2}x_r\frac{\partial}{\partial x_r} + \frac{1}{2}y_r\frac{\partial}{\partial y_r})$$

where

$$T_qQ = \{\cap_{j=k+1}^{n} Ker\, dx_j\} \cap \{\cap_{j=1}^{n} Ker\, dy_j\}.$$

A submanifold Q of P is said to be *isotropic* if the restriction of Ω to Q is identically zero, i.e. $\iota^*\Omega = 0$ where $\iota : Q \to P$ is the inclusion map. A smooth vector field of (P,Ω) is called *Liouville* if its flow φ_t satisfies $\varphi_t^*\Omega = e^t\Omega$ (or equivalently when $\mathcal{L}_\xi\Omega = \Omega$, where $\mathcal{L}$ denotes the Lie derivative).

Our goal is to prove

Theorem 1.1 *There is a C^1-smooth linearizing conjugation between the Liouville vector field ξ and its linear part on a neighborhood of each singular point q of ξ.*

This means that there is a local diffeomorphism on a neighborhood of q in P which carries the trajectories of the flow generated by the vector field ξ to the trajectories of the flow of its linear part, preserving the direction of motion.

A homeomorphism which linearizes a vector field in a neighborhood of a singular point can not always be chosen to be smooth, because of resonant eigenvalues of the linear part. (In our case the eigenvalues 1 and 1/2 are in resonance). See Definition 5.1. The situation becomes even more complicated when the linear part of the equation at the singular point has eigenvalues on the imaginary axis (for example 0 in our case). We overcome this obstacle by constructing Darboux coordinates adapted to the foliation W provided by the Center Manifold Theorem [6]. This allows us to treat the isotropic submanifold Q as a parameter set. Further, although the vector field ξ has resonant eigenvalues, using the fact that ξ is Liouville together with the recipe given by the Poincaré-Dulac theorem [2] we show that indeed it has no resonant terms in its Taylor expansion. Therefore we can apply Sell's algorithm [8], which guarantees that for each singular point q_0 the Liouville vector field ξ is at least C^1-conjugate to its linear part in a vicinity of q_0, and that the linearizing conjugation Φ depends smoothly on the parameter q.

This paper is organized as follows: in §2 we verify the hypothesis of the Center Manifold theorem and we show that the fibers W_q of the strong unstable foliation are coisotropic. In §3 we construct Darboux coordinates adapted to the foliation. We study the relation of the Liouville vector field ξ to the Darboux coordinates in §4. We show that ξ has no resonant quadratic monomials in its Taylor expantion and apply Sell's algorithm in §5.

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2 The Center Manifold Theorem

An *invariant manifold* of a vector field ξ and of the corresponding differential equation $\dot{x} = \xi(x)$ is a submanifold of the phase space P which is tangent to the vector field at each of its points. Let us denote by φ_t the flow generated by the vector field ξ. An invariant submanifold S is *normally hyperbolic* if the restriction of the tangent bundle of P to S splits into the continuous subbundles $TP|_S = N^u \oplus TS \oplus N^s$, each of which is invariant by the derivative of φ_t, $D\varphi_t$, and if for each q there is a number $\tau = \tau(q) > 0$ such that

a) $D\varphi_t$ expands N^u more sharply than it expands anything in TS.

b) $D\varphi_t$ contracts N^s more sharply than it contracts anything in TS.

More formally, the submanifold S is ***r*-*normally hyperbolic*** if φ_t is C^r, and for all q in S and $0 \leq k \leq r$

$$a)\ m(D\varphi_\tau|_{N^u_q}) > \|D\varphi_\tau|_{T_qS}\|^k$$

$$b)\ \|D\varphi_\tau|_{N^s_q}\|^k < m(D\varphi_\tau|_{T_qS})$$

where $m(L) = inf\{\|Lv\| : \|v\| = 1\}$ is the minimum norm of the linear map L and $\|L\| = sup\{\|Lv\| : \|v\| = 1\}$ is the norm of L.

Because the Liouville vector field ξ vanishes on the isotropic submanifold Q of (P, Ω), Q is an invariant manifold of the field ξ which consist of fixed points. Since at any point q in Q, the tangent space to P at q can be written as $T_qP = T_qQ \oplus N_q$, where $N_q = (T_qQ)^{\perp_\Omega}$ and the nonzero eigenvalues of ξ at a singular point $q \in Q$ are assumed to be 1 or 1/2 we have that $m(D\varphi_\tau|_{N_q^*}) = e^{1/2}$ and $\|D\varphi_\tau|_{T_qQ}\| = e^0$. It follows that Q is r-normally hyperbolic for all $r \in \mathbb{N}$ and all $q \in Q$. Therefore we can apply the Center Manifold Theorem, which states that under this conditions

i) P is smoothly foliated by strong unstable φ_t-invariant submanifolds W_q which are transverse to Q, i.e. $P = \bigcup_{q \in Q} W_q$.

ii) each W_q is a C^r-manifold and the map $\pi : P \to Q$ given by $\pi(W_q) = q$ is C^r. Points of W_q are characterized by the fact that the distance from $\varphi_t(p)$ to $\varphi_t(q)$ goes to zero exponentially fast as t goes to $-\infty$.

Here we are using the definition of r-normally hyperbolicity and the Center Manifold Theorem as stated by C.Robinson in [6], for a proof see [4].

We will show that the submanifolds W_q are coisotropic. Recall that a subspace W of a vector space V is said to be *coisotropic* if the orthogonal space with respect to the form Ω

$$W^{\perp_\Omega} = \{v \in V : \Omega(v, w) = 0\, \forall w \in W\}$$

is contained in W. A submanifold W of P is *coisotropic* if for each point p in W the tangent space T_pW to W at p is coisotropic.

Lemma 2.1 *N_q is a coisotropic vector subspace of T_qP.*

Proof: Consider a non-zero vector v in $N_q^{\perp_\Omega}$. Then for all vectors u in N_q we have $\Omega(v, u) = 0$ which implies that $G(Jv, u) = 0$. Thus Jv belongs to $N_q^{\perp_G} = T_qQ$. But $J(T_qQ) \subset T_qQ^{\perp_G} = N_q$ since T_qQ is an isotropic vector space. Hence $v = -J(Jv)$ belongs to N_q. Therefore $N_q^{\perp_\Omega} \subset N_q$, which means that N_q is coisotropic. □

The definition of the strong unstable foliation W_q implies that:

1. $\pi \circ \varphi_s = \pi$, (since if p belongs to W_q for some q so does $\varphi_s(p)$, therefore $\pi(\varphi_s(p)) = \pi(p) = q$);

2. if Y is a vector tangent to W_q at a point p, then $(\varphi_s)_* Y$ is tangent to W_q at $\varphi_s(p)$;

3. if $X \in (T_p W_q)^{\perp_\Omega}$ then $(\varphi_s)_* X \in (T_{\varphi_s(p)} W_q)^{\perp_\Omega}$, (since $\varphi_s^* \Omega = e^s \Omega$).

We now prove

Proposition 2.2 *The W_q are coisotropic submanifolds of (P, Ω).*

Proof: Assume by contradiction, that for some p in W_q there is a non-zero vector v in $(T_p W_q)^{\perp_\Omega}$ that is not tangent to $T_p W_q$. Let α be a smooth curve tangent to v at p, i.e. $\alpha : (-\epsilon, \epsilon) \to P$ is a smooth map such that $\alpha(0) = p$ and $\dot\alpha(0) = v$. Since $d(\phi_s(x), \pi(x)) \to 0$ as $s \to -\infty$,

$$d(\varphi_s(\alpha(t)), \pi(\varphi_s(\alpha(t)))) = d(\varphi_s(\alpha(t)), \pi(\alpha(t))) \to 0$$

Therefore

$$\lim_{s \to -\infty} \varphi_s(\alpha(t)) = \lim_{s \to -\infty} \pi(\varphi_s(\alpha(t))) = \pi(\alpha(t))$$

pointwise.

Define $\beta(t) = \pi(\varphi_s(\alpha(t)))$. This is a curve in Q whose tangent vector at $t = 0$ is $\pi_*(v)$ and hence is non-zero. Further

$$\lim_{s \to -\infty} (\varphi_s)_* \dot\alpha(0) = \lim_{s \to -\infty} (\pi \circ \varphi_s)_* \dot\alpha(0) = \lim_{s \to -\infty} \pi_*(\dot\alpha(0)) = \dot\beta(0).$$

By 2. and 3. above, $(\varphi_s)_*(v)$ does not belong to $T_{\varphi_s(p)} W_q$ but it belongs to $(T_{\varphi_s(p)} W_q)^{\perp_\Omega}$. By continuity $\dot\beta(0)$ $(= \lim_{s \to -\infty} (\varphi_s)_*(v))$ does not belong to N_q but it belongs to $(N_q)^{\perp_\Omega}$, however this is impossible since N_q is a coisotropic subspace. □

3 Construction of the Darboux coordinates

In this section we construct Darboux coordinates adapted to the W_q. More precisely, our aim is to construct coordinates $(U; x_1, ..., x_n, y_1, ..., y_n)$ near a point q in Q such that

- $$Q \cap U = \{y_i = 0\}_{i=1}^{k} \cap \{x_r = 0\}_{r=k+1}^{n} \cap \{y_r = 0\}_{r=k+1}^{n} ,$$
- the leaf through a point q in $Q \cap U$ is given by
$$W_q \cap U = \{p \in U \ : \ x_i(p) = x_i(q) \ i = 1, ..., k\},$$
- the linear part of ξ is
$$L(\xi) = \sum_{i=1}^{k}(0x_i\frac{\partial}{\partial x_i} + 1y_i\frac{\partial}{\partial y_i}) + \sum_{r=k+1}^{n}(\frac{1}{2}x_r\frac{\partial}{\partial x_r} + \frac{1}{2}y_r\frac{\partial}{\partial y_r})$$
- and
$$\Omega = \sum_{i=1}^{k} dx_i \wedge dy_i + \sum_{r=k+1}^{n} 2dx_r \wedge dy_r.$$

Let us denote by W the foliation of P by the strong unstable manifolds W_q, where W_q is the leaf through a point q in Q. Since W is a coisotropic foliation $(TW)^{\perp}$ considered as a subbundle of TW is integrable.(See [10] p.11). Consequently $TW^{\perp}$ is the tangent bundle to an isotropic foliation of W, which we denote by $W^{\perp}$. Now let q be a point in the isotropic submanifold Q, and let U be a neighborhood of q in P sufficiently small so that the foliations of U defined by $W^{\perp}$ and W are simple, i.e. the set of the leaves of the foliation are smooth manifolds and the correspondings projections

$$\Phi : U \longrightarrow \frac{P \cap U}{W^{\perp} \cap U} = B_{W^{\perp}}$$

and

$$\Psi : U \longrightarrow \frac{P \cap U}{W \cap U} = B_W$$

are submersions. Observe that the dimension of the quotient manifolds $B_{W^{\perp}}$ and B_W are $2n - k$ and k respectively. We have

Proposition 3.1 *1. There is a unique Poisson structure on $B_{W^{\perp}}$ such that Φ is a Poisson morphism, whose rank at $\Phi(x)$ is equal to the rank of the 2-form $\Omega|_W$ induced by Ω on the leaf W_x through x of the foliation defined by W, which equals $2n - 2k$.*

2. There is a unique Poisson structure on B_W such that Ψ is a Poisson morphism, whose rank at $\Psi(x)$ equals the rank of $\Omega|_{W_x^\perp}$, which is zero since $W_x^\perp$ is isotropic.

For a proof see [5] (Chapter 3 §9) or [3]. We will use these unique Poisson structures on $B_{W^\perp}$ and B_W to construct the desired coordinates by lifting functions from $B_{W^\perp}$ and B_W to P. Since $\dim B_W = k$ and the rank of its Poisson structure is zero there are k coordinates functions $\bar{x}_i$ $i = 1, ..., k$ on B_W such that their Poisson brackets vanish, i.e. $\{\bar{x}_i, \bar{x}_j\} = 0$; $i, j = 1, ..., k$. (See also [11]).

Definition of the functions x_i:

Define on P, the functions $x_i = \bar{x}_i \circ \Psi$; $i, j = 1, ..., k$. Observe that each of the functions x_i is constant when restricted to each leaf of the foliation defined by W. Thus if $x = \Psi(q) \in B$ we have

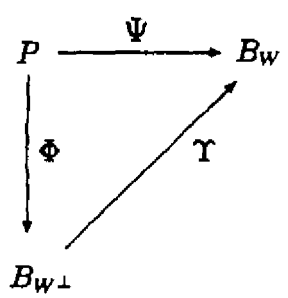
$$\Psi^{-1}(x) = W_q \cap U = \{p \in U \ : \ x_i(p) = x_i(q)\}.$$

Remark 3.2 *The functions x_i $i = 1, ..., k$ are first integrals of W and for each $i = 1, ..., k$ the Hamiltonian vector field ξ_{x_i} generated by x_i, is a section of $W^\perp$. Hence the $\{\xi_{x_i}\}_{i=1}^k$ span $TW^\perp$.*

Because the foliations of U defined by W and $W^\perp$ are simple and for each $p \in U$ we have

$$\frac{T_pP}{T_pW} \cong \frac{T_pP/T_pW^\perp}{T_pW/T_pW^\perp}$$

we can define a function $\Upsilon : B_{W^\perp} = P/W^\perp \to B_W = P/W$ such that the diagram



commutes. Therefore we also have k functions on $B_{W^\perp}$ defined by $\hat{x}_i = \tilde{x}_i \circ \Upsilon$ $i = 1, ..., k$.

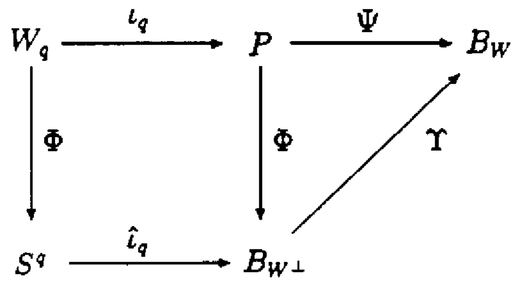
Now, for each q in $Q \cap U$ we may form the quotient manifold

$$S^q = \frac{W_q \cap U}{W_q^\perp \cap U}$$

contained in $B_{W^\perp}$. Observe that since $\Phi(W_q \cap U) = S^q$ we have

$$S^q = \{b \in B_{W^\perp} \ : \ \hat{x}_i(b) = \hat{x}_i(\Phi(q))\}$$

The tangent space to S^q at a point $b \in B_{W^\perp}$ may be identified with $T_pW_q/T_pW_q^\perp$ for some $p \in P$ such that $\Phi(p) = b$. Thus we have the following commutative diagram



Let σ^q be the 2-form defined on $T_pW_q/T_pW_q^\perp$ by

$$\sigma^q(X + W_q^\perp, Y + W_q^\perp) = \Omega|_{W_q}(X, Y)$$

where X, Y lie in TW_q. Note that

1. σ^q is well defined: (i.e. it does not depend on the point chosen on the leaf $TW_q^\perp$) Proof: For any section X of $TW_q^\perp$, we have

$$\mathcal{L}_X(\Omega|_{W_q}) = d(X \lrcorner \Omega|_{W_q}) + X \lrcorner d(\Omega|_{W_q}) = 0$$

since X lies in $TW_q^\perp$ and $d\Omega|_{W_q} = 0$. □

2. σ^q is non-degenerate: Proof: Given $Y \in TW_q - TW_q^\perp$ there exist $X \in TW_q - TW_q^\perp$ such that $\Omega(X, Y) \neq 0$ therefore $\sigma^q(X + W_q^\perp, Y + W_q^\perp) \neq 0$. □

3. σ^q is closed: Proof:

$$d\sigma^q(X + W_q^\perp, Y + W_q^\perp, Z + W_q^\perp) = \Phi^*(d\sigma^q)(X, Y, Z)$$
$$= d(\Phi^*\sigma^q)(X, Y, Z) = d\Omega|_{W_q}(X, Y, Z) = 0$$

□

It follows that there is a naturally determined symplectic structure on S^q for each $q \in Q \cap U$, whose pullback to W_q is $\Omega|_{W_q}$.

Proposition 3.3 *For each $q \in Q \cap U$, S^q is a Poisson submanifold of the Poisson manifold $B_{W^\perp}$.*

Proof: It is enough to show that all hamiltonian vector fields are tangent to S^q. (see [11] Lemma 1.1). Given any function $\hat{f}$ on $B_{W^\perp}$, the function $f = \hat{f} \circ \Phi$ defined on P is constant on $W_q^\perp$ for each q. Since the hamiltonian vector fields $\{\xi_{x_i}\}_{i=1}^k$ span $TW^\perp$ (Remark 3.2) we have that $\xi_{x_i} f = 0\ \forall i = 1, ..., k$, therefore $\{x_i, f\}_P = 0\ \forall i$. This implies that the derivative of the x_i in the direction of the hamiltonian vector field generated by f is zero, i.e. $\xi_f(x_i) = 0$, which implies that $dx_i(\xi_f) = 0$. Therefore the hamiltonian vector field ξ_f is tangent to W_q for each q. Hence the hamiltonian vector field generated by $\hat{f}$, $\xi_{\hat{f}} = \Phi_*(\xi_f)$ is tangent to $S^q = \Phi(W_q)$. Since $\hat{f}$ was an arbitrary function the proposition holds. □

We can consider q as a parameter and use Arnold's method to find Darboux coordinates on each S^q separately in such a way that they vary smoothly with respect to the parameter q. Denote them by x_r^q , y_r^q , $r = k+1, ..., n$. We can choose them such that $\{x_r^q, y_r^q\} = 2\delta_{rs}$.

Now define on $B_{W^\perp}$ functions $\widehat{x_r}, \widehat{y_r}$, $r = k+1, ..., n$ so that the restriction $\widehat{x_r}|_{S^q}$ of $\widehat{x_r}$ to S^q equals x_r^q and the restriction $\widehat{y_r}|_{S^q}$ of $\widehat{y_r}$ to S^q equals y_r^q.

Proposition 3.4 *The functions $\widehat{x_r}$, $\widehat{y_r}, r = k+1, ..., n$ together with the functions $\widehat{x_i}, i = 1, ..., k$ form a coordinate system which satisfy the following bracket relations: $\{\widehat{x_s}, \widehat{y_r}\} = 2\delta_{sr}$*

$\{\widehat{y_s}, \widehat{y_r}\} = \{\widehat{x_s}, \widehat{x_r}\} = \{\widehat{y_s}, \widehat{x_i}\} = \{\widehat{x_s}, \widehat{x_i}\} = \{\widehat{x_j}, \widehat{x_i}\} = 0$

for all $s, r = k+1, ..., n$ and $i, j = 1, ..., k$.

Proof: By the Proposition 3.3 , the hamiltonian vector fields $\xi_{\widehat{y_r}}$ and $\xi_{\widehat{x_r}}$ generated repectively by the functions $\widehat{y_r}, \widehat{x_r}$, $r = k+1, ..., n$ are tangent to S^q. Since each $\widehat{x_i}$ is constant on S^q we have that

$$0 = \xi_{\widehat{y_r}} \widehat{x_i} = \{\widehat{x_i}, \widehat{y_r}\}_{B_W^\perp}$$

and

$$0 = \xi_{\widehat{x_r}} \widehat{x_i} = \{\widehat{x_i}, \widehat{x_r}\}_{B_W^\perp}$$

for all $i = 1, ..., k$ and $r = k+1, ..., n$. Also

$$\{\widehat{x_j}, \widehat{x_i}\}_{B_W^\perp} = \{\check{x}_j, \check{x}_i\}_{B_W} \circ \Upsilon = 0 \, .$$

And finally

$$\{\widehat{x_r}, \widehat{y_s}\}_{B_W^\perp} \circ \iota_q = \{x_r^q, y_s^q\} = 2\delta_{sr}.$$

Therefore for all $b \in S^q$ and all $q \in Q \cap U$ we have $\{\widehat{x_r}, \widehat{y_s}\}_{B_W^\perp}(b) = 2\delta_{sr}$. The other brackets relations can be obtained in a similar way. □

Definition of the functions x_r, y_r :

We define the functions

$x_i = \widehat{x_i} \circ \Phi$, $i = 1, ..., k$

$x_r = \widehat{x_r} \circ \Phi$, $y_r = \widehat{y_r} \circ \Phi$; $r = k+1, ..., n$.

Since Φ is a Poisson morphism, it follows that for each p in U:

$\{y_s, x_r\}_P(p) = \{\widehat{y_s} \circ \Phi, \widehat{x_r} \circ \Phi\}_P(p) = \{\widehat{y_s}, \widehat{x_r}\}_{B_{W\perp}} \Phi(p) = 2\delta_{sr}\Phi(p)$

$\{y_s, y_r\}_P(p) = \{\widehat{y_s} \circ \Phi, \widehat{y_r} \circ \Phi\}_P(p) = \{\widehat{y_s}, \widehat{y_r}\}_{B_{W\perp}} \Phi(p) = 0$

In a similar manner we get the remaining bracket relations.

Summarizing we have, on the neighborhood U functions x_i, x_r, y_r ; $i = 1, ..., k$; $r = k+1, ..., n$ satisfying the relations:

$$\clubsuit \begin{cases} \{x_r, y_s\} = 2\delta_{rs} \\ \{y_s, y_r\} = \{x_s, x_r\} = \{y_s, x_i\} = \{x_s, x_i\} = \{x_j, x_i\} = 0 \\ \text{for all } s, r = k+1, .., n \text{ and } i, j = 1, ..., k. \end{cases}$$

Further the coisotropic leaf through the point $q \in Q \cap U$ is given by $W_q = \{p \in P \ : \ x_i(p) = x_i(q)\}$.

Let us denote by C the set of points in U whose x_r and y_r-coordinates $r = k+1, ..., n$ vanish. Notice that this is the union of the isotropic leaves $W_q^\perp$ intersecting the isotropic submanifold $Q \cap U$. We want to show that there is a neighborhood V of q_0 contained in C, such that the restriction of

Ω to V is symplectic. Since being symplectic is an open condition it suffices to show the following

Proposition 3.5 *The restriction of Ω to the tangent space to C at q_0 is nondegenerate.*

Proof: Let ξ_{x_r} and ξ_{y_r}; $r = k+1, ..., n$ be the hamiltonian vector fields corresponding to the hamiltonian functions x_r, y_r respectively, i.e. $\xi_{x_r} \lrcorner \Omega = dx_r$, $\xi_{y_r} \lrcorner \Omega = dy_r$ for all r. Let X be a vector tangent to C at q_0, and take any extension of X to a neighborhood of q_0. Since Ω depends only on the values of the vector field at the point, we have that $\Omega(X, \xi_{x_r}) = Xx_r = 0$ and $\Omega(X, \xi_{y_r}) = Xy_r = 0$ for all $r = k+1, ..., n$ because $x_r = y_r = 0$ on C and $X \in T_{q_0}C$. Therefore X is Ω-orthogonal to the vectors $\xi_{x_r}(q_0)$, $\xi_{y_r}(q_0)$ for all $r = k+1, ..., n$. Note that these vectors span a subspace complementary to $T_{q_0}C$ since their projections onto $B_{W^\perp}$ span $T(S_q)$. Therefore, if X were Ω-orthogonal to the whole $T_{q_0}C$, X would be orthogonal to $T_{q_0}P$, which is not possible because Ω is nondegenerate. □

By Darboux Theorem there are symplectic coordinates in a neighborhood V of the point q_0 in C. Let us denote them by $(\bar{x}_1, ..., \bar{x}_k, \bar{y}_1, ..., \bar{y}_k)$. It is possible to choose them such that $\bar{x}_i(p) = x_i(p)$, $p \in V$ and such that $\frac{\partial}{\partial \bar{y}_i}$ is tangent to $\frac{\partial}{\partial y_i}$.

Definition of the functions y_i:

Extend the coordinates ($\bar{x}_i$ and) $\bar{y}_i$, $i = 1, ..., k$ from V to U using the hamiltonian flows of x_r, y_r; $r = k+1, ..., n$ in the following way: Denote by $g^{t_r}_{x_r}$, $g^{s_r}_{y_r}$; $r = k+1, ..., n$ the hamiltonian flows with hamiltonian functions x_r, y_r respectively. Because the Poisson brackets of the hamiltonian functions are constant (♣-relations), their flows commute. Therefore every point p in some neighborhood $U_1 \subset U$ of the point $q_0 \in Q$, can be uniquely represented in the form

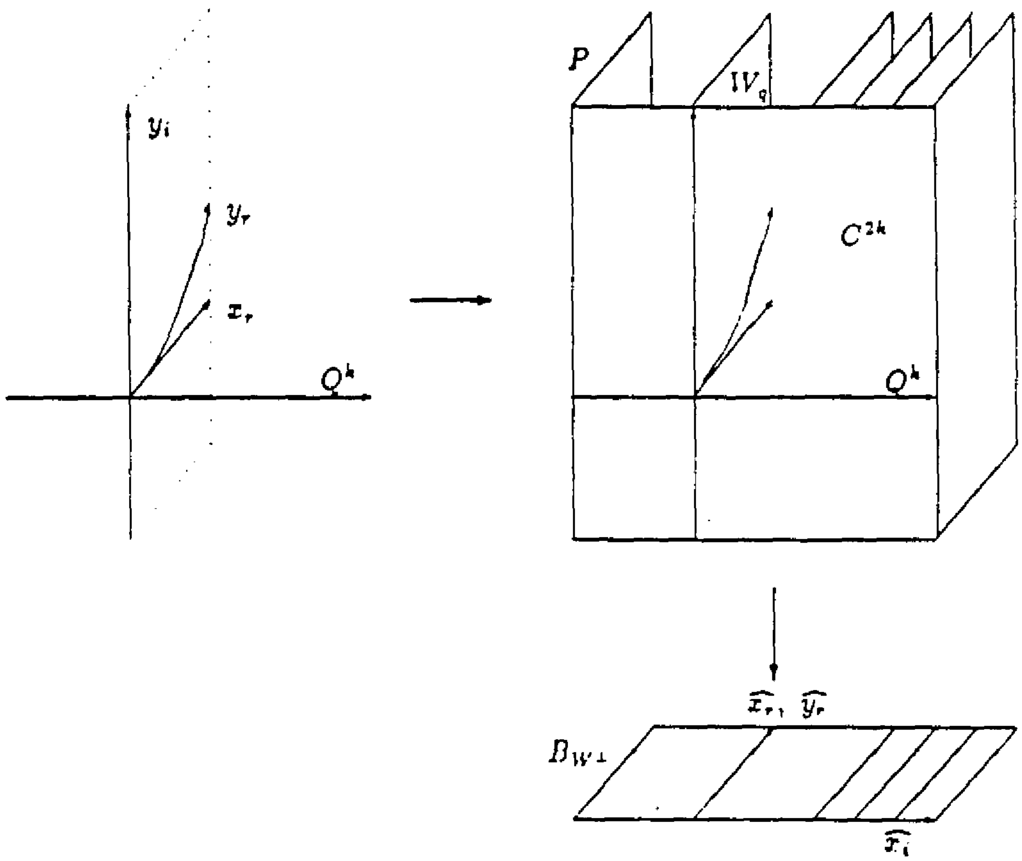
$$p = g^{t_{k+1}}_{x_{k+1}} \circ ... \circ g^{t_n}_{x_n} \circ g^{s_{k+1}}_{y_{k+1}} \circ ... \circ g^{s_n}_{y_n}(p') \stackrel{\text{def}}{=} \theta(p')$$

where p' belongs to the neighborhood $V \subset C$ and s_r , $t_r \in \mathbf{R}$; $r = k+1, ..., n$.

Define: $y_i(p) = \bar{y}_i(p')$ and $c_i(p) = \bar{x}_i(p')$, $i = 1, ..., k$.

We claim that the values of the extensions c_i of $\bar{x}_i$ agree with those of the old functions x_i Proof: For the ♣-relations $0 = \{c_i, x_r\} = \xi_{x_r} c_i$ and

$0 = \{c_i, y_r\} = \xi_{y_r} c_i \; i = 1, ..., k \; ; \; r = k+1, ..., n$ imply that each c_i is a first integral of the hamiltonian functions x_r, y_r; $r = k+1, ..., n$ and so is constant along the corresponding hamiltonian flows. Similar remarks apply to the x_i. Thus $x_i(p') = c_i(p') = c_i(\theta(p')) = c_i(p)$ and $x_i(p') = x_i(\theta(p')) = x_i(p)$ so that $x_i(p) = c_i(p)$. □



Proof that the coordinates are symplectic:

Because of the way the functions $\bar{y}_i$, $i = 1, ..., k$ were extended, each of this extensions y_i is invariant with respect to the flows $g^{t_r}_{x_r}$, $g^{s_r}_{y_r}$; $r = k+1, ..., n$. Thus the Poisson brackets of the y_i with the x_r, y_r are equal to zero: $\{y_i, x_r\} = \{y_i, y_r\} = 0$ $i = 1, ..., k$; $r = k+1, ..., n$. Therefore the hamiltonian flows $g^s_{y_i}$ with hamiltonian functions y_i $i = 1, ..., k$ commute with the $g^{t_r}_{x_r}$, $g^{s_r}_{y_r}$. Since the hamiltonian flows preserve the symplectic form Ω, we have that the values of Ω on the hamiltonian vector fields at p are the same of those at p', and these equal the values of the Poisson brackets, i.e.

$$\{y_i, y_r\}(p) = \Omega(\xi_{y_i}(p), \xi_{y_r}(p)) = \Omega(\xi_{y_i}(p'), \xi_{y_r}(p')) = \{y_i, y_r\}(p').$$

Further because $0 = \{x_r, y_i\} = \Omega(\xi_{x_r}, \xi_{y_i})$, and similarly $0 = \{y_r, y_i\} = \Omega(\xi_{y_r}, \xi_{y_i})$ $i = 1, ..., k$; $r = k+1, ..., n$, the functions x_r and y_r are first integrals of the hamiltonian vector fields ξ_{y_i}. Therefore the ξ_{y_i} restrict to hamiltonian vector fields on the symplectic manifold $(V, \Omega|_V)$ and the corresponding hamiltonian functions are $y_i|_V = \bar{y}_i$. Thus in the whole neighborhood $U_1 \subset U$, the Poisson brackets of the y_i with themselves is the same as the Poisson brackets of these coordinates in (V, Ω), i.e. $\{y_i, y_j\}(p') = \{\bar{y}_i, \bar{y}_j\}(p')$. Hence on the neighborhood $U_1 \subset U \subset P$ of q_0 $\{y_i, y_j\}(p) = \{\bar{y}_i, \bar{y}_j\}(p') = 0$. Similarly we get $\{y_i, x_j\}(p) = \delta_{ij}$. We already had $\{x_i, x_j\}(p) = 0$. We may assume $U_1 = U$.

The Poisson bracket of the coordinate functions determine the shape of the symplectic form uniquely, thus

$$\Omega = \sum_{i=1}^{k} dx_i \wedge dy_i + \sum_{r=k+1}^{n} 2dx_r \wedge dy_r$$

in the neighborhood U of q_0. (We could have choose x_i, y_i such that they vanish at q_0, so that all the coordinates vanish at q_0.)

4 The relation with the Liouville vector field

In this section we show that the matrix of the linear part of the Liouville vector field ξ at the point q_0 is diagonal with respect to the Darboux coordinates adapted to the coisotropic foliation W_q.

Let us write $\xi = L(\xi) + \eta$ where $L(\xi)$ denotes the linear part of ξ at q_0 and η contains terms of order ≥ 2 only. Then, for $q_0 \in Q$ we have that
$d(\eta \lrcorner \Omega)(q_0) = 0$ and $d(L(\xi) \lrcorner \Omega)(q_0) = \Omega(q_0)$. Therefore since
$d(\xi \lrcorner \Omega) = d(L(\xi) \lrcorner \Omega) + d(\eta \lrcorner \Omega) = \Omega$ and Ω is linear it follows that
$d(\eta \lrcorner \Omega) = 0$ and $d(L(\xi) \lrcorner \Omega) = \Omega$.

Now choose new coordinates $(z_1, ..., z_{2n})$ in a neighborhood of q_0 in P which depend linearly on the coordinates (x_i, y_i, x_r, y_r) such that $L(\xi)$ is diagonal with respect to the $(z_1, ..., z_{2n})$. We will take $z_i = x_i$ for $i = 1, ..., k$. Then

$$L(\xi) = \sum_{\ell=1}^{2n} \lambda_\ell z_\ell \frac{\partial}{\partial z_\ell}.$$

Further, because ξ preserves the submanifolds W_q and $W_q^\perp$ each of the subspaces $T_q(W_q)$ and $T_q(W_q^\perp)$ have a basis consisting of some of the $\frac{\partial}{\partial z_\ell}$ which diagonalize $L(\xi)$. We will assume that $\{\frac{\partial}{\partial z_j}\}_{j=k+1}^{2k}$ span $T_q(W_q^\perp)$. Therefore

$$\frac{\partial}{\partial y_i} = \sum_{j=k+1}^{2k} A_{ij} \frac{\partial}{\partial z_j}$$

and the transformation matrix has the form

$$\begin{pmatrix} \frac{\partial}{\partial x_i} \\ \frac{\partial}{\partial y_i} \\ \frac{\partial}{\partial w_s} \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z_i} \\ \frac{\partial}{\partial z_j} \\ \frac{\partial}{\partial z_\alpha} \end{pmatrix}$$

where

$$\left\{ \begin{array}{rcl} i & = & 1, ..., k \\ j & = & k+1, ..., 2k \\ s, \alpha & > & 2k \end{array} \right. \text{ and } \frac{\partial}{\partial w_s} = \left\{ \begin{array}{ll} \frac{\partial}{\partial x_r} & for\ s = k + r, \\ \frac{\partial}{\partial y_r} & for\ s = n + r, \\ where & r = k+1, .., n. \end{array} \right.$$

Thus

$$\begin{aligned} x_i &= z_i \\ y_i &= \sum_{k<j\leq 2k} a_j^i z_j + \sum_{\alpha>2k} a_\alpha^i z_\alpha \\ x_r &= \sum_{\alpha>2k} b_\alpha^r z_\alpha \end{aligned}$$

$$y_r = \sum_{\alpha>2k} c^r_\alpha z_\alpha$$

for some constants $a^i_j, a^i_\alpha, b^r_\alpha, c^r_\alpha$.

Thus with respect to the new coordinates

$$\Omega = \sum_{i=1,j=k+1}^{k,2k} a^i_j dz_i \wedge dz_j + \sum_{i=1,\alpha>k}^{k} a^i_\alpha dz_i \wedge dz_\alpha + 2\sum_{\alpha,\beta} b^r_\alpha c^r_\beta dz_\alpha \wedge dz_\beta$$

Hence we obtain

$$\begin{aligned} d(\mathrm{L}(\xi) \lrcorner \Omega) &= \sum(\lambda_i + \lambda_j) a^i_j dz_i \wedge dz_j + \sum(\lambda_i + \lambda_\alpha) a^i_\alpha dz_i \wedge dz_\alpha \\ &\quad +2\sum(\lambda_\alpha + \lambda_\beta) b^r_\alpha c^r_\beta dz_\alpha \wedge dz_\beta \end{aligned}$$

Because $\lambda_i = 0$ for $i = 1, ..., k$, $d(\mathrm{L}(\xi) \lrcorner \Omega) = \Omega$ implies that $\lambda_j = 1$ for all $a^i_j \neq 0$, $\lambda_\alpha = 1$ for all $a^i_\alpha \neq 0$ and $\lambda_\alpha + \lambda_\beta = 1$ for all $b^r_\alpha c^r_\beta \neq 0$. By hypothesis there are k eigenvalues equal to 1 and $2n - 2k$ equal to $1/2$. Since the matrix corresponds to a change of coordinates, the submatrix (a^i_j) is non-singular. Therefore all the $\lambda_j = 1$, and all a^i_α are zero.

Hence the eigenspace for $L(\xi)$ corresponding to the eigenvalue $\lambda = 1/2$ is exactly the space spanned by $\{\frac{\partial}{\partial z_r}, \frac{\partial}{\partial y_r}\}_{r=k+1}^{n}$ and the eigenspace corresponding to the eigevalue $\lambda = 1$ is exactly the space spanned by $\{\frac{\partial}{\partial y_i}\}$.

5 The linearizing conjugation

In order to prove that the Liouville vector field ξ is smoothly conjugate to its linear part, we shall use a linearization theorem due to G.Sell [8], which extends the linearization theorem of Sternberg [9] to the case of a vector field with resonant eigenvalues. We shall exploit the fact that the vector ξ is Liouville, together with the explicit algorithm that Sell's Theorem provides to compute a lower bound for the order of smoothness of the conjugacy. Samovol [7] proves a more general theorem which implies Sell's theorem, but we use Sell's version because it describes the dependence of the linearizing conjugation on the parameter set.

In the Darboux coordinate chart $(U, x_1, ..., x_n, y_1, ..., y_n)$ about a singular point q_0, adapted to the strong unstable coisotropic foliation the isotropic submanifold Q is given by

$$Q \cap U = \{y_i = 0\}_{i=1}^{k} \cap \{x_r = 0\}_{r=k+1}^{n} \cap \{y_r = 0\}_{r=k+1}^{n} ,$$

the leaf through a point q in $Q \cap U$ is given by

$$W_q \cap U = \{p \in U \ : \ x_i(p) = x_i(q) \ i = 1, ..., k\}$$

and all the coordinate functions vanish at the singular point q_0.

For each q in $Q \cap U$ the integral curves of the vector field $\xi|_{W_q}$ satisfy the following equation

$$(I) \qquad\qquad \dot{w} = A(q)w + F(w, q)$$

where $A(q)$ and $F(w, q)$ depend smoothly on $q \in Q \cap U$ and $w \in U$, where $w = (x_{k+1}, ..., x_n, y_1, ..., y_n)$. Furthermore the matrix $A(q)$ does not have zero eigenvalues, that is, $A(q)$ is hyperbolic. Because the eigenvalues are positive $A(q)$ is said to be unstable.

In order to go further we need the following

Definition 5.1 *A collection of non-zero eigenvalues is resonant if one of them is an integral linear combination (with nonnegative coefficients whose sum is at least two) of the others.*

i.e. Let $\lambda_1, ..., \lambda_N$ be a set of non-zero eigenvalues repeated with multiplicities and let $m = (m_1, ..., m_N)$ be nonnegative integers.

Define $|m| = \sum m_i$ and $\gamma(\lambda_i, m) = \lambda_i - \sum m_r \lambda_r$.

Then if a relation $\gamma(\lambda_i, m) = 0$ holds for $|m| \geq 2$ the eigenvalues are said to be in resonance , and $|m|$ is called the order of resonance.

If $(z^1, ..., z^N)$ are coordinates with respect to the basis $(e_1, ..., e_N)$, let z^m stand for $z_1^{m_1} ... z_N^{m_N}$.

Definition 5.2 *The vector valued monomial $z^m e_i$ is resonant if $\gamma(\lambda_i, m) = 0$ and $|m| \geq 2$.*

When A is hyperbolic, let $\sum^+(A)$ denote those eigenvalues λ of A with $Re\lambda > 0$ and $\sum^-(A)$ those with $Re\lambda < 0$. If $\sum^i(A) \neq \emptyset$ where $i = +$ or $-$, *the spectral spread* is defined to be

$$\rho^i = \frac{max\{|Re\lambda| \; : \; \lambda \in \sum^i(A)\}}{min\{|Re\lambda| \; : \; \lambda \in \sum^i(A)\}}.$$

Definition 5.3 *The r-smoothness of A is the largest integer $K \geq 0$ such that*

1. $r - K\rho^- \geq 0$, *if* $\sum^+(A) = \emptyset$
2. $r - K\rho^+ \geq 0$, *if* $\sum^-(A) = \emptyset$
3. *There exist positive numbers M, N with $r = M + N$, $M - K\rho^+ \geq 0$, $N - K\rho^- \geq 0$ if $\sum^+(A) \neq \emptyset$ and $\sum^-(A) \neq \emptyset$.*

Now suppose that the following condition holds for some integer $r \geq 2$. (This is condition "B" in [8]).

$$(II)\begin{cases} D^jF(q_0, q) = 0 & \text{for } 0 \leq j \leq r-1 \\ \text{and } Re\gamma(\lambda, m) \neq 0 & \text{for all } \lambda \in \sum A(q) = \sum^+ A(q) \cup \sum^- A(q), \\ \text{for all } m \text{ with } |m| = r & \text{and for all } q \in \hat{V} \text{ neighborhood of } q_0 \end{cases}$$

Then Sell's Theorem asserts that there is a C^K-smooth linearizing conjugation $w = z + \Phi(z, q)$ between (I) and $\dot{z} = A(q)z$, where Φ varies smoothly in terms of the parameter q and is of class C^K in z, where K is the r-smoothness of $A(q_0)$. The change of variable introduced by Sell gives rise to a related nonlinear differential equation on a different finite dimensional Banach space. The quantities $\gamma(\lambda_i, m) = \lambda_i - \sum m_r\lambda_r$, for $\lambda \in \sum A(q)$ and $|m| = r$, arise as the eigenvalues of the associated linear equation. Further the vector values monomials $z^m e_i$ are in one to one correspondence with the eigenvectors of the associated linear operator.

In the situation we are considering, the eigenvalues of the Liouville vector field ξ (i.e. the eigenvalues of $A(q)$) at the singular point q satisfy the integral relations

$$\lambda_i = 1\lambda_r + 1\lambda_s$$

where $\lambda_i = 1$, $i = 1, ..., k$ $\lambda_r = \lambda_s = \frac{1}{2}$, $r, s = k+1, ..., n$. According to definition 5.2 , the possible resonant monomials in each fiber are

$$(III) \qquad x_r x_s \frac{\partial}{\partial y_i} \ , \ y_r y_s \frac{\partial}{\partial y_i} \ , \ x_r y_s \frac{\partial}{\partial y_i}$$

for $i = 1, ..., k$; $r, s = k+1, ..., n$.

Recall that the linear part $L(\xi)$ of the Liouville vector field ξ in these coordinates is

$$L(\xi) = \sum_{i=1}^{k} y_i \frac{\partial}{\partial y_i} + \sum_{r=k+1}^{n} \left(\frac{1}{2} x_r \frac{\partial}{\partial x_r} + \frac{1}{2} y_r \frac{\partial}{\partial y_r}\right)$$

Thus

$$\xi = L(\xi) + \sum_{r=k+1}^{n} E_r \frac{\partial}{\partial x_r} + \sum_{\ell=1}^{n} F_\ell \frac{\partial}{\partial y_\ell}$$

where for each $q = (x_1, ..., x_k)$ the functions $E_r(q, \cdot), F_\ell(q, \cdot)$ vanish to higher order than $(\sum_{r=k+1}^{n} |x_r|^2 + \sum_{l=1}^{n} |y_\ell|^2)^{1/2}$. Denote by ξ^q the restriction of ξ to W_q

Proposition 5.4 *For each q the Taylor expansion of ξ^q contains no resonant quadratic terms.*

Proof: The Taylor expansions of ξ^q fit together to give a Taylor expansion of the vector field ξ in terms of the coordinates y_i, x_s, y_s, $i = 1, ..., k$; $s = k+1, ..., n$, with coefficients which are functions of the x_i, $i = 1, ..., k$. It suffices to show that this expansion has no terms of the form (III). Notice that the vector field $\eta = \xi - L(\xi)$ is hamiltonian since

$$\mathcal{L}_\eta \Omega = d([\xi - L(\xi)] \lrcorner \Omega) = \Omega - \Omega = 0.$$

Let H be a hamiltonian function for η (i.e. $\eta \lrcorner \Omega = dH$). If η (therefore ξ) had any resonant monomial, then the usual Taylor expansion of the hamiltonian function H in terms of all the x_l, y_l would contain nonzero terms of the type:

$$x_i x_r x_s, \ x_i x_r y_s, \ x_i y_r y_s$$

for $1 \le i \le k$; $k+1 \le s, r \le n$. Consequently η would also contain terms of the type

$$x_i x_s \frac{\partial}{\partial x_r}, \ x_i x_s \frac{\partial}{\partial y_r}, \ x_i y_s \frac{\partial}{\partial x_r} \text{ or } x_i y_s \frac{\partial}{\partial y_r}$$

which is impossible since the functions E_r and F_ℓ do not contain terms which depend linearly on x_s or y_s, $k+1 \leq s \leq n$. Note that for each leaf W_q, the x_i are constants and so are coefficients. □

Construction of the conjugacy

We have that $D^0F(q_0,q) = D^1F(q_0,q) = 0$. Even if the second part of condition (II) does not hold for $r=2$, by proposition 5.4 we still can apply Sell's algorithm. Since the eigenvalues of $A(q_0)$ are 1 or 1/2, the spectral spread ρ^+ equals 2. Thus, because $\sum^- A(q_0) = \emptyset$ the r-smoothness of $A(q_0)$ is the largest integer $K \geq 0$ such that $2 - K2 \geq 0$. Thus $K=1$. Hence Sell's theorem guarantees that the linearizing conjugation Φ is at least of class C^1.This completes the proof of Theorem 1.1.

Applications of this result will appear in a forthcoming paper.

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