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THE SUBGROUPS IN THE SPECIAL LINEAR GROUP OVER A SKEW FIELD THAT CONTAIN THE GROUP OF DIAGONAL MATRICES *

Bui Xuan Hai **
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

For an arbitrary skew field T we study the lattice of subgroups of the special linear group $\Gamma = SL(n,T)$ that contain the sugbroup $\Delta = SD(n,T)$ of diagonal matrices with Dieudonne's determinant equal to 1. We show that the description of these subgroups is standard in the following sense: For any subgroup $H, \Delta \leq H \leq \Gamma$ there exists a unique unital net such that $\Gamma(\sigma) \leq H \leq N(\sigma)$, where $\Gamma(\sigma)$ is the net subgroup that corresponds to the net σ and $N(\sigma)$ is the normalizer of $\Gamma(\sigma)$ in Γ.

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** Permanent address: Department of Mathematics, University of Ho Chi Minh City, Ho Chi Minh City, Vietnam.

§1. Introduction.

The problem on the standard description of subgroups in the special linear group $SL(n,K), n \geqslant 2$, over a field K that contain the group $SD(n,K)$ of diagonal matrices was studied by several authors. Seitz [9] considered the case $n \geqslant 2$, $K = F_q$ (F_q is the finite field that consists of q elements), where $q \geqslant 13$, $q \neq 2^m$. The case $n = 2$, K infinite, $char(K) \neq 2$ and $Card(K^*/K^{*2}) < Card(K^*)$ was studied by Vavilov and Dybcova (see [10]). Later in a series of articles (see [5]) Vavilov gave the standard description of these subgroups for the case $n \geqslant 3$ and K an arbitrary field with the condition that $Card(K) \geqslant 7$. In [11] there are several examples from which it follows that the condition $Card(K) \geqslant 7$ is necessary. Hence the result of Vavilov in [5] is the best possible for an arbitrary field K. In this article we continue with our study of the analogous problem for the case of skew fields began in [4] where we gave the standard description of these subgroups in the special linear group over a skew field with infinite center. Here we will discuss the case of a skew field with an arbitrary center which contains no less than seven elements. We adopt the following notation in our article: Λ is a associative ring with 1, Λ^* is the group of invertible elements of Λ. $G = GL(n,\Lambda)$ is the full linear group of degree n over Λ; $D = D(n,\Lambda)$ is the subgroup of diagonal matrices in G; $e = e_n$ is the identity matrix of degree n; e_{ij} is the matrix having 1 in position (i,j) and zero everywhere else; $t_{ij}(\alpha)$ is the elementary transvection $e + \alpha e_{ij}$, $\alpha \in \Lambda$ $(i \neq j)$; $d_r(\varepsilon)$ is the diagonal matrix $e + (\varepsilon - 1)e_{rr}$ ($\varepsilon \in \Lambda^*$, $1 \leqslant r \leqslant n$); T is a skew field; Z is the center of T; $[T^*,T^*]$ is the commutant of the multiplicative group T^* of T; $\Gamma = SL(n,T)$ is the special linear group over T of degree n; $\Delta = SD(n,T)$ is the subgroup of diagonal matrices with Dieudonne's determinant equal to 1; $d_{rs}(\varepsilon)$ is the diagonal matrix $e + (\varepsilon - 1)e_{rr} + (\varepsilon^{-1} - 1)e_{ss}$ for $r \neq s$ and $\varepsilon \in T^*$. If $a = (a_{ij})$ is an invertible matrix then we denote $a^{-1} = (a'_{ij})$.

§2. Net and net subgroups.

Let Λ be an arbitrary associative ring with identity 1. We consider, for a natural number n, the square array

$$\sigma = (\sigma_{ij}), \quad 1 \leq i,j \leq n$$

where all σ_{ij} are two-sided ideals of Λ. This array is called a net of ideals in Λ of degree n (in brief, a net of degree n over Λ) if

$$\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij} \qquad (1)$$

for all values of the indices i, j and r.

A net σ is called an unital net if $\sigma_{ii} = \Lambda$ for all i = 1, 2, ..., n. For a given net σ, we denote by $M(\sigma)$ the collection of all matrices $a = (a_{ij})$ in the ring $M(n,\Lambda)$ of matrices of degree n over Λ for which $a_{ij} \in \sigma_{ij}$ for all i and j. In addition we have the following concept which was introduced in [2].

DEFINITION. Let σ be an arbitrary net of degree n over a ring Λ. The largest subgroup of the full linear group $G = GL(n,\Lambda)$ contained in the multiplicative system $e + M(\sigma)$ is called the net subgroup in $GL(n,\Lambda)$ that corresponds to the net σ, and is denoted by $G(\sigma)$. If σ is an unital net then $G(\sigma)$ is also called an unital net subgroup.

Let R be a commutative ring with 1. The subgroup of the group GL(n,R) that consists of all invertible matrices with determinant equal to 1 is called the special linear group of degree n over R. This group is denoted by $\Gamma = SL(n,R)$. If R is a commutative semilocal ring, this subgroup coincides with the subgroup E(n,R) of GL(n,R) generated by all elementary transvections $t_{ij}(\alpha)$, $\alpha \in R$. Therefore for an arbitrary semilocal ring Λ, we will consider the subgroup $E(n,\Lambda)$ as the special linear group of degree n over Λ and denote it by $\Gamma = SL(n,\Lambda)$.

DEFINITION. For an arbitrary net σ over a semilocal ring Λ, the subgroup $\Gamma(\sigma) = \Gamma \cap G(\sigma)$ is called a net subgroup in the group Γ that corresponds to the net σ. If σ is an unital net then $\Gamma(\sigma)$ is also called an unital net subgroup in Γ.

§3. The associated net.

Let Λ be an associative ring with identity 1. We consider a subgroup H, $D(n,\Lambda) \leq H \leq GL(n,\Lambda)$. If Λ is generated by its invertible elements as a ring, i.e. if each element of Λ is represented by a sum of invertible elements, then the subgroup H corresponds to some unital net of degree n over Λ. In fact, if $i \neq j$ we put

$$\sigma_{ij} = \{\alpha \in \Lambda \ / \ t_{ij}(\alpha) \in H\}$$

Then σ_{ij} is the two-sided ideal in view of the following obvious formulas

$$d_i(\varepsilon)t_{ij}(\alpha)d_i(\varepsilon^{-1}) = t_{ij}(\varepsilon\alpha), \qquad \varepsilon \in \Lambda^*$$
$$d_j(\varepsilon)t_{ij}(\alpha)d_j(\varepsilon^{-1}) = t_{ij}(\alpha\varepsilon), \qquad \varepsilon \in \Lambda^*$$

If, moreover we put $\sigma_{ii} = \Lambda$ for all i = 1, 2, ..., n, we will have the unital net $\sigma = (\sigma_{ij})$. This net is called the unital net associated with the subgroup H.

Now let Λ be a semilocal ring and $\Gamma = SL(n,\Lambda)$ be the special linear group over Λ. Consider a subgroup H, $ED(n,\Lambda) \leq H \leq SL(n,\Lambda)$ where $ED(n,\Lambda) = E(n,\Lambda) \cap D(n,\Lambda)$. This subgroup H is called an intermediate subgroup. Put

$$\sigma_{ij} = \{\alpha \in \Lambda \ / \ t_{ij}(\alpha) \in H\}, \quad i \neq j$$

Then σ is the two-sided ideal in view of the following formulas

$$t_{ij}(\varepsilon\alpha) = d_{ik}(\varepsilon)t_{ij}(\alpha)d_{ik}(\varepsilon^{-1}), \quad \varepsilon \in \Lambda^*, \ k \neq i$$
$$t_{ij}(\alpha\varepsilon) = d_{jk}(\varepsilon)t_{ij}(\alpha)d_{jk}(\varepsilon^{-1}), \quad \varepsilon \in \Lambda^*, \ k \neq j$$

Moreover, for all i = 1, 2, ..., n we put $\sigma_{ii} = \Lambda$. Then the array $\sigma = (\sigma_{ij})$ is the unital net associated with the subgroup H.

Now, let T be a skew field, $\Gamma = SL(n,T)$ be the special linear group over T and $\Delta = SD(n,T)$ be the subgroup of diagonal matrices with Dieudonne's determinant equal to 1. Let Z be the center of T. If Z is infinite, then for all $n \geqslant 3$ the standard description of the intermediate subgroups was obtained in [4]. In fact, the following theorem was proved.

THEOREM 1(see [4]). Let T be a skew field with infinite center and $n \geqslant 3$. Let $\Gamma = SL(n,T)$ and $\Delta = SD(n,T)$. Then for each subgroup H, $\Delta \leqslant H \leqslant \Gamma$, there exists a unique unital net σ of degree n over T such that

$$\Gamma(\sigma) \leqslant H \leqslant N(\sigma)$$

where $\Gamma(\sigma)$ is the net subgroup in Γ that corresponds to the net σ and $N(\sigma)$ is the normalizer of the subgroup $\Gamma(\sigma)$ in Γ.

In this article we consider the case of a skew field T with a weaker condition where the center of T contains no less than seven elements.

§4. The subgroups containing an elementary transvection.

We say that the intermediate subgroup H belongs to the standard description if there exists a unique unital net σ such that H is between the net subgroup $\Gamma(\sigma)$ and its normalizer $N(\sigma)$ in Γ, i.e. $\Gamma(\sigma) \leqslant H \leqslant N(\sigma)$.

In this paragraph we consider the intermediate subgroups that contain an elementary transvection. We will prove that for all $n \geqslant 3$ and when the center of a skew field contains no less than seven elements each such subgroup belongs to the standard description.

THEOREM 2. Let T be a skew field, Card $(Z) \geqslant 7$ and $n \geqslant 3$. Suppose that an intermediate subgroup H satisfies at least one of the following conditions

(i) H is an irreducible subgroup

(ii) H is an imprimitive subgroup

(iii) H contains an elementary transvection

Then there exists a unique unital net of degree n over T such that

$$\Gamma(\sigma) \leqslant H \leqslant N(\sigma)$$

We will prove this theorem with the help of the following easily proved auxiliary lemmas. Everywhere for a given intermediate subgroup H, $\sigma = (\sigma_{ij})$ always is the net associated with H.

1. The auxiliary lemmas.

LEMMA 1. Let $\Delta \leqslant H \leqslant \Gamma$. The subgroup H is contained in the normalizer $N(\sigma)$ if and only if for each matrix $a = (a_{ij}) \in H$ and for all indices i,r,j the inclusions $a_{ir} a'_{rj} \in \sigma_{ij}$ are valid.

LEMMA 2. Let Card$(Z) \geqslant 5$, $n \geqslant 3$ and $\Delta \leqslant H \leqslant \Gamma$. If for some matrix $a = (a_{ij}) \in H$ there exists an index r, $1 \leqslant r \leqslant n$, such that $a_{rj} = 0$ for all $j \neq r$, then $a \in N(\sigma)$. In particular $a_{ir} \in \sigma_{ir}$ for all $i \neq r$.

LEMMA 3. Let Card$(Z) \geqslant 7$, $n \geqslant 3$ and $a = (a_{ij}) \in H$. If $a_{pr} = a_{ps} = 0$ for some p,r,s ($r \neq s$) then $a_{ir} a'_{rj}$, $a_{is} a'_{sj} \in \sigma_{ij}$ for all i and j.

REMARK: Note that the analogues of Lemmas 2 and 3 obtained by replacing the rows with the columns also hold.

2. The reducible subgroups.

We show that if H is a reducible subgroup then H satisfies the conclusion of our theorem.

Let $B = SB(n,T)$ be the subgroup of upper triangular matrices and $N = SN(n,T)$ be the subgroup of monomial matrices in Γ.

A standard parabolic subgroup is a subgroup in Γ containing B. A parabolic subgroup is a subgroup which is conjugated with some standard parabolic subgroup. Note that a subgroup is standard if and only if it is a subgroup of upper block-triangular matrices. A subgroup H of the group is irreducible if and only if it is not contained in any proper parabolic subgroup.

LEMMA 4. Let Card$(Z) \geqslant 5$ and $Q \geqslant \Delta = SD(n,T)$. If Q is a parabolic subgroup, then there exists a monomial matrix ω such that $\omega Q\omega^{-1}$ is a standard parabolic subgroup.

Proof. By the assumption above, there is an element $x\in\Gamma$ such that $xQx^{-1} = P$ is a standard parabolic subgroup. By Bruhat's decomposition $x = b_1\omega b_2$, where $b_1, b_2\in B, \omega\in N$ (see [8]). Because $b_1\in B\le P$ then we have $Q = b_2^{-1}\omega^{-1}P\omega b_2$ and hence $b_2^{-1}\Delta b_2 \le Q$ in view of $\omega^{-1}\Delta\omega = \Delta$. Moreover, by our assumption $\Delta\le Q$ and hence $\omega^{-1}P\omega = Q$.

Thus, if Card(Z) ≥ 5 and H is a reducible subgroup in Γ containing Δ, then there exists a monomial matrix ω such that $\omega H\omega^{-1}$ is contained in some group of block-triangular matrices of fixed type. From here and by Lemma 3 it is easy to show that any reducible subgroup in Γ that contains Δ satisfies the conclusion of Theorem 2. In fact, by Lemma 4 we can suppose that the subgroup H is contained in some group of block-triangular matrices of type, for example, (n_1, n_2), where $n_1+n_2=n$. This means, in any matrix $a=(a_{ij})\in H$, $a_{ij}=0$ for $i\ge n_1+1$, $j\le n_1$. Note that for the case $n_1=1$ or $n_2=1$ our conclusion comes from Lemma 2 or from its analogue (see remark after Lemma 3). Hence, it is possible to suppose that $n_1, n_2\ge 2$. But in this case we can use Lemma 3 to conclude that $a_{ir}a'_{rj}\in\sigma_{ij}$ for all $r\le n_1$. Using the analogue of Lemma 3 we conclude that $a_{ir}a'_{rj}\in\sigma_{ij}$ for all $r\ge n_1+1$. Now by Lemma 1, $H\le N(\sigma)$.

Thus, from now on we can suppose that H is an irreducible subgroup.

3. The imprimitive irreducible subgroups.

The group $\Gamma = SL(n,T)$ naturally acts on the left vector space T. Let $e_1, e_2, \dots, e_n$ be a standard basis in T. A subgroup H of Γ is called an imprimitive subgroup if T can be represented by a direct sum of subspaces $X_1, X_2,\dots,X_m$, $m\ge 2$, such that for each $g\in H$ and each k, $1\le k\le m$, there exists l, $1\le l\le m$, such that $g(X_k)\subseteq X_l$. The collection $\{X_k\}$ is called a system of imprimitivity and each X_k a block of the group H.

LEMMA 5. Let H be an irreducible subgroup in Γ that contains the subgroup $\Delta = SD(n,T)$. Then each block of any its system of imprimitivity admits some basis from the vectors e_i.

Proof. We must show that if X is an arbitrary block for H and $x=\sum x_k e_k\in X$, where $x_k\ne 0$, then $e_k\in X$.

Case 1: dim X ≥ 2. Suppose that $x, y \in X$ are linearly independent and $x_k\ne0$. We can assume that $y_k=0$. Let θ be an arbitrary element in $[T^*,T^*]$. It is clear that $d_k(\theta)y = y\in X$. Moreover, by our assumption the subgroup H is irreducible, whence it follows $d_k(\theta)X = X$. Take $\theta\in[T^*,T^*]$, $\theta\ne1$. Then $d_k(\theta)x - x = (\theta-1)x_ke_k\in X$ and $e_k\in X$.

Case 2: dim X = 1. Take $x \in X$ such that $x_k\ne0$. Suppose that there exists $x_l=0$ for some $l\ne k$. Consider ε, η where $\varepsilon\ne\eta$ and $\varepsilon,\eta \ne 0,1$. Let $y=d_{kl}(\varepsilon)x$, $z=d_{kl}(\eta)x$. If, for example, $y\in X$ then $y-x=(\varepsilon-1)x_ke_k\in X$ and $e_k\in X$. Hence, we can suppose that $y\notin X$ and $z\notin X$. If $d_{kl}(\varepsilon)X\ne d_{kl}(\eta)X$ then

$$e_k\in(X\oplus d_{kl}(\varepsilon)X)\cap(X\oplus d_{kl}(\eta)X)$$

and $e_k\in X$.

Now suppose that $x_l\ne0$ for all $l\ne k$. Let $n\geq 3$. Take ε, η where $\varepsilon\ne\eta$ and $\varepsilon,\eta\ne0,1$. Consider the vectors $y=d_{kl}(\varepsilon)x$, $z=d_{kl}(\eta)x$. The vectors x, y, z are linearly independent in pairs and, hence, they belong to three different blocks X, Y, W. Let $U = X \oplus Y \oplus W$. Then

$$y-x=(\varepsilon-1)x_ke_k+(\varepsilon^{-1}-1)x_le_l\in U$$
$$z-x=(\eta-1)x_ke_k+(\eta^{-1}-1)x_le_l\in U$$

Because the determinant $\begin{vmatrix}\varepsilon -1 & \varepsilon^{-1} -1\\ \eta -1 & \eta^{-1} -1\end{vmatrix}\ne0$, then $e_k, e_l\in U$.

It is now easy to show that the vectors e_k, e_l, $u = x - x_ke_k - x_le_l$ form a basis for the subspace U.

It is not difficult to verify that when $d_{kl}(\theta), \theta\in T$ acts on the vectors e ,e ,u, they are multiplied by a scalar. Hence, the subspace U is invariant under the action of $d_{kl}(\theta)$. In particular, $d_{kl}(\varepsilon\eta)X$ coincides with one of the subspace X, Y, W. But in this case it follows that all the coordinates of the vector x, except the k^{th} and l^{th} coordinates are equal to 0. This is a contradiction.

Now, suppose that n = 2. Take ε, η where $\varepsilon, \eta \neq 0,1$ and $\varepsilon^2 \neq \eta^2$. Consider the vectors

$$y = d_{12}(\varepsilon)x\ ,\ z = d_{12}(\eta)x$$

Clearly x, y, z are linearly independent in pairs. Hence, they belong to three different blocks. But in the case n = 2 only two such blocks exist. Hence, the lemma is proved.

It follows from this lemma that if H is an imprimitive subgroup in Γ that contains the subgroup Δ = SD(n,T) then there exists some monomial matrix ω such that $\omega H \omega^{-1}$ is contained in some group of block-monomial matrices. Using Lemma 3, as in the case for a reducible subgroup, we can see that the conclusion of Theorem 2 holds.

4. The subgroups containing an elementary transvection.

Let $t_{ij}(\alpha), \alpha \in T$ be an elementary transvection belonging to the subgroup H. Since H contains the subgroup Δ=SD(n,T), then in view of the formula

$$t_{ij}(\varepsilon\alpha) = d_{i2}(\varepsilon)t_{ij}(\alpha)d_{i2}(\varepsilon^{-1}), \varepsilon \in T^*$$

the subgroup L_{ij} generated by all $t_{ij}(\beta), \beta \in T$ is contained in H. Denote by L, the normal subgroup in H generated by all the subgroups L_{ij}. By Clifford's theorem (see [7]) this subgroup is irreducible. Now, by Theorem 1 from [6], L = SL(n,T). Hence, H = SL(n,T) and the theorem is proved.

§ 5.The subgroups contaning no elementary transvections.

In this paragraph we establish the standard description for all intermediate subgroups for the case $n \geq 5$ and Card(Z) ≥ 7.

THEOREM 3. Let T be skew field, Card(Z) $\geq$ 7 and $n \geq 5$. For any intermediate subgroup H, $\Delta \leq H \leq \Gamma$, there exists a unique unital net σ of degree n over T such that

$$\Gamma(\sigma) \leq H \leq N(\sigma).$$

From Theorem 2 it follows that each intermediate subgroup belongs to the standard description if H contains an elementary

transvection. Hence, to prove Theorem 3 we need only show that if H does not contain any elementary transvections, then H belongs to the standard description too.

From now on, we assume that the intermediate subgroup H does not contain any elementary transvections. This means that the net $\sigma = (\sigma_{ij})$ associated with H satisfies the condition $\sigma_{ij} = 0$ for all $i \neq j$. Our purpose is to prove that each such subgroup is monomial when $n \geq 5$ and Card(Z) $\geq$ 7. This will complete the proof of Theorem 3.

First, we formulate without the proofs the following two lemmas

LEMMA 6. Let Card (Z) $\geq$ 7 and $n \geq 3$. If in some row or in some column of a matrix $a \in H$ there is only one non-zero element, then a is a monomial matrix.

LEMMA 7. Let Card (Z) $\geq$ 7 and $n \geq 3$. If in some row or in some column of a matrix $a \in H$ there exist two zero elements, then a is a monomial matrix.

Now, we will prove a fact for skew fields which is interesting by itself.

THEOREM 4. There does not exist a skew field T, for which the commutant $[T^*,T^*]$ is a group of order 2.

The proof of this theorem is based on the following lemma

LEMMA 8.Let G be a group and C be a center of G. If Card([G ,G]) = 2 then

(i) $a^2 \in C$ for all $a \in G$

(ii) $[G,G] \leq C$.

Proof.(i) Let $x \in G$. Consider a commutator $[a^2,x]$.If $[a,x] = 1$ then clearly $[a^2,x] = 1$. Suppose that $[a,x] \neq 1$. If $[a^2,x] \neq 1$ then by our assumption $[a^2,x] = [a,x]$. Hence, $axa^{-1}x^{-1} = a^2xa^{-2}x^{-1}$. This implies xa = ax, i.e. $[a,x] = 1$ and we have a contradiction. We proved that $[a^2,x] = 1$ for all $x \in G$, i.e. $a^2 \in C$.

(ii). Let x, y be any two elements of G. We have $[x,y] = xyx^{-1}y^{-1} = xyxyy^{-1}x^{-2}y^{-1} = (xy)^2x^{-2}y^{-2}$ and $[x,y] \in C$ by (i)

REMARK. It is easy to show that the following non-commutative groups of order 8

$$G_1 = \{ a,b : a^4 = 1,\ b^4 = 1,\ a^2 = b^2,\ aba = b \}$$

$$G_2 = \{ a,b : a^2 = 1,\ b^4 = 1,\ ba = ab \}$$

satisfy the condition of Lemma 8.

Proof of Theorem 4. First, we note that the conclusion of the theorem holds trivially for a skew field of characteristic 2, because in this case there are no subgroups of order 2 in the multiplicative group T . Now, suppose that the characteristic of T is different from 2 and

$$[T, T] = \{ 1, -1 \} \qquad (1)$$

Let $x,\ y \in T$. Suppose that $[x,y] \neq 1$, then $[x,y] = -1$, i.e. $xy = -yx$. Let Z be the center of T. Because $x,y \notin Z$ and x^2, y^2 are contained in Z (by Lemma 8), then $Z + Zx$ and $Z + Zy$ are quadratic extensions of the field Z. For the non-zero elements $1+x$ and $1+y$, by (1) we have $[1+x,1+y] = \pm 1$. If $[1+x,1+y] = 1$ then $(1+x)(1+y) = (1+y)(1+x)$, and so $xy = yx$ and we have a contradiction. If $[1+x,1+y] = -1$ then $(1+x)(1+y) = -(1+y)(1+x)$, and so $1+x+y = 0$, $y \in Z[x]$. Hence $xy = yx$ and again a contradiction. The theorem is proved.

The following lemma holds for an arbitrary subgroup H of a group .

LEMMA 9. Let $Card(Z) \geq 5$ and $n \geq 3$. If in a matrix $a = (a_{ij}) \in H$ $a_{pr} = 0$ for some p,r then $a_{ir} a'_{rj} \in \sigma_{ij}$ for all i,j.

Proof. Take $1 \neq \theta \in [T^*,T^*]$ and consider the matrix $d_r(\theta)$. Clearly, the matrix $b = ad_r(\theta)a^{-1}$ belongs to H and

$$b_{ij} = \delta_{ij} + a_{ir}(\theta - 1)a'_{rj}$$

Since $a_{pr} = 0$, it follows $b_{pj} = \delta_{pj}$. Therefore we can use Lemma 2 to conclude that $b \in N(\sigma)$. In particular $b_{ij} b'_{jj} \in \sigma_{ij}$. Since $b'_{jj} = 1 + a_{jr}(\theta - 1)a'_{rj}$ and $Card([T^*,T^*]) \geq 3$ (see Theorem 4), we can choose θ such that $b'_{jj} \neq 0$. Then $b_{ij} \in \sigma_{ij}$. Since

$$b_{ij} = \delta_{ij} + a_{ir}(\theta - 1)a'_{rj} ,$$

it follows $a_{ir} a'_{rj} \in \sigma_{ij}$ for all i, j.

LEMMA 10. Let Card $(Z) \geq 7$ and $n \geq 3$. If in a matrix $a = (a_{ij}) \in H$

there exists one zero element, then a is monomial.

Proof. Let $a_{pr} = 0$ for some p,r. By Lemma 9 $a_{ir} a'_{rj} = 0$ for all $i \neq j$. Since $n \geq 3$, we can choose $i \neq p$. If $a_{ir} \neq 0$ then $a'_{rj} = 0$ for all $i \neq j$ and the matrix a^{-1} is monomial by Lemma 6. If $a_{ir} = 0$ then a is monomial by Lemma 7.

LEMMA 11. Let $n \geq 4$, $\varepsilon \neq 0,1$ and $y = xd_{rs}(\varepsilon)x^{-1}$ $(r \neq s)$ be a non-diagonal monomial matrix. Then all the rows of the matrix y, except for two, three or four of them, coincide with the corresponding rows of the identity matrix.

Proof. For a matrix y there is a formula

$$y_{ij} = \delta_{ij} + x_{ir}(\varepsilon - 1)x'_{rj} + x_{is}(\varepsilon^{-1} - 1)x'_{sj}$$

Since y is a monomial non-diagonal matrix, there exists an index k for which $y_{kk} = 0$. Hence $y_{kl} \neq 0$ for some $l \neq k$.

If $y_{lk} \neq 0$ then

$$y_{lt} = 0 \text{ for all } t \neq k$$
$$y_{kt} = 0 \text{ for all } t \neq l$$
$$y_{tk} = 0 \text{ for all } t \neq l$$
$$y_{tl} = 0 \text{ for all } t \neq k$$

Thus, for $t \neq k,l$ we have

$$\left.\begin{aligned} x_{lr}(\varepsilon - 1)x'_{rt} + x_{ls}(\varepsilon^{-1} - 1)x'_{st} &= 0 \\ x_{kr}(\varepsilon - 1)x'_{rt} + x_{ks}(\varepsilon^{-1} - 1)x'_{st} &= 0 \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} x_{tr}(\varepsilon - 1)x'_{rk} + x_{ts}(\varepsilon^{-1} - 1)x'_{sk} &= 0 \\ x_{tr}(\varepsilon - 1)x'_{rl} + x_{ts}(\varepsilon^{-1} - 1)x'_{sl} &= 0 \end{aligned}\right\} \qquad (2)$$

Suppose that the following determinants are not equal to 0 at the same time

$$\begin{vmatrix} x_{lr} & x_{ls} \\ x_{kr} & x_{ks} \end{vmatrix} \neq 0 \quad \text{or} \quad \begin{vmatrix} x'_{rk} & x'_{sk} \\ x'_{rl} & x'_{sl} \end{vmatrix} \neq 0$$

Then at least one of the following conditions is satisfied

$$x'_{rt} = x'_{st} = 0 \quad \text{or} \quad x_{tr} = x_{ts} = 0$$

Whence it follows that for any $t \neq k,l$

$$y_{tt} = 1 + x_{tr}(\varepsilon - 1)x'_{rt} + x_{ts}(\varepsilon^{-1} - 1)x'_{st} = 1$$

Now, If $y_{lk} = 0$ then $y_{lq} \neq 0$ for some $q \neq k,l$. We show that $y_{qk} \neq 0$. In fact, if $y_{qk} = 0$ then $y_{qp} \neq 0$ for some $p \neq k,q,l$ and hence

$$y_{kp} = y_{lp} = y_{pp} = y_{qk} = y_{ql} = y_{qq} = 0$$

From the conditions $y_{qk} = y_{ql} = 0$ it follows

$$\left.\begin{array}{l} x_{qr}(\varepsilon-1)x'_{rk} + x_{qs}(\varepsilon^{-1}-1)x'_{sk} = 0 \\ x_{qr}(\varepsilon-1)x'_{rl} + x_{qs}(\varepsilon^{-1}-1)x'_{sl} = 0 \end{array}\right\} \qquad (3)$$

Since x_{qr} and x_{qs} are not equal to 0 at the same time then from (3) it follows that

$$\begin{vmatrix} x'_{rk} & x'_{sk} \\ x'_{rl} & x'_{sl} \end{vmatrix} = 0$$

Similarly, from the conditions $y_{kp} = y_{lp} = y_{pp} = 0$, it follows that

$$\begin{vmatrix} x_{kr} & x_{ks} \\ x_{lr} & x_{ls} \end{vmatrix} = 0$$

But this contradicts our supposition that both these determinants are not equal to 0 at the same time. Thus, $y_{qk} \neq 0$, whence it follows $y_{tt} = 1$ for any $t \neq k,l,q$.

Now, suppose that both the above determinants equal to 0. Then

$$\begin{vmatrix} x_{l'r} & x_{l's} \\ x_{k'r} & x_{k's} \end{vmatrix} \neq 0 \quad \text{for some } k', l' \qquad (4)$$

Suppose that $l' \neq l$, $k' \neq k$ and $l' \neq k$. If $y_{k'k} = 0$ then since $y_{k'k} = y_{l'k} = 0$ it follows from (4) that $y_{kk} = 1$. But this contradicts the condition $y_{kk} = 0$. Hence $y_{k'k} \neq 0$, whence it follows $k' = l$.

If $k' = k$ then from (4) and the condition

$$\begin{vmatrix} x_{kr} & x_{ks} \\ x_{lr} & x_{ls} \end{vmatrix} = 0 \text{ it follows}$$

$$\begin{vmatrix} x_{lr} & x_{ls} \\ x_{l'r} & x_{l's} \end{vmatrix} \neq 0 \text{ and } l \neq l'.$$

Similarly, if $l' = k$ then

$$\begin{vmatrix} x_{lr} & x_{ls} \\ x_{k'r} & x_{k's} \end{vmatrix} \neq 0 \text{ and } k' \neq l.$$

As we saw from the above, in these cases we have the same situation. Therefore we can suppose that

$$\begin{vmatrix} x_{lr} & x_{ls} \\ x_{l'r} & x_{l's} \end{vmatrix} \neq 0 \text{ for some } l' \neq l.$$

Now, it is easy to see that $y_{l'h} \neq 0$ and $y_{hl'} \neq 0$ for some $h \neq k,l,l'$. Thus, for any $t \neq k,l,h,l'$ we have $y_{tt} = 1$. The lemma is proved.

REMARK. There is a possibility that Lemma 11 holds for an arbitrary commutative ring, for which there is the usual theory of determinants ,since in our proof we only use the properties of determinants.

LEMMA 12. Let $n \geq 5$ and Card(Z) ≥ 7. If a matrix $b = ad_{hp}(\eta)a^{-1}$ where $\eta \neq 0,1$ and $h \neq p$,is monomial,then the matrix a is monomial too.

Proof.By Lemma 11 some row of the matrix b, for example, the k-th row, coincides with the corresponding row of the identity matrix, i.e.

$$a_{kh}(\eta-1)a'_{hj} + a_{kp}(\eta^{-1}-1)a'_{pj} = 0 \text{ for all } j$$

If both elements a_{kh} and a_{kp} are not equal to 0, then the h-th and p-th rows of the matrix a^{-1} are proportional. But this is impossible. If only one of these elements is equal to 0, then a^{-1} is the matrix with a zero row. But this is impossible too. Thus, $a_{kp} = a_{kh} = 0$ and we can use Lemma 7 to conclude that the matrix a is monomial.

LEMMA 13. Let a be an arbitrary matrix in Γ and k,l,r,s be the indices such that $k \neq l$ and $a_{rl}, a'_{ls}, a_{rk}, a'_{ks} \neq 0$. Put

$$\varepsilon = \varepsilon^{rs}_{kl} = (a_{rl}\, a'_{ls})(a_{rk}\, a'_{ks})^{-1}$$

$$c = ad_{kp}(\theta)d_{lp}(\eta)$$

where $\theta = a^{-1}_{rk}$, $\eta = a^{-1}_{rl}$, $p \neq k,l$.

Then for the matrix $b = cd_{kl}(\varepsilon)c^{-1}$ we have $b_{rs} = \delta_{rs}$.

Proof. We have

$$b_{ij} = \delta_{ij} + c_{ik}(\varepsilon-1)c'_{kj} + c_{il}(\varepsilon^{-1}-1)c'_{lj}$$

Then

$$b_{rs} = \delta_{rs} + a_{rk}\theta(\varepsilon-1)\theta^{-1}a'_{ks} + a_{rl}\eta(\varepsilon^{-1}-1)\eta^{-1}a'_{ls} =$$
$$= \delta_{rs} + (\varepsilon-1)(a_{rk}\, a'_{ks} - \varepsilon^{-1}a_{rl}\, a'_{ls}) = \delta_{rs}.$$

Now, we consider for a given matrix a the following determinant

$$\Delta^{pq}_{kl}(a) = \begin{vmatrix} a_{pk} & a_{pl} \\ a_{qk} & a_{ql} \end{vmatrix}$$

We also write $\Delta_{k\ell}^{pq}$ if the matrix a is understood from the context.

LEMMA 14. Let all elements $a_{pk}, a_{p\ell}, a_{qk}, a_{q\ell}, a'_{ks}, a'_{\ell s}$ be non-zero. Then

(i) $\Delta_{k\ell}^{pq} = 0 \Longrightarrow \varepsilon_{k\ell}^{ps} = \alpha\, \varepsilon_{k\ell}^{qs} \alpha^{-1}$, $\alpha \in T^*$

(ii) $\varepsilon_{k\ell}^{ps} = \varepsilon_{k\ell}^{qs} \longrightarrow \Delta_{k\ell}^{pq} = 0$

Proof. It is clear.

Proof of Theorem 3. Let $a = (a_{ij})$ be an arbitrary matrix of the subgroup H. If in the matrix a there exists some zero element, then by Lemma 10 the matrix a is monomial. Hence we can suppose that $a_{ij} \neq 0$ for all i,j. If $\varepsilon = \varepsilon_{pq}^{rs} \neq 1$ for some $p \neq q$, $r \neq s$ then by Lemma 13 the matrix $b = c d_{pq}(\varepsilon) c^{-1}$, where $c = a d_{ph}(a_{hp}^{-1}) d_{qh}(a_{hq}^{-1})$ ($h \neq p,q$), is monomial and clearly a is monomial too. Let r,s,t be pairwise distinct. Suppose $\varepsilon_{pq}^{rt} = \varepsilon_{pq}^{st} = 1$. Then $\Delta_{pq}^{rs} = 0$ and hence for each j we have $\varepsilon_{pq}^{rj} = \alpha\, \varepsilon_{pq}^{sj} \alpha^{-1} = \varepsilon_j$ (see Lemma 14). It is clear that there exists some index h, for which $\varepsilon_h \neq 1$ (otherwise the p-th and q-th rows of a^{-1} are linearly dependent). Hence we have the same situation as before (since $h \neq r$ or $h \neq s$). The proof is now complete.

REMARK: Our proofs do not give the standard description of the subgroups when n = 3,4 and the center of a skew field with no less than seven elements. However, in our opinion, it is clear from the proofs of Lemmas 11 and 12 that a method which is different from our's is necessary to study these cases.

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