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ON THE CONCIRCULAR CURVATURE TENSOR OF RIEMANNIAN MANIFOLDS *

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ABSTRACT

Definition of the concircular curvature tensor, Z_{hijk}, along with Z–tensor, Z_{ij}, is given and some properties of Z_{hijk} are described. Tensors identical with Z_{hijk} are shown. A necessary and sufficient condition that a Riemannian V_n has zero Z–tensor is found. A number of theorems on concircular symmetric space, concircular recurrent space (Z_n–space) and Z_n–space with zero Z–tensor are deduced.

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1. INTRODUCTION

The *concircular curvature tensor* of a Riemannian manifold $V_n(n > 2)$ was introduced by Yano (1940). This is defined by

$$Z_{hijk} = R_{hijk} - \frac{R}{n(n-1)}(g_{hk}\, g_{ij} - g_{hj}\, g_{ik}) . \qquad (1.1)$$

The tensor, since its discovery, has come into great prominence and proved of considerable interest in Riemannian geometry.

Transvection of (1.1) with g^{hk} gives rise to another tensor which we call Z*–tensor* and denote by Z_{ij}.
We then have

$$Z_{ij} = R_{ij} - \frac{R}{n}\, g_{ij} . \qquad (1.2)$$

We write the projective and conformal curvature tensors:

$$P_{hijk} = R_{hijk} - \frac{1}{n-1}(g_{hk}\, R_{ij} - g_{hj}\, R_{ik}) ,$$

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{hk}\, R_{ij} - g_{hj}\, R_{ik} + g_{ij}\, R_{hk} - g_{ik}\, R_{hj}) + \frac{R}{(n-1)(n-2)}(g_{hk}\, g_{ij} - g_{hj}\, g_{ik}) .$$

We note in the following another tensor studied by Yano and Sawaki (1968):

$$W_{hijk} = a\, R_{hijk} + b(g_{hk}\, Z_{ij} - g_{hj}\, Z_{ik} + g_{ij}\, Z_{hk} - g_{ik}\, Z_{hj}) ,$$

where a and b are constants.

Desai and Amur (1975, pp.98–102) expressed P_{hijk}, C_{hijk} and W_{hijk} in terms of Z_{hijk} and Z_{ij}.

We now see that the above tensors are also expressible in terms of Z_{hijk} and P_{ij} where the latter is defined as follows:

$$P_{ij} = g^{hk}\, P_{ihkj} = \frac{n}{n-1}(R_{ij} - \frac{R}{n}\, g_{ij}) . \qquad (1.3)$$

We, however, write out Z_{hijk} as asserted:

$$Z_{hijk} = P_{hijk} + \frac{1}{n}(g_{hk}P_{ij} - g_{hj}\,P_{ik}) \,, \tag{1.4a}$$

$$Z_{hijk} = C_{hijk} + \frac{n-1}{n(n-2)}(g_{hk}\,P_{ij} - g_{hj}\,P_{ik} + g_{ij}\,P_{hk} - g_{ik}\,P_{hj}) \,, \tag{1.4b}$$

$$aZ_{hijk} = W_{hijk} - \frac{n-1}{n}b(g_{hk}\,P_{ij} - g_{hj}\,P_{ik} + g_{ij}\,P_{hk} - g_{ik}\,P_{hj}) \,. \tag{1.4c}$$

The concircular curvature tensor Z_{hijk} has the following properties as possessed by the Riemann symbols (Rahman, 1988):

$$Z_{hijk} = -Z_{ihjk} \,,$$
$$Z_{hijk} = -Z_{hikj} \,,$$
$$Z_{hijk} = Z_{jkhi} \,,$$
$$Z_{hijk} + Z_{hjki} + Z_{hkij} = 0 \,.$$

The paper aims at presenting some results on concircular curvature tensor Z_{hijk} and on related class of spaces characterized by Z_{hijk}.

2. TENSORS IDENTICAL WITH Z_{hijk}

We shall first turn to the question of determining the conditions for which Z_{hijk} is identical with P_{hijk}, C_{hijk} and W_{hijk}.

If a Riemannian V_n is Einstein then it is obvious from (1.3) that $P_{ij} = 0$. Hence the conditions to be determined follow from (1.4a), (1.4b) and (1.4c):

In an Einstein V_n, $Z_{hijk} = P_{hijk} = C_{hijk} = W_{hijk}$ provided a is unity.

3. ZERO Z-TENSOR

We shall here investigate into how the Z-tensor of a V_n becomes a zero tensor.

Let the Z-tensor of a V_n be a zero tensor. Eq.(1.2) then gives $R_{ij} = \frac{R}{n}\, g_{ij}$.

This means that V_n is *Einstein*.

Conversely, if a V_n is an Einstein space then $R_{ij} = \frac{R}{n}\, g_{ij}$ which reduces (1.2) to $Z_{ij} = 0$.

We thus summarize the result as follows:

A necessary and sufficient condition that a V_n has zero Z-tensor is that the space be Einstein.

4. CONCIRCULAR SYMMETRIC SPACE

Definition 4.1 A Riemannian V_n for which $Z_{hijk,l} = 0$ is called a *concircular symmetric space.*

We are in a position to deduce

Theorem 4.1 *The scalar curvature of a concircular symmetric space is constant.*

Proof Suppose a V_n is concircular symmetric. Then, by Definition 4.1, $Z_{hijk,l} = 0$.

That is,

$$R_{hijk,l} = \frac{1}{n(n-1)} R_{,l}\,(g_{hk}\,g_{ij} - g_{hj}\,g_{ik}) \,.$$

As a consequence of the Bianchi identity we have

$$R_{,l}\,(g_{hk}\,g_{ij} - g_{hj}\,g_{ik}) + R_{,j}\,(g_{hl}\,g_{ik} - g_{hk}\,g_{il}) + R_{,k}\,(g_{hj}\,g_{il} - g_{hl}\,g_{ij}) = 0 \,.$$

Transvecting this with g^{hk} we get for $n > 2$,

$$(R_{,l}\,g_{ij} - R_{,j}\,g_{il}) = 0 \,.$$

The above one, on contracting with g^{ij}, yields

$$(n-1)\,R_{,l} = 0 \,.$$

Hence the *scalar curvature R is constant.*

This completes the proof.

It is known (Desai and Amur, 1975; pp.119–124) that a *Riemannian V_n is concircularly symmetric if and only if it is symmetric in the sense of Cartan.*

We shall give below a proof of the theorem in a simpler method than that employed by Desai and Amur.

If a V_n is symmetric in the sense of Cartan, it is evidently concircular symmetric.

Conversely, if a V_n is concircular symmetric then $Z_{hijk,l} = 0$ which, because of $R_{,l} = 0$ as in Theorem 4.1, readily gives $R_{hijk,l} = 0$.

We are next able to present

Theorem 4.2 *A Riemannian space V_n is projective symmetric if and only if it is concircular symmetric.*

Proof Let a Riemannian V_n be concircular symmetric. Therefore, in view of (1.4a) and Definition 4.1, we have

$$P_{hijk,l} + \frac{1}{n}(g_{hk}\,P_{ij,l} - g_{hj}\,P_{ik,l}) = 0 \,. \tag{4.1}$$

This, on transvection with g^{ij}, yields

$$P_{hk,l} - \frac{1}{n} P_{hk,l} = 0$$

whence

$$P_{hk,l} = 0 \ . \tag{4.2}$$

Hence it follows from (4.1), in virtue of (4.2), that $P_{hijk,l} = 0$.

Suppose now $P_{hijk,l} = 0$. This leads to $P_{ij,l} = 0$.

We thus conclude from (1.4a) that $Z_{hijk,l} = 0$.

We next note in the following a part from Theorems 2.1 and 2.2 respectively (Desai and Amur, 1975; pp.119–124) and make some observations:

If the covariant derivative of the Ricci tensor is symmetric then $Z^h_{ijk,h} = 0$.

Suppose the covariant derivative of the Ricci tensor is symmetric. Then the concircular curvature tensor satisfies the Bianchi identity.

In proving the former theorem the authors made use of a lemma which states:

The covariant derivative of the Ricci tensor is symmetric implies that the scalar curvature is constant.

Observations

Since the concircular symmetric space has constant scalar curvature R, use is not required of the lemma to prove the former theorem.

In view of the constancy of R, the latter theorem follows without being the covariant derivative of Ricci tensor symmetric.

The theorem may therefore be rephrased as

The concircular curvature tensor satisfies the Bianchi identity $Z_{hijk,l} + Z_{hikl,j} + Z_{hilj,k} = 0$ *analogous to the one for curvature tensor.*

5. CONCIRCULAR RECURRENT SPACE

Definition 5.1 A Riemannian V_n–space is said to be *concircular recurrent* if its concircular curvature tensor satisfies the relation

$$Z_{hijk,l} = \kappa_l Z_{hijk} \tag{5.1}$$

for some non–zero vector κ_l.

We call κ_l the *recurrence vector*, and denote a concircular recurrent V_n by Z_n.

We are able to establish

Theorem 5.1 *A necessary and sufficient condition that a Riemannian space be concircular recurrent is that* $P_{hijk,l} = \kappa_l P_{hijk}$.

Proof Consider those Riemannian V_n which are concircular recurrent.

Eq.(5.1), together with (1.4a), gives

$$P_{hijk,l} + \frac{1}{n}(g_{hk} P_{ij,l} - g_{hj} P_{ik,l}) = \kappa_l P_{hijk} + \frac{1}{n}\kappa_l(g_{hk} P_{ij} - g_{hj} P_{ik}) \ . \tag{5.2}$$

It follows from (5.1) that

$$Z_{ij,l} = \kappa_l Z_{ij} \ . \tag{5.3}$$

Since $P_{ij} = \frac{n}{n-1} Z_{ij}$ we also have

$$P_{ij,l} = \kappa_l P_{ij} \ .$$

On applying this to (5.2) we obtain

$$P_{hijk,l} = \kappa_l P_{hijk} \ .$$

Conversely, if $P_{hijk,l} = \kappa_l P_{hijk}$ then

$$Z_{hijk,l} - \frac{1}{n-1}(g_{hk} Z_{ij,l} - g_{hj} Z_{ik,l}) = \kappa_l Z_{hijk} - \frac{1}{n-1}\kappa_l(g_{hk} Z_{ij} - g_{hj} Z_{ik})$$

whence, by use of (5.3), we find that $Z_{hijk,l} = \kappa_l Z_{hijk}$.

Theorem 5.2 *Every symmetric* Z_n *is a space of constant curvature.*

Proof It follows from Theorem 4.1 and (5.1) that

$$R_{hijk,l} = \kappa_l \left[R_{hijk} - \frac{R}{n(n-1)}(g_{hk} g_{ij} - g_{hj} g_{ik}) \right] \ .$$

If Z_n is symmetric then $R_{hijk,l} = 0$, and the above one, because of $\kappa_l \neq 0$, yields

$$R_{hijk} = \frac{R}{n(n-1)}(g_{hk} g_{ij} - g_{hj} g_{ik}) \ .$$

This, R being constant, shows that *every symmetric* Z_n *is a space of constant curvature.*

Theorem 5.3 *A* Z_n*–space has a gradient recurrence vector or has constant curvature.*

Proof It is evident from (5.1) that

$$Z_{hijk,lm} - Z_{hijk,ml} = R_{hijk,lm} - R_{hijk,ml}$$

$$= \kappa_{lm} Z_{hijk}$$

where

$$\kappa_{lm} = \kappa_{l,m} - \kappa_{m,l} \; .$$

On applying the above second relation to Lemma 1 (Walker, 1950) we find

$$\kappa_{lm}\, Z_{hijk} + \kappa_{hi}\, Z_{jklm} + \kappa_{jk}\, Z_{lmhi} = 0 \; .$$

This is of the form of Lemma 2 (Walker, 1950).

Thus two cases arise:

Either $\kappa_{lm} = 0$ which means that the *recurrence vector* κ_l *is a gradient* or $Z_{hijk} = 0$, that is,

$$R_{hijk} = \frac{R}{n(n-1)}(g_{hk}\, g_{ij} - g_{hj}\, g_{ik}) \; .$$

In this case, the space has *constant curvature*.

We now turn our attention to

Definition 5.2 A Riemannian space $V_n (n > 2)$ whose non–zero curvature tensor satisfies the relation

$$R_{hijk,l} = \kappa_l\, R_{hijk} \tag{5.4}$$

for some non–zero vector κ_l is called a *recurrent space*.
We designate a recurrent space by K_n. It is obvious that a K_n is a Z_n. But the condition that a Z_n be a K_n is cast into the form of

Theorem 5.4 *A Z_n is a K_n if it satisfies the condition $R_{,l} = \kappa_l\, R$.*

Proof A Z_n has, on inserting (1.1) into (5.1), the relation:

$$R_{hijk,l} - \frac{1}{n(n-1)} R_{,l}\,(g_{hk}\, g_{ij} - g_{hj}\, g_{ik}) = \kappa_l \left[R_{hijk} - \frac{R}{n(n-1)}(g_{hk}\, g_{ij} - g_{hj}\, g_{ik}) \right] \; .$$

That is,

$$R_{hijk,l} - \kappa_l\, R_{hijk} = \frac{1}{n(n-1)}(R_{,l} - \kappa_l\, R)(g_{hk}\, g_{ij} - g_{hj}\, g_{ik}) \; . \tag{5.5}$$

Assume that a Z_n is a K_n. Then (5.5), on account of (5.4), yields

$$\frac{1}{n(n-1)}(R_{,l} - \kappa_l R)(g_{hk}\, g_{ij} - g_{hj}\, g_{ik}) = 0$$

which leads to

$$R_{,l} = \kappa_l\, R \; .$$

This ends the proof.

We finally touch upon this section with the conclusion that a Z_n is also a PK_n (projective recurrent space) and a CK_n (conformally recurrent space).

The converse whether a PK_n and also a CK_n is a Z_n is true on the condition that the space be *P–recurrent*, i.e. $P_{ij,l} = \kappa_l\, P_{ij}$. The proofs follow readily.

6. Z_n–SPACE WITH ZERO Z–TENSOR

In this section we study Z_n–space with zero Z–tensor and arrive at a result which is formulated as

Theorem 6.1 *A Z_n–space with zero Z–tensor has either null recurrence vector or constant curvature.*

Proof We write, from the concluding part of Section 4, the identity:

$$Z_{hijk,l} + Z_{hikl,j} + Z_{hilj,k} = 0 \; .$$

The identity, with the aid of (5.1), yields

$$\kappa_l Z_{hijk} + \kappa_j Z_{hikl} + \kappa_k Z_{hilj} = 0 \; . \tag{6.1}$$

Transvecting (6.1) with g^{hl} and using $Z_{ij} = 0$ we have

$$\kappa^h Z_{hijk} = 0 \; . \tag{6.2}$$

Transvecting (6.1) with κ^l and applying (6.2) we see that

$$\kappa^l\, \kappa_l\, Z_{hijk} = 0 \; . \tag{6.3}$$

It follows from (6.3) that $\kappa^l\, \kappa_l = 0$ or $Z_{hijk} = 0$.
The former one implies that κ_l is *null* while the latter means

$$R_{hijk} = \frac{R}{n(n-1)}(g_{hk}\, g_{ij} - g_{hj}\, g_{ik}) \; . \tag{6.4}$$

We resume our discussion of Section 3 and note that for a zero Z–tensor, R is constant. Consequently, the Z_n–space has, as appears in (6.4), *constant curvature*.

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