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Abstract

A path integral representation is derived for the Wigner distribution function of a nonequilibrium system coupled with heat bath. Under appropriate conditions, the Wigner distribution function approaches an equilibrium distribution, which manifests shifting and broadening of spectral lines due to the interaction with heat bath. It is shown that the equilibrium distribution becomes the quantum canonical distribution in the vanishing coupling constant limit.

1 Introduction

The aim of this article is to study how a *quantum* dissipative system approaches an equilibrium state and to clarify under what conditions such state becomes the canonical distribution. Although the subject was discussed over many years, the problems have not been studied to enough generality and clarity. We consider here the quantum distribution function (or the Wigner distribution function) of a dissipative system in the form of path integral, which will give full informations about the nonequilibrium process.

To give an outlook on the subject, let us first summarize the known results in the classical statistical mechanics of the nonequilibrium process. It is well known that classically the dynamics of a particle system coupled with a heat bath of temperature T can, under appropriate circumstances, be described by the Langevin equation,

$$\begin{aligned} \frac{dq}{dt} &= \frac{p}{m}, \\ \frac{dp}{dt} &= -\gamma p - V'(q) + E(t). \end{aligned} \tag{1.1}$$

In the above γ and $V(q)$ denote the friction coefficient and the potential respectively. And $E(t)$ is the purely random Gaussian force characterized by

$$\langle E(t_1)E(t_2)\rangle = 2\gamma m k_B T \delta(t_1 - t_2). \tag{1.2}$$

The stochastic process described by eqs (1.1) and (1.2) can, after a standard procedure [1], be transformed into the Fokker-Planck equation (or Kramers equation)

$$\frac{\partial F}{\partial t} = -\frac{p}{m}\frac{\partial F}{\partial q} + V'(q)\frac{\partial F}{\partial p} + \gamma\frac{\partial}{\partial p}(p + m k_B T\frac{\partial}{\partial p})F, \tag{1.3}$$

for the classical distribution function $F(q, p, t|q_0, p_0, t_0)$, the probability density of q and p at time t with the initial condition q_0 and p_0 at time t_0. It is not difficult to show that in the limit of $t - t_0 \to \infty$, the distribution function F approaches an equilibrium one (the canonical distribution)

$$\begin{aligned} F_{eq}(q,p) &= \frac{\exp[-H_0(q,p)/k_B T]}{\int dq dp \exp[-H_0(q,p)/k_B T]}, \\ H_0(q,p) &= \frac{p^2}{2m} + V(q), \end{aligned} \tag{1.4}$$

under plausible assumptions on the behavior of $V(q)$. Such conditions are: (C.1) boundedness of $V(q)$ from below and (C.2) $V(q) \to \infty$ when $q \to \pm\infty$. Note that these assumptions are also sufficient conditions for the denominator in eq.(1.4) to be finite.

Several years ago, Benguria and Kac [2] raised a question if there exists an analogous theory within the realm of quantum mechanics. They started with the quantum Langevin equation devised by Ford, Kac and Mazur [3]. It is the operator version of eq.(1.1) with

$$\begin{aligned} \langle \mathrm{E}(t_1)\mathrm{E}(t_2)\rangle_s &\equiv \frac{1}{2}\langle \mathrm{E}(t_1)\mathrm{E}(t_2) + \mathrm{E}(t_2)\mathrm{E}(t_1)\rangle \\ &= \frac{m\gamma}{\pi}\int_0^\infty d\omega \frac{\hbar\omega}{\tanh(\hbar\omega/2k_B T)} \cos[\omega(t_1 - t_2)], \end{aligned} \tag{1.5}$$

which becomes eq.(1.2) in the classical limit of $\hbar \to 0$. Benguria and Kac restricted the potential $V(q)$ to be the harmonic potential plus a small perturbation for mathematical reasons so that they could express the moment generating function $\langle \exp(bq(t))\rangle$ recursively by perturbation. Fully using the Gaussian nature of eq.(1.5), they analysed this recursion formula and succeeded in showing that $\lim_{\gamma\to 0}\lim_{t-t_0\to\infty}\langle \exp(bq(t))\rangle$ becomes the expectation value of $\exp(bq)$ in the quantum canonical distribution. It is important to take the additional limit of $\gamma \to 0$ after $t - t_0 \to \infty$ for the proof of the approach to the canonical distribution. Here *canonical* implies that the distribution is specified only by the system Hamiltonian. In the case of nonvanishing friction coefficient, one should expect in general shifting and broadening of the spectral lines.

We would like to generalize the work by Benguria and Kac in two respects: first for wider class of heat bath and second by non-perturbative treatment of the Wigner distribution function (instead of moment generating function), which is a quantum counterpart of the classical distribution function F mentioned earlier. For this purpose we first devise a path integral formulation of Wigner distribution function, so that we can discuss in quasi-classical language without being involved with complicated operator algebra.

In §2 a path integral representation of the Wigner distribution function is introduced and the nonequilibrium transition function is derived after elimination of heat bath variables. In §3 we prove that the Wigner distribution function approaches an equilibrium distribution by the use of analytic properties of system variable and interaction kernel in the complex time domain. The last section §4 is devoted to conclusions and discussions. In appendix is given an exact nonequilibrium distribution function of harmonic oscillator which is newly derived by the quantum Langevin equation method.

2 Path integral representation of Wigner distribution function

Since fundamentals and diversities of the Wigner distribution function (Wdf) can be found in the literature [4], we take here the simplest example, whose generalizations are straightforward. Let us consider a particle system whose Hamiltonian is given by

$$\mathrm{H}_0 = \frac{\mathrm{p}^2}{2m} + \mathrm{V}(\mathrm{q}), \tag{2.1}$$

where canonical coordinates (q, p) are quantum operators. The density matrix obeys the Liouville equation

$$i\hbar\frac{\partial}{\partial t}\rho = [\rho, \mathrm{H}_0], \tag{2.2}$$

whose solution is given by

$$\rho(t) = e^{\frac{i}{\hbar}\mathrm{H}_0(t-t_0)}\rho(t_0)e^{-\frac{i}{\hbar}\mathrm{H}_0(t-t_0)}. \tag{2.3}$$

Now the Wdf is defined by the off-diagonal element of density matrix,

$$f(q,p,t) = \int_{-\infty}^{\infty} \mathrm{dQ}e^{-ip\mathrm{Q}/\hbar}\langle q + \frac{\mathrm{Q}}{2}|\rho(t)|q - \frac{\mathrm{Q}}{2}\rangle, \tag{2.4}$$

which is the quantum counterpart of the distribution function F in §1. Substituting eq.(2.3) into eq.(2.4) and following the standard procedure to introduce path integral, we obtain

$$f(q,p,t) = \int\int_{-\infty}^{\infty} \frac{dq_0 dp_0}{2\pi\hbar}\mathrm{K}(q,p,t|q_0,p_0,t_0)f(q_0,p_0,t_0). \tag{2.5}$$

Here the transition function K is given by

$$K(q,p,t|q_0,p_0,t_0) = \int_{-\infty}^{\infty} dQ e^{-ipQ/\hbar} \int_{-\infty}^{\infty} dQ_0 e^{+ip_0Q_0/\hbar}$$
$$\times \int \mathcal{D}q_+(\tau) \int \mathcal{D}q_-(\tau) \exp[\frac{i}{\hbar}\int_{t_0}^{t} d\tau(\mathcal{L}_+ - \mathcal{L}_-)], \qquad (2.6)$$

where Lagrangians $\mathcal{L}_+$ and $\mathcal{L}_-$ are defined by

$$\mathcal{L}_+ = \frac{m}{2}\dot{q}_+^2 - V(q_+), \qquad \mathcal{L}_- = \frac{m}{2}\dot{q}_-^2 - V(q_-). \qquad (2.7)$$

It should be noted that the boundary conditions for the path integral in eq.(2.6) are taken as

$$q_+(t) = q + \frac{Q}{2}, \qquad q_-(t) = q - \frac{Q}{2},$$
$$q_+(t_0) = q + \frac{Q_0}{2}, \qquad q_-(t_0) = q - \frac{Q_0}{2}. \qquad (2.8)$$

Equations (2.6) $\sim$ (2.8) are the fundamental results of the path integral representaion for the Wigner distribution function.

Next we consider total Hamiltonian of system and heat bath

$$H = H_0 + H_B + H_{int}. \qquad (2.9)$$

Here H_0 is the same as in eq.(2.1), and H_B is an assembly of infinite harmonic oscillators whose thermodynamic property is characterized by spectral density $J(\omega)$ introduced later. The interaction part H_{int} is chosen as

$$H_{int} = -\mathbf{q}\cdot\mathbf{X}, \qquad (2.10)$$

where $\mathbf{X}$ is a certain linear combination of the heat bath coordinates. Applying the procedure explained above we obtain the path integral form of the transition function for the total system. Then in accordance with Schwinger [5] and Feynman-Vernon [6] we eliminate the heat bath variables by integrating them out. In the end we obtain the transition function K of the nonequilibrium dissipative system,

$$K(q,p,t|q_0,p_0,t_0) = \int_{-\infty}^{\infty} dQ e^{-ipQ/\hbar} \int_{-\infty}^{\infty} dQ_0 e^{+ip_0Q_0/\hbar}$$
$$\times \int \mathcal{D}q_+(\tau) \int \mathcal{D}q_-(\tau) \exp[\frac{i}{\hbar}\int_{t_0}^{t} d\tau(\mathcal{L}_+ - \mathcal{L}_-)$$
$$-\frac{1}{2\hbar^2}\int_{t_0}^{t} d\tau \int_{t_0}^{t} d\tau' (q_+, -q_-)_\tau A(\tau-\tau') \begin{pmatrix} q_+ \\ -q_- \end{pmatrix}_{\tau'}], (2.11)$$

which has the same structure obtained by Schwinger. In eq.(2.11) Lagrangians $\mathcal{L}_+$, $\mathcal{L}_-$ and boundary conditions for q_+, q_- are as given in eqs (2.7), (2.8). The matrix element of 2×2 matrix A is defined by (see Schwinger [5])

$$\begin{aligned} A_{++}(\tau-\tau') &= \langle (X(\tau)X(\tau'))_+ \rangle_T \\ A_{+-}(\tau-\tau') &= \langle X(\tau')X(\tau) \rangle_T \\ A_{-+}(\tau-\tau') &= \langle X(\tau)X(\tau') \rangle_T \\ A_{--}(\tau-\tau') &= \langle (X(\tau)X(\tau'))_- \rangle_T, \end{aligned} \tag{2.12}$$

where $()_+$ and $()_-$ imply the ordinary time ordered products.

It is convenient for later discussion to write matrix A by a spectral representation. They are

$$\begin{aligned} A_{+-}(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} J(\omega) e^{-i\omega t} \\ A_{-+}(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} J(\omega) e^{\hbar\beta\omega} e^{-i\omega t} \\ A_{++}(t) &+ A_{--}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} J(\omega)(1+e^{\hbar\beta\omega}) e^{-i\omega t} \\ A_{++}(t) &- A_{--}(t) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} P\int \frac{d\omega'}{\omega-\omega'} J(\omega')(e^{\hbar\beta\omega'}-1) e^{-i\omega t} \end{aligned} \tag{2.13}$$

where we denote $\beta = 1/k_B T$, P the principal value. The spectral density $J(\omega)$ is defined by

$$\begin{aligned} J(\omega) &= Z^{-1} \sum_{\mu,\nu} \langle \nu | X(0) | \mu \rangle \langle \mu | X(0) | \nu \rangle e^{-\beta E_\mu} \delta\left(\frac{E_\mu - E_\nu}{\hbar} - \omega\right), \\ Z &\equiv \sum_\mu e^{-\beta E_\mu}, \end{aligned} \tag{2.14}$$

where E_μ' s and $|\mu\rangle$' s are eigenvalues and eigenstates of H_B. Since the spectral density $J(\omega)$ satisfies

$$J(-\omega) = J(\omega) e^{\hbar\beta\omega}, \tag{2.15}$$

we have

$$\begin{aligned} A_{+-}^*(t) &= A_{-+}(t) \\ A_{+-}(t+i\hbar\beta) &= A_{-+}(t), \end{aligned} \tag{2.16}$$

which manifests the Kubo-Martin-Schwinger (KMS) condition. By the use of eqs (2.13) and (2.16), we can rewrite eq.(2.11) as

$$
\begin{aligned}
\mathrm{K}(q,p,t|q_0,p_0,t_0) &= \int_{-\infty}^{\infty} \mathrm{d}Q e^{-ipQ/\hbar} \int_{-\infty}^{\infty} \mathrm{d}Q_0 e^{+ip_0Q_0/\hbar} \\
&\times \int \mathcal{D}q_+(\tau) \int \mathcal{D}q_-(\tau) \exp[\frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau(\mathcal{L}_+ - \mathcal{L}_-) - \frac{1}{\hbar^2}\int_{t_0}^{t}\mathrm{d}\tau \int_{t_0}^{\tau}\mathrm{d}\tau' \\
&(q_+(\tau), -q_-(\tau)) \begin{vmatrix} A_{+-}^*(\tau-\tau'), & A_{+-}(\tau-\tau') \\ A_{+-}^*(\tau-\tau'), & A_{+-}(\tau-\tau') \end{vmatrix} \begin{pmatrix} q_+(\tau') \\ -q_-(\tau') \end{pmatrix}], \quad (2.17)
\end{aligned}
$$

which has the same structure given by Feynman-Vernon. If we write $A_{+-}(t) = A(t) + iB(t)$, eq.(2.16) is expressed as

$$A(t+i\hbar\beta) = A(t), \qquad B(t+i\hbar\beta) = -B(t). \qquad (2.18)$$

This relation will play an important role in the next section.

Some remarks on previous works are in order. Schwinger [5] considered a harmonic oscillator system coupled with not necessarily harmonic heat bath. Taking the Born approximation for the heat bath action, he derived a similar formula as eq.(2.11). For the harmonic oscillator heat bath the Born approximation becomes exact, as Feynman-Vernon [6] found. Although Feynman-Vernon's result is essentially the same as Schwinger's in this sense, the analyticity such as Kramers-Kronig relations and KMS condition (eq.(2.18)) is not so clearly seen. Finally both works did not refer the relation to the Wigner distribution function, which is one of the subjects of the present paper.

3 Approach to an equilibrium distribution

Let us examine what happens for the Wdf when we take $t - t_0 \to \infty$ limit. For the sake of simplicity we consider $t = 0$ and $t_0 \to -\infty$. Recalling the results by Benguria and Kac, we expect that we should obtain an equilibrium distribution which contains shifting and broadening in the spectral lines. This is indeed the case, as is shown below.

First let us give sufficient conditions for the existence of such an equilibrium distribution.

(A.1) The coordinate functional $q(\tau)$ can be analytically continuated to the whole strip $\mathcal{S}$ in the complex τ plane:

$$\mathcal{S} = \{\tau | -\hbar\beta \le \mathrm{Im}\ \tau \le 0, -\infty < \mathrm{Re}\ \tau < +\infty\}. \qquad (3.1)$$

(A.2) The measure $\mathcal{D}q(\tau)$ is invariant under the analytic continuation on $\mathcal{S}$:

$$\mathcal{D}q(\tau) = \mathcal{D}q(\tau - i\hbar\beta). \tag{3.2}$$

(A.3) The generalized Kubo-Martin-Schwinger condition holds:

$$q_+(\tau - i\hbar\beta) = q_-(\tau). \tag{3.3}$$

Some subsidiary comments are due on these conditions. Implication of (A.1) is not trivial. Firstly the potential $V(q)$ cannot be arbitrary in order that (A.1) holds. The simplest sufficient conditions for $V(q)$ are those given in §1: (C.1) boundedness of $V(q)$ from below and (C.2) $V(q) \to +\infty$ when $q \to \pm\infty$. Secondly one must specify the way of analytic continuation, which is defined by

$$q_+(\tau - i\hbar\beta) = U(\beta)^{-1} q_+(\tau) U(\beta). \tag{3.4}$$

Here the *time* translation operator $U(\beta)$ is connected to the equilibrium distribution which is described by a complex and nonlocal Hamiltonian incorporating shifting and broadening, and thus $U(\beta) \neq e^{-\beta H_0}$. It is not always guaranteed that condition (A.2) holds. And it should be proved or disproved for each model. Here we only state that it can be proved to hold for the potential $V(q)$ satisfying (C.1) and (C.2). Condition (A.3) is, despite appearance, not the well known KMS condition, because $q_+(\tau - i\beta\hbar)$ is defined by eq.(3.4). This is why we call it the generalized KMS condition.

Under these conditions (A.1) $\sim$ (A.3) which hold for the potential satisfying (C.1) and (C.2), the proof that the Wdf approaches an equilibrium distribution goes as follows. Using (A.2) and (A.3) we can firstly express the path integral (2.17) (shown graphically at the left in Fig.1) by using only $q_+(\tau)$ as a contour integration along the path shown at the center in Fig.1. Using the analyticity conditions (A.1) and (A.2) we can deform the contour to the right-most one in Fig.1.

To be explicit, the transformations are performed as follows. Firstly we have

$$\begin{aligned}
&\frac{i}{\hbar}\int_{-\infty}^{0} dt(\mathcal{L}_+ - \mathcal{L}_-) \\
&\qquad = \frac{i}{\hbar}\int_{-\infty}^{0} dt(\mathcal{L}(q_+(t)) - \mathcal{L}(q_+(t - i\hbar\beta)) \\
&\qquad = -\frac{1}{\hbar}\int_{0}^{\hbar\beta} d\tau(\frac{\dot{q}^2}{2m} + V(q)),
\end{aligned} \tag{3.5}$$

where we have made a change of variable $t \to -i\tau$. Subtle is the deformation of the interaction part,

$$-\frac{1}{\hbar^2}\int_{-\infty}^{0} dt \int_{-\infty}^{t} dt' \quad \{A(t-t')(q_+(t)-q_-(t))(q_+(t')-q_-(t')) \\ -iB(t-t')(q_+(t)-q_-(t))(q_+(t')+q_-(t'))\}. \quad (3.6)$$

By using eq.(2.18) and by changing the contour, it is transformed into

$$\frac{1}{\hbar^2}\int_0^{\hbar\beta} d\tau \int_0^{\tau} d\tau' q(\tau)\kappa(\tau-\tau')q(\tau'), \quad (3.7)$$

where

$$\kappa(\tau) = A(-i\tau) - iB(-i\tau). \quad (3.8)$$

Then the resulting path integral is

$$\begin{aligned} K_{eq}(q,p) &= \int_{-\infty}^{\infty} dQ e^{-ipQ/\hbar} \int \mathcal{D}q(\tau) \\ &\quad \times \exp[-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau (\frac{m}{2}\dot{q}^2 - V(q)) \\ &\quad + \frac{1}{\hbar^2}\int_0^{\hbar\beta} d\tau \int_0^{\tau} d\tau' q(\tau)\kappa(\tau-\tau')q(\tau')]. \end{aligned} \quad (3.9)$$

The boundary condition is taken as

$$q(0) = q - \frac{Q}{2}, \qquad q(\hbar\beta) = q + \frac{Q}{2}. \quad (3.10)$$

The kernel $\kappa(\tau)$ is the thermal Green function

$$\kappa(\tau) = k_B T \sum_n e^{i\omega_n \tau} \hat{\kappa}(\omega_n), \quad (3.11)$$

with Matsubara frequency $\omega_n = 2\pi n k_B T/\hbar$. And the corresponding double time Green function $A_{-+}(t)$ satisfies the Lehmann-Landau relation

$$\begin{aligned} \mathrm{Re}\hat{A}_{-+}(\omega) &= \frac{1}{2}J(\omega)(1+e^{\hbar\beta\omega}), \\ \mathrm{Im}\hat{A}_{-+}(\omega) &= -\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega-\omega'}J(\omega')\tanh(\frac{1}{2}\hbar\beta\omega'). \end{aligned} \quad (3.12)$$

Let us note that $\kappa(\tau)$, being complex, is a manifestation of *shifting* and broadening of the spectral lines. When we take the limit of vanishing coupling with heat bath, we have vanishing $\kappa(\tau)$ and we obtain the canonical distribution, as is also expected from the work by Benguria and Kac.

Before closing this section, we would like to give a comment on the related work by Caldeira and Leggett [7]. They employed from the begining the Euclidean path integral of the total system, which implies that both the system and the bath are in *equilibrium* from the outset. And then they eliminated the bath variables. By doing this way, they did not find the imaginary part of $\kappa(\tau)$. But, we think, it is necessary from the analyticity point of view. On the other hand we eliminated the heat bath variables in the real time (nonequilibrium) stage and then after taking $t_0 \to -\infty$ we arrived at the imaginary time (Euclidean or equilibrium) stage. The imaginary part of $\kappa(\tau)$ is due to the secular effect of heat bath. Therefore the dissipation discussed in Caldeira-Leggett is caused not by the heat bath but by the instability of the system itself. For an unstable system one cannot expect an equilibrium state since the conditions (C.1) and (C.2) do not hold. The Euclidean path integral for an unstable system is justified only in a *formal* sense. Some remarks on the extension of the present result to unstable cases are given in the next section.

4 Conclusions and discussions

The Wigner distribution function is represented in a path integral form. After elimination of heat bath variables we obtain the Wigner distribution function of nonequilibrium system. Under the analyticity properties of the path integral, we proved in a non-perturbative way that the Wigner distribution function approaches to an equilibrium one. It is also shown that the equilibrium distribution becomes the quantum canonical distribution in the limit of vanishing coupling with heat bath. The proof is restricted to the stable potential $V(q)$. If $V(q)$ is unstable (such as q^3), the analyticity properties (A.1) $\sim$ (A.3) break down in general.

There are seemingly two ways to generalize our results to such unstable case. The first is to introduce an external parameter α which controls the instability ($\alpha > 0$ stable, $\alpha < 0$ unstable). And in accordance with Langer

[8], one considers the analytic continuation with respect to α. It is to this prescription that Callan-Coleman [9] and Caldeira-Leggett [7] resorted. It is stated as

$$\text{decay rate} = \frac{2}{\hbar}\text{Im(ground state energy)}. \qquad (4.1)$$

The second is to consider the stationary (steady state) situation of the distribution function, as Langer [8] did in the classical mechanics realm. One obtains the decay rate by the analyses of a saddle point solution and its quantum fluctuation. Both methods originate from the ingenious works by Langer. And to prove the equivalence of both methods would be almost identical to extend eq.(4.1) to finite temperature. We would like to discuss these issues in future publications.

Appendix

Here we calculate the nonequilibrium distribution function of the quantum harmonic oscillator. Although the expectation value such as $\langle n(t)\rangle$ and $\langle \epsilon(t)\rangle$ were computed before, the distribution function itself is derived here for the first time. We begin with the quantum Langevin equation (we take $m=1$)

$$\begin{aligned} \dot{q} &= u, \\ \dot{u} &= -\gamma u - \omega^2 q + \mathrm{E}(t), \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} \langle \mathrm{E}(t_1)\mathrm{E}(t_2)\rangle_{\mathrm{S}} &\equiv \frac{1}{2}\langle \mathrm{E}(t_1)\mathrm{E}(t_2) + \mathrm{E}(t_2)\mathrm{E}(t_1)\rangle \\ &= \frac{\gamma}{\pi}\int_0^\infty \mathrm{d}\omega \frac{\hbar\omega}{\tanh(\hbar\omega/2k_{\mathrm{B}}\mathrm{T})}\cos[\omega(t_1-t_2)], \end{aligned} \tag{A.2}$$

Let us introduce new variables (see Chandrasekhar [10])

$$\xi = (q\mu_1 - u)\mathrm{e}^{-\mu_2 t}, \qquad \eta = (q\mu_2 - u)\mathrm{e}^{-\mu_1 t}, \tag{A.3}$$

where

$$\mu_1 = \frac{1}{2}(-\gamma + \sqrt{4\gamma^2 - \omega^2}), \qquad \mu_2 = \frac{1}{2}(-\gamma - \sqrt{4\gamma^2 - \omega^2}). \tag{A.4}$$

Then eq.(A.1) is transformed to

$$\dot{\xi} = -\mathrm{e}^{-\mu_2 t}\mathrm{E}(t), \qquad \dot{\eta} = -\mathrm{e}^{-\mu_1 t}\mathrm{E}(t), \tag{A.5}$$

which is solved as

$$\begin{aligned} \xi(t) &= \xi_0 - \int_0^t \mathrm{d}\tau \mathrm{e}^{-\mu_2 \tau}\mathrm{E}(\tau), \\ \eta(t) &= \eta_0 - \int_0^t \mathrm{d}\tau \mathrm{e}^{-\mu_1 \tau}\mathrm{E}(\tau). \end{aligned} \tag{A.6}$$

Now the distribution function $f(\xi,\eta,t)$ is defined by (see also Kubo [4])

$$\begin{aligned} f(\xi,\eta,t) &= \langle \Delta(\xi - \xi(t), \eta - \eta(t))\rangle_{\mathrm{S}} \\ &= \int\int \frac{\mathrm{d}a\mathrm{d}b}{(2\pi)^2}\langle \mathrm{e}^{i(\xi-\xi(t))a + i(\eta-\eta(t))b}\rangle_{\mathrm{S}}, \end{aligned} \tag{A.7}$$

where S denotes the symmetric average defined in eq.(A.2). The exponent of eq.(A.7) is rewritten as

$$i\{(\xi-\xi_0)a+(\eta-\eta_0)b\}+i\int_0^t d\tau(ae^{-\mu_2\tau}+be^{-\mu_1\tau})E(\tau). \tag{A.8}$$

By using the formula

$$\langle e^{i\int_0^t d\tau G(\tau)E(\tau)}\rangle_S=\exp[-\frac{1}{2}\int_0^t d\tau\int_0^t d\tau' G(\tau)G(\tau')\langle E(\tau)E(\tau')\rangle_S], \tag{A.9}$$

we arrive at the final result

$$\begin{aligned} f(\xi,\eta,t) &= \int\int\frac{dadb}{(2\pi)^2}e^{i(\xi-\xi_0)a+i(\eta-\eta_0)b}\exp[-\frac{1}{2}Ba^2+Hab-\frac{1}{2}Ab^2] \\ &= \frac{1}{2\pi\sqrt{\Delta}}\exp[-\frac{1}{2\Delta}\{A(\xi-\xi_0)^2+2H(\xi-\xi_0)(\eta-\eta_0)+B(\eta-\eta_0)^2\}], \end{aligned} \tag{A.10}$$

where

$$\begin{aligned} A &= \frac{\gamma}{\pi}\int_0^\infty d\omega P(\beta\hbar\omega)\int_0^t d\tau\int_0^t d\tau' e^{-\mu_1(\tau+\tau')}\cos\omega(\tau-\tau') \\ B &= \frac{\gamma}{\pi}\int_0^\infty d\omega P(\beta\hbar\omega)\int_0^t d\tau\int_0^t d\tau' e^{-\mu_2(\tau+\tau')}\cos\omega(\tau-\tau') \\ H &= \frac{\gamma}{2\pi}\int_0^\infty d\omega P(\beta\hbar\omega)\int_0^t d\tau\int_0^t d\tau'(e^{-\mu_2\tau-\mu_1\tau'}+e^{-\mu_1\tau-\mu_2\tau'})\cos\omega(\tau-\tau') \\ \Delta &= AB-H^2, \end{aligned} \tag{A.11}$$

with

$$P(x)=\frac{k_B T x}{\tanh(\frac{1}{2}x)}. \tag{A.12}$$

Although the double time integrations can be explicitly performed (which we omit since it is too long), the remaining ω integration cannot be done in a closed form. It is, however, amusing to realize that as $t\to\infty$ eq.(A.10) becomes the canonical distribution. It is the exceptional nature of harmonic oscillator that the additional limit of $\gamma\to 0$ is not needed to realize the canonical distribution.

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Figure Caption

Fig. 1 Integration contours in complex time domain.

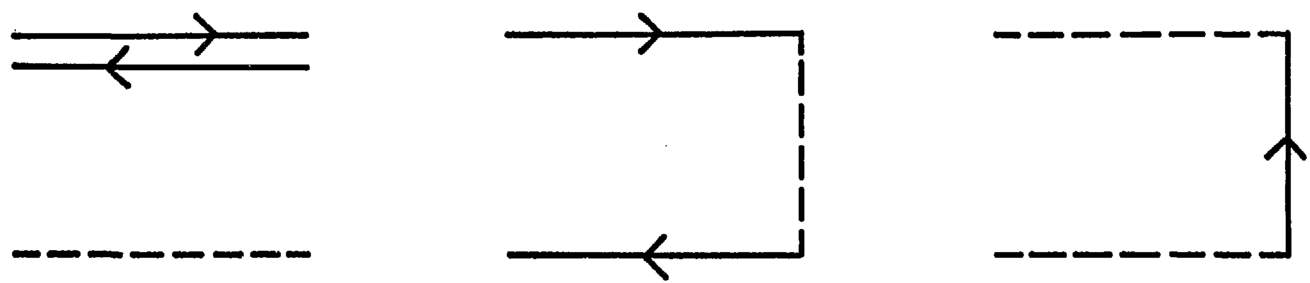

Fig.1