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REDUCTION OF 4-dim SELF DUAL SUPER YANG-MILLS ONTO SUPER RIEMANN SURFACES *

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ABSTRACT

Recently self dual super Yang–Mills over a super Riemann surface was obtained as the zero set of a moment map on the space of superconnections to the dual of the super Lie algebra of gauge transformations. We present a new formulation of 4–dim Euclidean self dual super Yang–Mills in terms of constraints on the supercurvature. By dimensional reduction we obtain the same set of superconformal field equations which define self dual connections on a super Riemann surface.

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The problem of Yang–Mills connections over Riemann surfaces has been recently [1] related to the string theory in a very fundamental way:

(i) The space of self dual connections over Riemann surfaces is homeomorphic to Teichmüller space. Its extension to the SUSY case [2] gives a very interesting approach to analyze super Teichmüller spaces, where it is well–known there are ambiguities [3] in defining superstring amplitudes [4].

(ii) It provides a geometrical structure to conformal field theories [1]. Yang–Mills field equations over a Riemann surface define a topological theory [5]. Its geometric quantization, by using a Kählerian polarization, allow the construction of Hilbert spaces of physical states, which provide the projective varieties inherent to conformal field theories.

Self dual Yang–Mills equations over Riemann surfaces have been recently extended to the SUSY case [2], via the construction of a moment map over super Riemann surfaces. We show now that those equations are also obtained by dimensional reduction of 4–dim $N = 1$ self dual super Yang–Mills equations.

Self dual Yang–Mills equations are relevant to the non–perturbative analysis of quantum field theories and quantum string theories. It has been extensively studied from a physical and a mathematical point of view with very remarkable success. Recently Yang–Mills instantons have been used to construct soliton solutions to the low energy approximations of heterotic string field equations [6], and may be relevant to the non–perturbative analysis of these theories. Self dual super Yang–Mills equations were discussed in [7] and more rigorously in [8] based on the work of Osterwalder and Schraeder.

We present a new set of superfield constraints which describe $d = 4, N = 1$ self dual super Yang–Mills equations that allows to obtain by a simple dimensional reduction the self dual super Yang–Mills equations over super Riemann surfaces.

In [2] we consider a vector bundle V with a super Riemann surface M as as base space. A superconnection on V, in local coordinates is given by

$$\begin{aligned} d_A &= d + i d\theta\, A_\theta + i\, d\bar{\theta}\, A_{\bar{\theta}} + A_z\, \eta + A_{\bar{z}}\, \bar{\eta}\,, \\ d &= d\theta\, D_\theta + d\bar{\theta}\, D_{\bar{\theta}} + \eta\, \frac{\partial}{\partial z} + \bar{\eta}\, \frac{\partial}{\partial \bar{z}}\,, \end{aligned} \tag{1}$$

where $\eta = dz + i\theta\, d\theta$ and $\bar{\eta} = d\bar{z} + i\,\bar{\theta}\, d\bar{\theta}$. We assume [2] it satisfies

$$F_{\theta\theta} = 0\,, \qquad F_{\bar{\theta}\bar{\theta}} = 0\,. \tag{2}$$

The differential operator d_A splits into the left and right operators

$$d_A = d'_A + d''_A\,.$$

Let $T^*\mathcal{M}$ be the cotangent bundle to the space $\mathcal{M}$ of superholomorphic structures. The local coordinates of $T^*\mathcal{M}$ are $(A_\theta, \Phi_{\bar\theta})$. In [2] we introduce the moment map $J : \mathcal{M} \to \mathcal{G}^*$ the dual of the Lie algebra of G, the group of gauge transformations,

$$J(A,\Phi)(\psi) = \int_M dz d\bar z\ D_\theta D_{\bar\theta}\ Str(\psi_\theta\, \mathcal{D}''_\theta\, \Phi_{\bar\theta}) \ . \tag{3}$$

The zero set of this map is

$$\mathcal{D}''_\theta\, \Phi_{\bar\theta} = 0 \ . \tag{4}$$

It defines a supermanifold $\mathcal{N}$, in which we introduce a close 2–form

$$\omega = \int_M dz d\bar z\ D_\theta D_{\bar\theta}\ Str(\delta B^*_{\bar\theta} \wedge \delta B_\theta + \delta\Phi^*_{\bar\theta} \wedge \delta\Phi_\theta)\,, \tag{5}$$

invariant under unitary automorphisms of V. The associated moment map is

$$J(A,\Phi)(\psi) = \int_M dz d\bar z\ D_\theta D_{\bar\theta}\ Str(\psi\{F_{\theta\bar\theta} + [\Phi_\theta, \Phi^*_{\bar\theta}]\})\,, \tag{6}$$

and its zero set

$$F_{\theta\bar\theta} + [\Phi_\theta, \Phi^*_{\bar\theta}] = 0 \ . \tag{7}$$

Eqs.(2), (4) and (7) define super self dual Yang–Mills equations over a super Riemann surface. In [2] we show that they are a SUSY generalization of Hitchin's equations [1].

We are now going to show that (2), (4) and (7) are obtained by dimensional reduction from 4–dim self dual super Yang–Mills equations.

The $N = 1$ SUSY algebra, with zero central charge, in 4–dim Euclidean space in terms of two components spinors is given by [8]

$$\begin{aligned}\{Q_\alpha, Q_\beta\} &= \{Q_{\dot\alpha}, Q_{\dot\beta}\} = \{Q^\alpha, Q^\beta\} = \{Q^{\dot\alpha}, Q^{\dot\beta}\} = \{Q_\alpha, Q^{\dot\beta}\} = \{Q_{\dot\alpha}, Q^\beta\} = \\ &= \{Q_\alpha, Q^\beta\} = \{Q_{\dot\alpha}, Q^{\dot\beta}\} = 0 \ ,\end{aligned}$$

$$\{Q_\alpha, Q_{\dot\beta}\} = -\not P_{\alpha\dot\beta}, \qquad \{Q^\alpha, Q^{\dot\beta}\} = -\not P^{\dot\beta\alpha}\,, \tag{8}$$

where

$$\begin{aligned} \not P_{\alpha\dot\beta} &= i\, e^\mu_{\alpha\dot\beta}\partial_\mu\,, \qquad e^\mu_{\alpha\dot\beta} = (\sigma^0_{\alpha\dot\beta}, -i\,\sigma^i_{\alpha\dot\beta})\,, \\ \not P^{\dot\beta\alpha} &= i\, e^{\mu\dot\beta\alpha}\partial_\mu\,, \quad e^{\mu\dot\beta\alpha} = (\sigma^0_{\alpha\dot\beta}, i\,\sigma^i_{\alpha\dot\beta})\,, \end{aligned}$$

$$e^\mu_{\alpha\dot\beta}\, e^{w\dot\beta\alpha} = 2\,\delta^{\mu w} \ . \tag{9}$$

The SUSY covariant derivatives, as usual, anticommute with the SUSY generators and satisfy the algebra (8) with a change of sign on the left member of (8). In terms of the local coordinates

$x^\mu, \theta^\alpha, \theta_\beta, \theta^{\dot\alpha}, \theta_{\dot\beta}$, where θ_α and $\theta_{\dot\alpha}$ are the complex conjugate to θ^α and $\theta^{\dot\alpha}$ respectively, they may be written:

$$
\begin{aligned}
Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}\, e^\mu_{\alpha\dot\alpha}\, \theta^{\dot\alpha}\, \frac{\partial}{\partial x^\mu} ,\\
Q^\alpha &= -\frac{\partial}{\partial\theta_\alpha} - \frac{i}{2}\, e^{\mu\dot\alpha\alpha}\, \theta_{\dot\alpha}\, \frac{\partial}{\partial x^\mu} ,\\
Q_{\dot\alpha} &= -\frac{\partial}{\partial\theta^{\dot\alpha}} - \frac{i}{2}\, \theta^\alpha\, e^\mu_{\alpha\dot\alpha}\, \frac{\partial}{\partial x^\mu} ,\\
Q^{\dot\alpha} &= \frac{\partial}{\partial\theta_{\dot\alpha}} + \frac{i}{2}\, e^{\mu\dot\alpha\alpha}\, \theta_\alpha ,
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - \frac{i}{2}\, e^\mu_{\alpha\beta}\, \theta^\beta\, \frac{\partial}{\partial x^\mu} ,\\
D^\alpha &= -\frac{\partial}{\partial\theta_\alpha} + \frac{i}{2}\, e^{\mu\dot\alpha\alpha}\, \theta_{\dot\alpha}\, \frac{\partial}{\partial x^\mu} ,\\
D_{\dot\alpha} &= -\frac{\partial}{\partial\theta^{\dot\alpha}} + \frac{i}{2}\, \theta^\alpha\, e^\mu_{\alpha\dot\alpha}\, \frac{\partial}{\partial x^\mu} ,\\
D^{\dot\alpha} &= \frac{\partial}{\partial\theta_{\dot\alpha}} - \frac{i}{2}\, e^{\mu\dot\alpha\alpha}\, \theta_\alpha ,
\end{aligned} \tag{11}
$$

where the fermionic derivatives are left derivatives.

Under a SUSY infinitesimal transformation

$$
\begin{aligned}
x^{\mu'} &= x^\mu - \frac{i}{2}\, \theta^\alpha\, e^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha} + \frac{i}{2}\epsilon^\alpha\, e^\mu_{\alpha\dot\alpha}\, \theta^{\dot\alpha} + \frac{i}{2}\epsilon_{\dot\alpha}\, e^{\mu\dot\alpha\alpha}\, \theta_\alpha - \frac{i}{2}\, \theta_{\dot\alpha}\, e^{\mu\dot\alpha\alpha}\epsilon_\alpha \\
\delta\theta^\alpha &= \epsilon^\alpha, \quad \delta\theta_\alpha = \epsilon_\alpha, \quad \delta\theta^{\dot\alpha} = \epsilon^{\dot\alpha}, \quad \delta\theta_{\dot\alpha} = \epsilon_{\dot\alpha} ,
\end{aligned} \tag{12}
$$

a superfield transforms as

$$
-\delta\phi = \epsilon^\alpha\, Q_\alpha\phi + Q^\alpha\phi \cdot \epsilon_\alpha + \epsilon_{\dot\alpha}\, Q^{\dot\alpha}\phi + Q_{\dot\alpha}\phi \cdot \epsilon^{\dot\alpha} . \tag{13}
$$

The SUSY generators satisfy $(Q^\alpha\phi)^+ = Q_\alpha\phi, (Q^{\dot\alpha}\phi)^+ = Q_{\dot\alpha}\phi$ if ϕ is a real superfield.

We now introduce a superconnection one form Lie algebra valued A and define self dual super Yang–Mills equations by imposing the following constraints on the supercurvature

$$F_{\alpha\beta} = F_{\alpha\dot\beta} = F_{\dot\alpha\dot\beta} = 0 , \tag{14a}$$

$$F^{\alpha\beta} = F^{\alpha\dot\beta} = F^{\dot\alpha\dot\beta} = 0 , \tag{14b}$$

$$F_\alpha{}^{\dot\beta} = F_{\dot\alpha}{}^\beta = 0 , \tag{14c}$$

$$F_\alpha{}^\beta = \delta^\beta_\alpha\, W , \tag{14d}$$

$$F_{\dot\alpha}{}^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}\, \overline{W} = 0 , \tag{14e}$$

where W and $\overline{W}$ are real independent scalar superfields.

In [9] we discuss the set of constraints with $\overline{W} \neq 0$, which lead to Euclidean super Yang–Mills theory, and its relation to the version in [8], based on the work of Osterwalder and Schraeder.

From (14a), (14b) and (14c) we obtain

$$A_\alpha = V^{-1} D_\alpha V \,, \qquad B^{\dot\alpha} = V^{-1} D^{\dot\alpha} V \,,$$
$$A_{\dot\alpha} = U^{-1} D_{\dot\alpha} U \,, \qquad B^{\alpha} = U^{-1} D^{\alpha} U \,, \tag{15}$$

where A_α, $A_{\dot\alpha}$, $B^{\dot\alpha}$ and B^α are the odd components of the superconnection A. We may fix partially the gauge, by imposing the SUSY conditions

$$A_{\dot\alpha} = 0 \,, \qquad B^\alpha = 0 \,. \tag{16}$$

In this gauge

$$D^\beta A_\alpha = \delta^\beta_\alpha W \,,$$
$$D_{\dot\alpha} B^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta} \overline{W} \,, \tag{17}$$

which imply

$$D^\alpha W = D_{\dot\beta} W = 0$$
$$D^{\dot\beta} \overline{W} = D_\beta \overline{W} = 0 \,. \tag{18}$$

To analyze the physical content of the superfield W we linearize the prepotential V. We obtain

$$A_\alpha = D_\alpha V \,, \qquad B^{\dot\alpha} = D^{\dot\alpha} V \,, \tag{19a}$$
$$V = D^\gamma D_\gamma \Omega = D^{\dot\gamma} D_{\dot\gamma} \overline{\Omega} \,. \tag{19b}$$

The components of W and $\overline{W}$ are related by (19b). After imposing the self dual condition, $\overline{W} = 0$, the independent components of W reduce to

$$W, \quad D_{\dot\alpha} W, \quad D^{\dot\beta} W, \quad \epsilon^{(\dot\beta\dot\gamma} D^{\dot\alpha)} D_{\dot\gamma} W \equiv \mathcal{F}^{\dot\beta\dot\alpha} \tag{20}$$

which satisfy the on–shell conditions

$$P^2 W = 0 \,, \quad \not{P}^{\dot\alpha\beta} D_{\dot\alpha} W = 0 \,, \quad \not{P}_{\alpha\dot\beta} D^{\dot\beta} W = 0 \,,$$
$$\not{P}_{\gamma\dot\alpha} \mathcal{F}^{\dot\alpha\dot\beta} = 0 \,. \tag{21}$$

In general, without any partial gauge fixing, we obtain for the curvature superfield $F_{\mu w}$,

$$F_{\mu w} = e_{\mu w}{}^{\dot\alpha\dot\beta} \quad \mathcal{D}_{\dot\alpha} \epsilon_{\dot\beta\dot\gamma} \mathcal{D}^{\dot\gamma} W \equiv e_{\mu w}{}^{\dot\alpha\dot\beta} \mathcal{F}_{\dot\alpha\dot\beta} \,. \tag{22}$$

Its θ independent component gives the standard self dual conditon for the potential A_μ.

In order to obtain 2–dim self dual super Yang–Mills field equations on a super Riemann surface, those equations must be superconformal invariant. We thus introduce new, more appropriate, local coordinates and rewrite (14) in a non–manifest 4–dim Euclidean version. The dimensional reduction to 2–dim is then going to be straightforward. We complexify the local coordinates x^μ, that is, we work on a complex superspace. After doing the dimensional reduction we may reduce again to real x^μ coordinates.

We introduce

$$D_\theta \equiv D_1 + \lambda_1 D_{\dot{1}}, \qquad D_\varphi \equiv D_2 + \lambda_2 D_{\dot{2}} \ , \tag{23}$$

which in terms of

$$\theta \equiv \theta^1 + \frac{\theta^{\dot{1}}}{\lambda_1}\ , \quad \varphi = \theta^2 + \frac{\theta^{\dot{2}}}{\lambda_2}\ , \tag{24}$$

may be written, when applied to a space of functions of $z = x^0 + ix^3, \tilde{z} = x^0 - ix^3, z^*$ and $\tilde{z}^*$,

$$D_\theta = \frac{\partial}{\partial\theta} + i\theta\lambda_1 \frac{\partial}{\partial z}\ , D_{\bar{\theta}} = \frac{\partial}{\partial\theta^*} - i\theta^*\lambda_1 \frac{\partial}{\partial z^*}\ , \{D_\theta, D_\theta\} = 2i\lambda_1 D_z\ , \tag{25a}$$

$$D_\varphi = \frac{\partial}{\partial\varphi} + i\varphi\lambda_2 \frac{\partial}{\partial \tilde{z}}\ , D_{\bar{\varphi}} = \frac{\partial}{\partial\varphi^*} - i\varphi^*\lambda_2 \frac{\partial}{\partial \tilde{z}^*}\ , \{D_\varphi, D_\varphi\} = 2i\lambda_2 D_{\tilde{z}}\ , \tag{25b}$$

* indicates complex conjugation.

We also introduce

$$\begin{aligned}
A_\theta &\equiv A_1 + \lambda_1\ A_{\dot{1}}\ , \\
A_\varphi &\equiv A_2 + \lambda_2\ A_{\dot{2}}\ , \\
A_{\bar{\theta}} &\equiv B^1 + \lambda_1^*\ B^{\dot{1}}\ , \\
A_{\bar{\varphi}} &\equiv B^2 + \lambda_2^*\ B^{\dot{2}}\ .
\end{aligned} \tag{26}$$

$\lambda_i, i = 1,2$ are even parameters.

Constraints (14) may now be rewritten as

$$F_{\theta\theta} = 0\ , \quad F_{\bar{\theta}\bar{\theta}} = 0\ , \tag{27a}$$

$$F_{\theta\varphi} = 0\ , \quad F_{\bar{\theta}\bar{\varphi}} = 0\ , \tag{27b}$$

$$F_{\varphi\varphi} = 0\ , \quad F_{\bar{\varphi}\bar{\varphi}} = 0\ , \tag{27c}$$

$$F_{\theta\bar{\varphi}} = 0\ , \quad F_{\bar{\theta}\varphi} = 0\ , \tag{27d}$$

$$F_{\theta\bar{\theta}} = F_{\varphi\bar{\varphi}}\ . \tag{27e}$$

In fact

$$\begin{aligned}
F_{\theta\theta} &= F_{11} + 2\lambda_1\ F_{1\dot{1}} + \lambda_1^2 F_{\dot{1}\dot{1}} \\
F_{\theta\varphi} &= F_{12} + \lambda_1\ F_{2\dot{1}} + \lambda_2 F_{1\dot{2}} + \lambda_1\lambda_2\ F_{\dot{1}\dot{2}} \\
F_{\varphi\varphi} &= F_{22} + 2\lambda_2\ F_{2\dot{2}} + \lambda_2^2 F_{\dot{2}\dot{2}} \\
F_{\theta\bar{\varphi}} &= F_1{}^2 + \lambda_1\ F_{\dot{1}}{}^2 + \bar{\lambda}_2 F_1{}^{\dot{2}} + \lambda_1\bar{\lambda}_2 F_{\dot{1}}{}^{\dot{2}} \\
F_{\theta\bar{\theta}} &= F_1{}^1 + \lambda_1\ F_{\dot{1}}{}^1 + \bar{\lambda}_1 F_1{}^{\dot{1}} + \lambda_1\bar{\lambda}_1\ F_{\dot{1}}{}^{\dot{1}} \\
F_{\varphi\bar{\varphi}} &= f_2{}^2 + \lambda_2\ F_{\dot{2}}{}^2 + \bar{\lambda}_2 F_2{}^{\dot{2}} + \lambda_2\bar{\lambda}_2 F_{\dot{2}}{}^{\dot{2}}
\end{aligned} \tag{28}$$

and the corresponding expressions for $F_{\bar\theta\bar\theta}, F_{\bar\theta\bar\varphi}, F_{\bar\varphi\bar\varphi}, F_{\bar\theta\varphi}$. Notice that $F_{21}=0$ abd $F_{1\dot 2}=0$ define A_y and $A_{\bar y}$ where $y=\frac{1}{2}(x^1+ix^2), \bar y=\frac{1}{2}(x^1-ix^2)$. Given a solution to (14) it defines, via (26), a solution to (27) for any value of λ.

We now dimensional reduce to 2–dim by taking

$$D_\varphi \cdot = 0 \ ,$$
$$D_{\bar\varphi} \cdot = 0 \ . \tag{29}$$

They imply $\frac{\partial}{\partial z}=\frac{\partial}{\partial z^*}=0$.

Eqs.(27a), (27d) and (27e) then reduce to (2), (4) and (7). Moreover, the algebra (25a) reduces to the superconformal algebra if

$$\lambda_1 = 1 \ . \tag{30}$$

We have thus shown that any solution of 4–dim self dual Yang–Mills which satisfies (29) for $\lambda_1=1$ defines a solution of self dual super Yang–Mills, Eqs.(2), (4) and (7), over a super Riemann surface. We notice that constraints (27b) and (27c) have no geometrical meaning in terms of connections on a vector bundle over a super Riemann surface. (27a), (27d) and (27e) instead are intrinsically related to the geometry of vectors bundles over super Riemann surfaces as we have shown in [2]. In [9] we show the relation between super Teichmüller spaces and self dual connections over super Riemann surfaces.

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