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THE NORMAL HOLONOMY GROUP

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1. Introduction.

The restricted holonomy group of a Riemanniann manifold is a compact Lie group and its representation on the tangent space is a product of irreducible representations and a trivial one. This product unique up to order (see e.g. [K-N,sec.5]). Each one of the non-trivial factors is either an orthogonal representation of a connected compact Lie group which acts transitively on the unit sphere or it is the isotropy representation of a simple Riemannian symmetric space of rank ≥ 2 (see [B] and [S]).

We prove that, surprisingly, all these properties are also true for the representation on the normal space of the restricted normal holonomy group of any submanifold of a space of constant curvature. Moreover, the non trivial part of this representation is the isotropy representation of a semisimple Riemannian symmetric space.

In order to prove this fact we define a tensor

$$\mathcal{R}^{\perp} : C^{\infty}(N(M))^{3} \longrightarrow C^{\infty}(N(M))$$

which provides the same geometric information as the normal curvature tensor $R^{\perp}$ and has the algebraic properties of a Riemannian curvature tensor. The methods used here are then a slight modification of those of Simons in [S].

2. Normal curvature.

Let $(M^n, \langle\ ,\ \rangle)$ be a Riemannian connected manifold and let $i: M^n \longrightarrow Q^N$ be an isometric immersion, where Q^N is of constant curvature.

Let $N(M) \xrightarrow{\pi} M$ be the normal boundle over M induced by i. For the sake of simplicity the Riemannian metric on Q^N, as well as the usual metric on the fibres of $N(M)$, will also be denoted by $\langle\ ,\ \rangle$. By $C^\infty(N(M))$ we denote the C^∞ sections from M into $N(M)$.

Define the tensor

$$\mathcal{R}^\perp : C^\infty(N(M))^3 \longrightarrow C^\infty(N(M))$$

by putting

$$\mathcal{R}^\perp_p(\xi_1,\xi_2)\xi_3 = \sum_{j=1}^{n} R^\perp_p(A_{\xi_1}(e_j),A_{\xi_2}(e_j))\xi_3$$

$p \in M$, $\xi_1, \xi_2, \xi_3 \in N(M)_p$; where A is the shape operator, $R^\perp$ is the curvature operator of the normal connection $\nabla^\perp$ and $\{e_1,\dots,e_n\}$ is an arbitrary orthonormal basis of T_pM.

The above tensor was just defined in [S-O].

Given an Euclidean space $\mathbb{V}$ we will denote by $\mathcal{A}(\mathbb{V})$ the vector space of skew-symmetric endomorphisms of $\mathbb{V}$, with the usual inner product $(\ ,\ )$, i. e., $(A,B) = -\mathrm{trace}(A\circ B)$.

Lemma 2.1. In the hypothesis and notation of this section. Then, for all $\xi_1, \xi_2, \xi_3, \xi_4 \in C^\infty(N(M))$ it is verified:

i) $\mathcal{R}^\perp(\xi_1,\xi_2) = -\mathcal{R}^\perp(\xi_2,\xi_1)$

ii) $\mathcal{R}^\perp(\xi_1,\xi_2)\xi_3 + \mathcal{R}^\perp(\xi_2,\xi_3)\xi_1 + \mathcal{R}^\perp(\xi_3,\xi_1)\xi_2 = 0$

iii) $\langle\mathcal{R}^\perp(\xi_1,\xi_2)\xi_3,\xi_4\rangle = -\langle\xi_3,\mathcal{R}^\perp(\xi_1,\xi_2)\xi_4\rangle$

iv) $\langle\mathcal{R}^\perp(\xi_1,\xi_2)\xi_3,\xi_4\rangle = \langle\mathcal{R}^\perp(\xi_3,\xi_4)\xi_1,\xi_2\rangle =$

$$= -1/2\ ([A_{\xi_1},A_{\xi_2}],[A_{\xi_3},A_{\xi_4}])$$

Proof. It was just given in [S-O], but we reproduce it completly. Let $p \in M$ and let $\{e_1,\dots,e_n\}$ be an orthonormal basis of T_pM

$$\langle \mathcal{R}_p^{\perp}(\xi_1,\xi_2)\xi_3,\xi_4\rangle = \langle \sum_{j=1}^{n} R_p^{\perp}(A_{\xi_1}(e_j),A_{\xi_2}(e_j))\xi_3,\xi_4\rangle =$$

$$= \sum_{j=1}^{n} \langle [A_{\xi_3},A_{\xi_4}](A_{\xi_1}(e_j)),A_{\xi_2}(e_j)\rangle =$$

by the well known formula, when the ambient space is of constant curvature,

$$= \sum_{j=1}^{n} \langle A_{\xi_2} \circ [A_{\xi_3},A_{\xi_4}] \circ A_{\xi_1}(e_j),e_j\rangle =$$

$$= \mathrm{trace}_p(A_{\xi_2} \circ [A_{\xi_3},A_{\xi_4}] \circ A_{\xi_1}) =$$

$$= \frac{1}{2}\,\mathrm{trace}_p(A_{\xi_2} \circ [A_{\xi_3},A_{\xi_4}] \circ A_{\xi_1}) +$$

$$+ \frac{1}{2}\,\mathrm{trace}_p((A_{\xi_2} \circ [A_{\xi_3},A_{\xi_4}] \circ A_{\xi_1})^t) =$$

$$= \frac{1}{2}\,\mathrm{trace}_p(A_{\xi_1} \circ A_{\xi_2} \circ [A_{\xi_3},A_{\xi_4}]) -$$

$$- \frac{1}{2}\,\mathrm{trace}_p(A_{\xi_1} \circ [A_{\xi_3},A_{\xi_4}] \circ A_{\xi_2}) =$$

$$= \frac{1}{2}\,\mathrm{trace}_p\{ A_{\xi_1} \circ A_{\xi_2} \circ [A_{\xi_3},A_{\xi_4}] - A_{\xi_2} \circ A_{\xi_1} \circ [A_{\xi_3},A_{\xi_4}] \}$$

$$= \frac{1}{2}\,\mathrm{trace}_p\,([A_{\xi_1},A_{\xi_2}] \circ [A_{\xi_3},A_{\xi_4}])$$

which proves i), iii) and iv).

As we have seen above

$$\langle \mathcal{R}_p^{\perp}(\xi_1,\xi_2)\xi_3,\xi_4\rangle = \mathrm{trace}_p(A_{\xi_2} \circ [A_{\xi_3},A_{\xi_4}] \circ A_{\xi_1}) =$$

$$= \mathrm{trace}_p(A_{\xi_2} \circ A_{\xi_3} \circ A_{\xi_4} \circ A_{\xi_1}) -$$

$$- \mathrm{trace}_p(A_{\xi_2} \circ A_{\xi_4} \circ A_{\xi_3} \circ A_{\xi_1}) =$$

$$= \mathrm{trace}_p(A_{\xi_1} \circ A_{\xi_2} \circ A_{\xi_3} \circ A_{\xi_4}) -$$

$$- \mathrm{trace}_p(A_{\xi_3} \circ A_{\xi_1} \circ A_{\xi_2} \circ A_{\xi_4})$$

Summing over all cyclic permutations of {1, 2, 3} we clearly obtain ii).

■

Observe that from iv) we have that $R_p^{\perp} = 0 \Leftrightarrow \mathcal{R}_p^{\perp} = 0$.

We have much more than this. The following result tells us that $\mathcal{R}^{\perp}$ carries the same geometrical information as $R^{\perp}$.

Proposition 2.2. In the notation and assumptions of this section. Then, for all $p \in M$, the linear space of skew-symmetric endomorphisms of $N(M)_p$ spanned by the set $\{ R^{\perp}_p(X,Y) : X, Y \in T_pM \}$ coincides with that spanned by the set $\{ \mathcal{R}^{\perp}_p(\xi,\eta) : \xi, \eta \in N(M)_p \}$.

In order to prove last fact we shall next define $\mathcal{R}^{\perp}$ in an equivalent but convenient way.

First of all observe that, by i) of Lemma 2.1, we can see, for each $p \in M$,

$$\mathcal{R}^{\perp}_p : \Lambda^2(N(M)_p) \longrightarrow \mathcal{A}(N(M)_p)$$

by putting

$$\mathcal{R}^{\perp}_p(\xi \wedge \eta) = \mathcal{R}^{\perp}_p(\xi,\eta)$$

Similarly we can see

$$R^{\perp}_p : \Lambda^2(T_pM) \longrightarrow \mathcal{A}(N(M)_p)$$

Consider now

$$\Lambda^2(N(M)_p) \xrightarrow{L_p} \mathcal{A}(T_pM) \xrightarrow{h_p} \Lambda^2(T_pM) \xrightarrow{R^{\perp}_p} \mathcal{A}(N(M)_p)$$

where

$$L_p(\xi \wedge \eta) = [A_\xi, A_\eta]$$

and h_p is the isomorphism given by

$$\langle h_p^{-1}(x \wedge y)(u),v\rangle = \langle x,u\rangle.\langle y,v\rangle - \langle y,u\rangle.\langle x,v\rangle$$

By a straightforward calculation we have

Lemma 2.3. $\mathcal{R}^{\perp}_p = - R^{\perp}_p \circ h_p \circ L_p$.

■

Put on $\Lambda^2(T_pM)$ the inner product defined by

$$((v,w)) = (h_p^{-1}(v), h_p^{-1}(w)) = -\,\mathrm{trace}(h_p^{-1}(v) \circ h_p^{-1}(w))$$

Then we have

Lemma 2.4. $\ker R_p^{\perp} = (h_p \circ L_p(\Lambda^2(N(M)_p))^{\perp}$

Proof. Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of T_pM and let $u = \sum_{k<l} a_{kl}.e_k \wedge e_l \in \Lambda^2(T_pM)$. If $\xi, \eta \in N(M)_p$ are arbitrary

$$< R_p^{\perp}(u)\xi, \eta > = < \sum_{k<l} a_{kl}. R_p^{\perp}(e_k, e_l)\xi, \eta > =$$

$$= \sum_{k<l} a_{kl}.< [A_\xi, A_\eta](e_k), e_l >$$

But

$$((u, h_p \circ L_p(\xi \wedge \eta))) = -\,\mathrm{trace}(h_p^{-1}(u) \circ L_p(\xi \wedge \eta)) =$$

$$= \sum_{s,t} < h_p^{-1}(u)(e_s), e_t >.< [A_\xi, A_\eta](e_s), e_t > =$$

$$= 2. \sum_{k<l} a_{kl}.< [A_\xi, A_\eta](e_k), e_l >$$

Then

$$< R_p^{\perp}(u)\xi, \eta > = 1/2.((u, h_p \circ L_p(\xi \wedge \eta)))$$

which clearly implies the Lemma. ■

Proof of the Proposition. It is immediate from Lemma 2.4. ■

3. The main result.

Theorem 3.1. Let M^n be an immersed submanifold of a Riemannian manifold Q^N of constant curvature. Let $p \in M$ and let Φ^* be the restricted holonomy group of the normal connection at p. Then Φ^*

is compact, there exists an unique (up to order) orthogonal decomposition of the normal space at p $N(M)_p = \mathbb{V}_0 \oplus \ldots \oplus \mathbb{V}_k$ into Φ^*-invariant subspaces and there exist $\Phi_0, \ldots, \Phi_k$ normal Lie subgroups of Φ^* such that:

i) $\Phi^* = \Phi_0 \times \ldots \times \Phi_k$ (direct product).

ii) Φ_i acts trivially in $\mathbb{V}_j$ if $i \neq j$.

iii) $\Phi_0 = \{1\}$ and if $i \geq 1$ Φ_i acts irreducibly in $\mathbb{V}_i$ as the isotropy representation of a simple Riemannian symmetric space.

We keep the notation and assumptions of section 2.

Let $p \in M$ be fixed and let $\gamma : [0,1] \longrightarrow M$ be a piece-wise differentiable curve with $\gamma(1) = p$. Denote by $\gamma^*(\mathcal{R}^\perp)$ the tensor of type (1,3) in $N(M)_p$ defined by

$$\gamma^*(\mathcal{R}^\perp)(v,w)z = P_\gamma(\mathcal{R}^\perp_q(P_\gamma^{-1}(v),P_\gamma^{-1}(w))P_\gamma^{-1}(z))$$

where $\gamma(0) = q$ and P_γ denotes the parallel displacement along γ with the normal connection.

Denote by $\mathcal{S}$ the linear subspace, of the tensors of type (1,3) of $N(M)_p$, generated by all the $\gamma^*(\mathcal{R}^\perp)$, where γ runs over all piece-wise differentiable curves ending at p.

From the theorem of Ambrose-Singer and Proposition 2.2 we have that the Lie algebra $\mathfrak{g}$ of the restricted normal holonomy group Φ^* at p coincides with the linear span of the set $\{ R(u,v) : R \in \mathcal{S},\ u,\ v \in N(M)_p \}$.

From Lemma 2.1 we have that if $R \in \mathcal{S}$:

i) $R(u,v) = - R(v,u)$

ii) $R(u,v)w + R(v,w)u + R(w,u)v = 0$

iii) $\langle R(u,v)w,z\rangle = - \langle w,R(u,v)z\rangle$

iv) $\langle R(u,v)w,z\rangle = \langle R(w,z)u,v\rangle$

Decompose orthogonally

$$N(M)_p = \mathbb{V}_o \oplus \ldots \oplus \mathbb{V}_k$$

into Φ^*-invariant subspaces such that Φ^* acts trivially in $\mathbb{V}_o$ and if $i \geq 1$ Φ^* acts irreducibly in $\mathbb{V}_i$ ($\dim \mathbb{V}_i \geq 2$).

If $u \in N(M)_p$ denote by u_i the projection of u into $\mathbb{V}_i$.

Lemma 3.2. Let x, $y \in N(M)_p$ and let $R \in \mathcal{S}$. Then

i) $R(x_i, y_j) = 0$ if $i \neq j$.

ii) $R(x,y) = \sum_{i=o}^{k} R(x_i, y_i)$

iii) $R(x_i, y_i)\mathbb{V}_j = \{0\}$ if $i \neq j$.

iv) $R(x_i, y_i)\mathbb{V}_i \subset \mathbb{V}_i$

Proof. It is the same of that given in [S, p.217, 218], but as it is very easy we write it down.

If $i \neq j$ and u, $v \in N(M)_p$ then

$$\langle R(x_i, y_j)u, v\rangle = \langle R(u,v)x_i, y_j\rangle = 0$$

because $R(u,v) \in \mathfrak{g}$ and $\mathfrak{g}$ leaves $\mathbb{V}_i$ invariant. This proves i) and therefore ii).

Let $v_j \in \mathbb{V}_j$, then

$$R(x_i, y_i)v_j = - R(y_i, v_j)x_i - R(v_j, x_i)y_i = 0$$

by part i), which proves iii).

Part iv) in immediate.

■

Let $\mathfrak{g}_i$ be the vector subspace of $\mathfrak{g}$ generated by all the $R(x_i, y_i)$, $R \in \mathcal{S}$ and x_i, $y_i \in \mathbb{V}_i$. From the above lemma it is easily derived (see [S, p.218])

Lemma 3.3.

i) $\mathfrak{g}_0 = \{0\}$ and each $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$, for $i = 0, \ldots, k$

ii) $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$, with $[\mathfrak{g}_i, \mathfrak{g}_j] = \{0\}$ if $i \neq j$.

iii) $\mathfrak{g}_i \mathbb{V}_i = \mathbb{V}_i$ for $i = 0, \ldots, k$.

iv) $\mathfrak{g}_i \mathbb{V}_j = \{0\}$ if $i \neq j$.

v) $\mathfrak{g}_i$ acts irreducibly in $\mathbb{V}_i$, for $i = 1, \ldots, k$.

■

Proof of theorem 3.1. We keep the notation of this section. For $i = 0, \ldots, k$, let Φ_i be the connected Lie subgroup of Φ^* (which is also connected) with Lie algebra $\mathfrak{g}_i$. Lemma 3.3 implies that we have the direct product $\Phi^* = \Phi_0 \times \ldots \times \Phi_k$, that Φ_i acts trivially in $\mathbb{V}_j$ if $i \neq j$ and that Φ_i acts irreducibly in $\mathbb{V}_i$ if $i \geq 1$. The uniqueness part of the theorem is now clear.

Now, a connected Lie subgroup of orthogonal transformations of a vector space which acts irreducibly in it must be compact (see [K-N, appendix 5]). Then, each Φ_i is compact and therefore Φ^* is compact.

Now, for $i \geq 1$, choose $R_i \in S$ such that R_i is not identically zero in $\mathbb{V}_i^3$. Each $[\mathbb{V}_i, R_i, \Phi_i]$ is an irreducible holonomy system, in the notation of [S]. Using [S, th.5] we finish the proof, since, by Lemma 2.1 iv), R_i must have negative scalar curvature.

■

In a next paper we will use the above results to establish the relation between isoparametric submanifolds in the sense of Terng and those in the sense of Strübing.

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