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PHASE TRANSITIONS INDUCED BY THE AHARONOV–BOHM FIELD *

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ABSTRACT

The influence of the Aharonov–Bohm flux (ϕ) on the order parameters of the 3–dimensional Gross–Neveu model and CP^N–model in $R^2 \times S^1$ space is considered. It is shown that the variation of flux causes the order parameter oscillations and for the small enough length of circular coordinate $\ell < \ell_c$ these oscillations attended with re–ordering phase transitions (i.e. the repeating transitions between the ordered and the disordered phases of the models in question).

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1. Introduction

Recent progress in superconductivity has initiated a burst of activity on (2 + 1)-dimensional field theories. Basically, most of high temperature superconductors have been found to be layer-structured antiferromagnets. Further investigations have exposed intimate relations between planar Hubbard systems and quantum field models in three dimensions [1, 2]. Particularly, it is widely believed that the nonlinear $O(3)$ sigma model provides quite a satisfactory basis for studying quantum antiferromagnets in the continuum limit [3].

The most peculiar feature of the three-dimensional $O(3)$ sigma model is the existence of two stable phases: the ordered (Goldstone) and the disordered (massive) ones [2-4]. The dynamically generated mass depends on the variety of external parameters, e.g., temperature [5], chemical potentials [6], background electric fields [7], stochastic sources [8]*.

It seems reasonable that the Aharonov-Bohm (A-B) background field A_μ should affect vacuum in a striking fashion: in multiply connected spaces, like $R^{d-1} \times S^1$, the charged fields spectra depend on the Aharonov-Bohm (A-B) flux

$$\Phi = \oint_{S^1} A_\mu dx_\mu \tag{1.1}$$

so that thermodynamic potentials are periodic in Φ (oscillations in condensed matter are discussed in Ref. 9; for relativistic models, see Ref. 10).

In this paper we present a detailed analysis of the A-B flux influence on the vacuum properties of the Gross-Neveu and the CP^{N-1} in the cylindrical space $R^2 \times S^1$ (for a motivation

* Note that Refs. 5-8 are concerned with d = 2.

of the choice made, see below).

Specifically, we denote ℓ the length of S^1 and assume that

$$A_\mu = (0, 0, A) \quad A = const \tag{1.2}$$

for the flux to take the form

$$\Phi = \ell A \tag{1.3}$$

We show that, at ℓ less than certain $\ell_c^{(max)}$, the A-B flux causes phase transitions (which will be referred to as re-ordering A-B oscillations). We derive an explicit formula for the critical flux value and present the corresponding phase diagrams.

The d = 3 Gross-Neveu model (unlike the d = 2 one, [11]) does not seem to have any applications to the solid state physics (especially when the A-B flux is treated by hand), it can still be considered as a good toy model.

On the contrary, the d = 3 CP^{N-1} model is much more realistic (just recall the equivalence between the CP^{N-1} and the $O(3)$ sigma models). Although its fundamental fields z_j, j = 1 ... N, are electrically neutral and thus do not couple to the background field A_μ, one can incorporate fermions [12] carrying both electric and internal $U(1)$ charges (see Sec. 3 for details). The resulting model provides a reliable description of doped antiferromagnetic Mott insulators [2]. Recently the corresponding hole-driving Lagrangian has been derived from the microscopic spin exchange Hamiltonian [13].

The paper is arranged as follows. In Sec. 2 we set up a formalism and derive the advertized results for the Gross-Neveu model. The treatment, very simple in the case, offers a good guide to much more cumbersome Sec. 3 in which we analyze the CP^{N-1} model.

2. A Simple Example of the Aharonov-Bohm Oscillations.

In this section we wish to set an instructive example of how the A-B oscillation occur. Let us consider the d = 3 Gross-Neveu model in the background A-B field given by Eq. (1.2).

The action thus reads:

$$S = \int_{R^2 \times S^1} d^3x \left\{ \bar{\Psi}_j \left[i\gamma_\mu(\partial_\mu + ieA_\mu) - g\sigma \right] \psi_j - \frac{N}{2}\sigma^2 \right\}, \tag{2.1}$$

where $j = 1 \dots N$ is a flavor index, g is a coupling constant, $\sigma(x)$ is an auxiliary field introduced to make the Lagrangian quadratic in ψ.

The analysis proceeds by first, integrating out fermions, and then employing the 1/N-approach to expand the resultant effective action in powers of $\sigma(x)$. One easily finds that the auxiliary field develops a non-zero vacuum expectation value $\langle\sigma\rangle = const$, with the fermions acquiring the mass $m = g\langle\sigma\rangle$.

Consider the most common boundary condition on S^1:

$$\psi(t, x, y+\ell) = e^{i\alpha}\psi(t, x, y) \tag{2.2}$$

then the fermionic spectrum becomes

$$\lambda_n(p) = m^2 + p^2 + \left(\frac{2\pi}{\ell} - eA + \frac{\alpha}{\ell}\right)^2, \quad n \text{ integer.} \tag{2.3}$$

Notice that Eq. (2.2) includes both ordinary ($\alpha = 0, \bmod 2\pi$) and twisted ($\alpha = \pi, \bmod 2\pi$) fermions [14].

Now, we use Eq. (2.3) to obtain the effective action stationary equation with respect to the mass gap m:

$$\frac{1}{g^2} = \frac{2}{\ell} \sum_{n=-\infty}^{\infty} \int_m^\infty \frac{dE}{2\pi} \frac{1}{E^2 + (2\pi n/\ell - eA + \alpha/\ell)^2} \tag{2.4}$$

Since the integral on the right-hand side is linear UV-divergent, we introduce a cut-off Λ. Proceeding along the lines of Ref. 15 one finds Eq. (2.4) to have two different branches subject to the value of the bare coupling g_Λ:

(i) the "classical" phase: $m_o = 0,$ at $g_\Lambda \leq g_c = \sqrt{\frac{2\pi}{\Lambda}}$;

(ii) the "quantum" phase: $m_o > 0,$ at $g_\Lambda > g_c$;

where m_o is the "vacuum" mass (in the sense that $A=0, \ell \to \infty$). Recall that the "vacuum" part of the effective action is only used for the renormalization procedure.

As the "quantum" phase is of primary interest, we choose $g_\Lambda > g_c$. Then the divergence on the right-hand side of Eq. (2.4) can be absorbed into m_o via

$$\frac{1}{g_\Lambda^2} = \int_{m_o}^{\Lambda} \frac{dE}{2\pi}, \quad m_o = g_c^{-2} - g_\Lambda^{-2}. \tag{2.5}$$

No other divergences will bother us since the model under consideration is renormalizable within the 1/N expansion [15].

The renormalized fermionic mass as a function of the A-B flux Φ can be evaluated explicitly:

$$m(\ell,\alpha;\Phi) = \frac{1}{\ell}\cosh^{-1}\left[\frac{1}{2}e^{\ell m_o} + \cos(2\pi\Phi/\Phi_o - \alpha)\right], \tag{2.6}$$

where $\Phi_o = 2\pi/e$ is the unit flux. Eq. (2.6) clearly exposes the effect of A-B oscillations /10, 16/: varying Φ leads to "breathing" of the induced mass gap m.

For large ℓ (i.e. $\ell m_o \gg 1$) we obtain

$$m(\ell,\alpha;\Phi) \simeq m_o\left[1 + \frac{2\cos(2\pi\Phi/\Phi_o - \alpha)}{\ell m_o} e^{-\ell m_o}\right]. \tag{2.7}$$

Consider, for instance, $\Phi = 0$. One immediately finds that m vs ℓ behavior is sufficiently different for ordinary

($\alpha = 0$) and twisted ($\alpha = \pi$) fermions. In the former case $m \propto 1/\ell$ as ℓ tends to zero. The situation changes drastically for $\alpha = \pi$: the mass m decreases along with ℓ, and there exists the critical length

$$\ell_c^{(max)} = 2\ln 2/m_o, \quad m(\ell_c^{(max)}, \pi; 0) = 0. \tag{2.8}$$

It is no surprise because twisted fermions can be interpreted as those at temperature $T = 1/\ell$ /17/. The critical temperature for the $d=3$ Gross-Neveu model was found in Ref. 18; also note the similar m vs ℓ behavior for $d=2$ /10, 19/.

How does the A-B flux Φ affect the mass gap? It simply produces an extra "twist" giving rise to the "Casimir heating". Consequently, at any α, there exists a region $\ell \leq \ell_c^{(max)}$ where varying Φ causes oscillations between the two phases. These can be viewed as the A-B oscillations of the critical length $\ell_c(\Phi)$ *. Setting $m(\ell = \ell_c, \alpha; \Phi) = 0$ we arrive at

$$\ell_c(\Phi) = \frac{2}{m_o}\ln\left|2\sin(\pi\Phi/\Phi_o - \alpha/2)\right|. \tag{2.9}$$

The corresponding phase diagram for ordinary ($\alpha = 0$) fermions is shown in Fig. 1. The maximum of the phase curve corresponds to $\ell_c^{(max)} = 2\ln 2/m_o$. At $\ell > \ell_c^{(max)}$ only the massive quantum phase is available.

Fig. 2 exhibits the dependence m vs Φ for two typical values of ℓ. The derivative of m with respect to Φ becomes infinitely large at the phase transition point

$$\Phi_c = \frac{\Phi}{2\pi}\cos^{-1}\left(1 - \frac{1}{2}e^{\ell m_o}\right), \quad \alpha = 0. \tag{2.10}$$

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*Compare with the fact that in thin-walled superconducting hollow cylinders, the flux quantization causes oscillations of the critical temperature [20].

Now we are ready to start our assault on the CP^{N-1} model.

3. Re-ordering Oscillations in the CP^{N-1} Model.

Let us briefly recall the main features of the model (see Ref. 21 for a good review). It contains complex N -vector fields $z_j(x)$ and an auxiliary $U(1)$ gauge potential $V_\mu(x)$. Classically, the latter is just a composite field; at the quantum level, however, it behaves like a true gauge field. Besides, we introduce the background electromagnetic potential A_μ (viz. Eq. (1.2)) coupled to the fermionic field ψ, and denote g_f and e the charges corresponding to V_μ and A_μ respectively.

Since long-range forces are only relevant to the possible applications, we start with the following truncated action:

$$S=\int_{R^2\times S^1} d^3x \left\{ |(\partial+iV_\mu)z|^2+M^2\left(z^+z-\frac{N}{g}\right)+ \right. \qquad (3.1)$$

$$\left. +\bar{\psi}\left[i\gamma_\mu(\partial_\mu+ig_fV_\mu+ieA_\mu)-m\right]\psi\right\}.$$

In Eq. (3.1), we have ignored the short-range interaction (which enforces $z^+z=\frac{N}{g}$ at the classical level) save that the mass M be found at the stationary point.

As it was mentioned in the introduction, the action (3.1) provided $N=2$, $m=0$, $g_f=\pm 1$ gives a satisfactory description of a doped planar antiferromagnet [2, 13], where a three-vector $\vec{n}=z^+\vec{\sigma}z$ should be identified with a antiferromagnetism vector ($\vec{\sigma}$ being the Pauli matrices).

We assume that the topology $R^2\times S^1$ is fixed by the presence of a thin solenoid which creates the flux $\Phi=\ell A$. The fields ψ and $\vec{n}$ must obey the periodic boundary conditions

on S^1. The latter requirement is, however, compatible with

$$\psi(t,x,y+\ell)=\psi(t,x,y),\quad z(t,x,y+\ell)=e^{i\alpha}z(t,x,y). \qquad (3.2)$$

Using Eqs. (3.1), (3.2) the effective potential $V_{eff}(M,V;\Phi)$ can be evaluated in a standard fashion; the resultant expression is, however, too cumbersome to be presented here. (In fact, we need the stationary equations rather than V_{eff} itself).

The stationary equation with respect to M implies

$$M(\ell)=\frac{1}{\ell}\cosh^{-1}\left[\frac{1}{2}e^{\ell m_0}+\cos(\ell V-\alpha)\right]. \qquad (3.3)$$

The "vacuum" mass gap M_0 (i.e. M at $V=0,\ \ell\to\infty$) is given by

$$\frac{4\pi}{g_\Lambda}=\Lambda-M_0 \qquad (3.4)$$

where g_Λ is the bare coupling, Λ is a cut-off introduced to eliminate the UV-divergences from V_{eff} [2,4].

The previous two expressions are almost identical to those of the Gross-Neveu model (cf. Eqs. (2.6), (2.4)). It seems quite natural that the conclusions drawn in Sec. 2 are still valid for the CP^{N-1} model and this is really the case. We will be brief on this point rather focussing on the distinctions to arise.

Two different phases occur depending on the value of g_Λ [1, 2]:

(i) the ordered phase: $M_0=0$, at $g_\Lambda\leq g_c=\frac{4\pi}{\Lambda}$;

(ii) the disordered (quantum paramagnetic) phase: $M_0>0$, at $g_\Lambda>g_c$. Of course, there are oscillations between the two, although the arguments are not so trivial as in Sec. 2.

In everything to follow it is essential to realize that V in Eq. (3.3) is by no means at our disposal. Recall that the field

V_μ appears to be dynamical under the quantum treatment. Due to the non-trivial topology, it condenses [21]:

$$\langle V_\mu \rangle = (0, 0, V), \quad V = \mathrm{const} \tag{3.5}$$

The mean value V, mentioned in Eq. (3.3), is fixed by the requirement that V_{eff} be minimized in V.

Thus no trace of the external flux Φ is seen in Eq. (3.3). Where do the A-B oscillations, if any, emerge? The answer to this is simple: the mean value V is driven by the flux Φ and other fermionic parameters, via the minimum equations with respect to V.

To make things more clear let us, for instance, switch off the fermions. Since the background field A_μ is coupled to Ψ (viz. Eq. (3.1)), we do not expect the mass gap to "breath". The minimum conditions in V yield

$$\ell V - \alpha = 2\pi n, \quad n \ \text{integer} \tag{3.6}$$

so as to cancel out the fictitious A-B flux $-\alpha$ produced by the "twisted" boundary conditions (3.2). Hence the physical properties of the model are independent of any specific α choice, as it should be by the gauge invariance.

Inserting Eq. (3.6) into Eq. (3.3) we obtain

$$M(\ell) \simeq \begin{cases} M_0 + \frac{2}{\ell} e^{-\ell m_0}, & \text{at } \ell M_0 \gg 1, \\ \frac{1}{\ell} \ln \frac{3+\sqrt{5}}{2}, & \text{at } \ell M_0 \ll 1, \end{cases} \tag{3.7}$$

i.e. the mass gap increases monotonically along with $1/\ell$. The reason for this is quite transparent: one can interprete as the "Casimir temperature" [17], and temperature is found to make z -field heavier [5, 21].

Therefore, to allow phase transitions one should incorporate

fermions into the model. The consequent extremum equation with respect to V is:

$$N \sin(\ell V - \alpha) \int_{\ell M(\ell)}^{\infty} \frac{t\,dt}{\cosh t - \cos(\ell V - \alpha)} =$$

$$= \sum_f g_f \sin(g_f \ell V + 2\pi \Phi/\Phi_0) \int_{\ell m}^{\infty} \frac{t\,dt}{\cosh t - \cos(g_f \ell V + 2\pi\Phi/\Phi_0)}, \tag{3.8}$$

where $\Phi_0 = 2\pi/e$ is the unit flux, and the sum $\sum_f$ is taken over the fermionic flavors.

We restrict ourselves to the following cases:

(a) N massless ($m=0$) fermion singlets (with $g_f = 1$ each);

(b) a massless (internal-) charge doublet ($g_f = \pm 1$) attached to the CP^1 model ($N = 2$)*.

Likewise in Sec. 2, we wish to obtain phase diagrams. The critical length ℓ_c can easily be derived from Eq. (3.3) provided $\ell = \ell_c$, $M(\ell_c) = 0$. After some algebra we arrive at

$$L(y+x) - L(y) - x \ln 2 = 0, \tag{3.9a}$$

$$L(y+x) - L(y-x) - 2L(y) = 0, \tag{3.9b}$$

where

$$L(z) = -\int_0^z \ln \cos t \, dt \tag{3.10}$$

is the Lobachevsky function [22]. The variables x and y are given by

$$x = \pi \Phi/\Phi_0 + \alpha/2, \quad y = \frac{\pi}{2} + (\ell V - \alpha)/2. \tag{3.11}$$

The analysis proceeds in the following fashion. One should, first,

* As it has been discussed at the beginning of this section, the latter case coincides with a model of a doped planar antiferromagnet.

find the solution $y=y_\ell(x)$ of Eq. (3.9) in order to define V as the function of the A-B flux Φ ; and then insert $V(\Phi)$ into Eq. (3.3) to get $\ell_c(\Phi)$

Using the identity [22]

$$L(\pi \pm z) = \pm L(z) + \pi \ln 2 \tag{3.12}$$

one easily verifies that $y_\ell(x)$ are π - periodic functions for either case, (3.9 a,b), so that one can restrict himself to a single period: $-\pi/2 \le x \le \pi/2$. Furthermore, the positiveness of ℓ_c requires $|y| \le \pi/3$.

Eqs. (3.9) apparently have to be solved numerically. The respective phase curves are shown in Fig. 3 (a, b).

As it is seen directly from Eq. (3.3), the two phases co-exist up to the length

$$\ell_c^{(max)} = 2\ln 2/M_0 \tag{3.13}$$

(cf. Eq. (2.8)) which corresponds to the maximum of each curve, namely

$$y_\ell(x=\pi/2) = 1, \tag{3.14a}$$

$$y_\ell(x \to 0) \to 1. \tag{3.14b}$$

Hence, Eqs. (3.9) do have physical (i.e. at $|y| \le \pi/3$) solutions for the whole x range, so the critical length $\ell_c(\Phi)$ exists at any external flux Φ , as distinct from the case of the Gross-Neveu model (Fig. 1).

To summarize, we have shown that varying Φ leads to re-ordering oscillations, i.e. the oscillations between the ordered and the disordered phases.

Note also that the critical flux value Φ_c depends on the "twisting" parameter α which is left arbitraty. The α - dependence vanishes if we imply "twisted" boundary conditions for

the fermions as well

$$\psi(t,x,y+\ell) = e^{-i\alpha}\psi(t,x,y) \tag{3.15}$$

It seems reasonable for the case of the supersymmetric model.

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FIGURE CAPTIONS

FIG.1
The phase diagram for the d=3 Gross-Neveu model with ordinary (α = 0) fermions.

FIG.2
Typical m(Φ) behavior for the d=3 Gross-Neveu model at:

1) $l = \ln 2 / m_0$

2) $l = 2 \ln 2 / m_0$

FIG.3
The phase diagrams for the d=3 CP^{N-1} - model. Aforementioned (i) and (ii) cases are shown in (a) and (b) respectively.

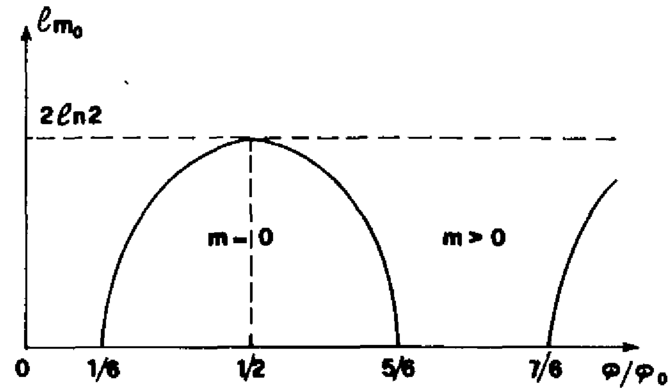
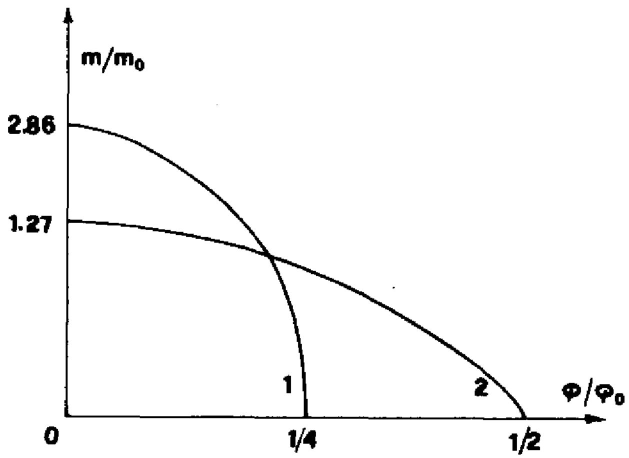


Fig.1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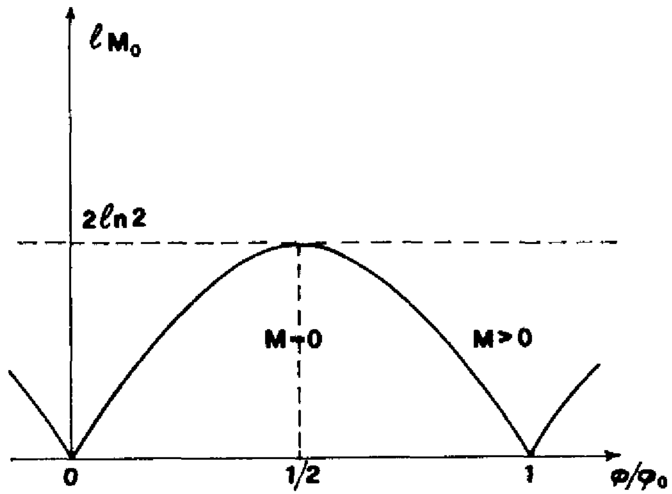
2.86
1.27
1
2
φ/φ_0
0
1/4
1/2

Fig.2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$2\ell n 2$
M=0
M>0
0
1/2
1
φ/φ_0

Fig.3a

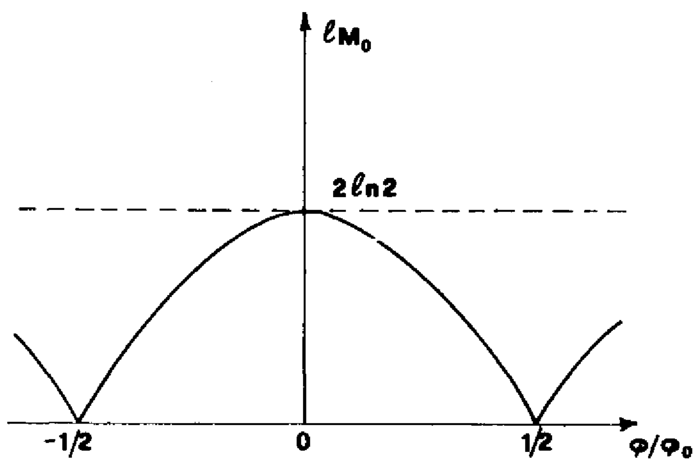

Fig.3b

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