IC/90/218

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

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INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1990 MIRAMARE-TRIESTE

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

MATRIX ORDERINGS AND THEIR ASSOCIATED SKEW FIELDS *

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ABSTRACT

Matrix orderings on rings are investigated. It is shown that in the commutative case they are essentially positive cones. This is proved by reducing it to the field case; similarly one can show that on a skew field, matrix positive cones can be reduced to positive cones by using the Dieudonné determinant. Our main result shows that there is a natural bijection betwen the matrix positive cones on a ring R and the ordered epic R-fields.

MIRAMARE – TRIESTE
August 1990

* Submitted for publication.
** On sabbatical leave from Sharif University of Technology, P.O. Box 11365-8639, Tehran, Iran.

1 Introduction

As is well known, a commutative ring R with a positive cone P is necessarily an integral domain, and P extends in just one way to a positive cone of the field of fractions of R. When we drop commutativity, this may not hold, but such a ring need not even be embeddable in a skew field. Now in the construction of skew fields from rings in chapter 7 of [2] square matrices play an essential role and in order to define orderings on skew fields of fractions we shall need the idea of a matrix positive cone. Our object in this note is to show how to define a matrix positive cone on a ring R, and to see how it can be used to describe the orderings on the various epic R-fields.

Matrix positive cones are defined (in analogy to matrix valuations, cf. [3] or [7]) in section 2 and it is shown that in the commutative case they are essentially positive cones (Cor.2.2). This is verified by reducing it to the field case; in a similar way one can prove that on a skew field, matrix positive cones may be reduced to positive cones by using the Dieudonne determinant (Th.2.1). Our main result, in section 3, shows that there is a natural bijection between the matrix positive cones on R and the ordered epic R-fields.

2 Matrix Positive Cones

In this note all rings occurring are associative (but not necessarily commutative), with a unit element, denoted by 1, which is preserved by homomorphisms and inherited by subrings. By a field we understand a not necessarily commutative division ring; sometimes the prefix "skew" is used for emphasis. We write K^* for the group of non-zero elements of a field K and K^{*ab} for the group K^* made abelian. If R is any ring, we denote by $GL_n(R)$ the group of all invertible $n \times n$ matrices over R, and by $M(R)$ the set of all square matrices over R.

Let R be a ring. By a *positive cone* on R we understand a subset P of R such that

p.1. $P \cap -P = \emptyset$,

p.2. $P + P \subseteq P$,

p.3. $PP \subseteq P$,

p.4. $P \cup -P \cup \{0\} = R$.

We recall that according to the above definition , P in fact defines a total ordering on R (cf.[4]); and R becomes an integral domain.

In analogy with matrix valuations (cf. [3] or [7]) we now introduce the notion of a matrix positive cone. We first need to recall some concepts from [2]. For any matrices A, B over a ring R we define the *diagonal sum* as

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

If A, B are matrices of the same order n, whose entries agree except possibly in the first column, say $A = (A_1, A_2, \ldots, A_n), B = (B_1, A_2, \ldots, A_n)$, then the *determinantal sum* of A and B with respect to the first column is defined as

$$A \nabla B = (A_1 + B_1, A_2, \ldots, A_n).$$

It is clear how the determinantal sum with respect to another column, or a row, is defined for a suitable pair of matrices. We shall use the same notation in these cases, indicating in words the relevant row or column when this is necessary.

A square matrix A is called *non-full* if it can be written as $A = PQ$, where P is $n \times r, Q$ is $r \times n$ and $r < n$; in the contrary case A is said to be *full*. If $\mathcal{P}$ is a subset of $M(R)$, we put $\mathcal{P}^- = \{A^- \in M(R); A \in \mathcal{P}\}$, where A^- is obtained from A by multiplying one row (or column) by -1.

DEFINITION. A *matrix positive cone* on a ring R is a set $\mathcal{P}$ of full matrices in $M(R)$ such that

MPC.1. $A, B \in \mathcal{P} \Longrightarrow A \oplus B \in \mathcal{P}$,

MPC.2. $A, B \in \mathcal{P} \Longrightarrow A \nabla B \in \mathcal{P}$ whenever $A \nabla B$ is defined,

MPC.3. $1 \in \mathcal{P}$,

MPC.4. $\mathcal{P} \cap \mathcal{P}^- = \emptyset$,

MPC.5. $A \in \Pi = M(R) \setminus \mathcal{P} \cup \mathcal{P}^- \Longrightarrow A \oplus B \in \Pi$ for all square matrices B in $M(R)$,

MPC.6. $A \in \mathcal{P}, B \in \Pi \Longrightarrow A \nabla B \in \mathcal{P}$ whenever $A \nabla B$ is defined.

We observe that axioms MPC.5-6, merely say that each element of Π acts as zero if we think of ∇ as addition and $\oplus$ as multiplication. The restriction of Π to R is precisely 0. This situation will be more clarified in

the following consequences of MPC.1-6, which are easily verified (cf. [5]). We recall that a matrix is called *elementary* if it differs from the unit matrix in one off-diagonal entry; two matrices A, B are said to be *E-associated* if $AU = VB$, where U, V are products of elementary matrices, and A, B are *stably E-associated* if $A \oplus I$ is E-associated to $B \oplus I$, for unit matrices of suitable order (not necessarily the same on both sides).

MPC.7. $E \in \mathcal{P}$ for any elementary matrix $E \in M(R)$.

MPC.8. If A' is obtained from $A \in \mathcal{P}$ by multiplying on the left (or right) by an elementary matrix, then $A' \in \mathcal{P}$. Hence if B is stably E-associated to $A \in \mathcal{P}$, then $B \in \mathcal{P}$.

MPC.9. If A' is obtained from $A \in \mathcal{P}$ by exchanging two adjacent rows (or columns), then $A' \in \mathcal{P}^-$.

MPC.10. If $A, B \in \mathcal{P}$, then

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{P}, \begin{pmatrix} A & 0 \\ D & B \end{pmatrix} \in \mathcal{P},$$

for any C, D of appropriate size.

MPC.11. If $A, B \in \mathcal{P}$, then $AB \in \mathcal{P}$ for square matrices A, B of the same order.

MPC.12. The restriction of $\mathcal{P}$ to R is a positive cone of R.

We note that the axioms for a prime matrix ideal (cf. Chapter 7 of [2])may be obtained by writing down the conditions for the set $\Pi = M(R) \setminus \mathcal{P} \cup \mathcal{P}^-$. In the commutative case matrix positive cones can be reduced entirely to positive cones by the use of determinants (cf. the Cor. below). In the general case we no longer have the determinant at our disposal, but for a skew field there is the Dieudonne determinant. We recall (cf. e.g. [1] Th.4.7, p.163) that for any skew field K and any $n \geq 1$, the Dieudonne determinant is a homomorphism $GL_n(K) \to K^{*ab}$ which is universal for homomorphisms of $GL_n(K)$ into abelian groups (except when $n = 2, K = F_2$). It is clear that stably E-associated matrices have the same determinant. To prove our next result we need the following lemma. For a proof see [7, p.151].

LEMMA. *Let A be a non-singular matrix over a skew field and B a matrix such that the determinantal sum $C = A \nabla B$ is defined with respect to the first column, say. Then there exist non-singular matrices P, Q such that Q*

agrees with the unit matrix in the first row and column and

$$A = P\begin{pmatrix} a_{11} & 0 \\ a & I \end{pmatrix}Q, B = P\begin{pmatrix} b_{11} & 0 \\ b & I \end{pmatrix}Q, C = P\begin{pmatrix} c_{11} & 0 \\ c & I \end{pmatrix}Q,$$

where $c = a + b, c_{11} = a_{11} + b_{11}$.

Now using this we can prove

THEOREM 2.1. *Let K be a skew field with a positive cone P, then P may be extended to a matrix positive cone $\mathcal{P}$ on K by*

$$A \in \mathcal{P} \Longleftrightarrow \bar{a} \in P, \tag{1}$$

where $\bar{a}$= $DetA$ is the Dieudonne determinant of A, and the correspondence $P \leftrightarrow \mathcal{P}$ is a bijection between positive cones and matrix positive cones on K.

PROOF. We note that $\mathcal{P}$ in (1) is well-defined as each element of the commutator subgroup $[K^*, K^*]$ is positive (cf. [5]). Now it is clear by the lemma that the set $\mathcal{P}$ defined by (1) is a matrix positive cone whose restriction to K is just P. The converse is clear by MPC.12. This shows that each matrix positive cone is uniquely determined by its restriction to K and satisfies (1).

In the commutative case Dieudonne determinants reduce to the usual sort and we have

COROLLARY.2.2. *Let R be a commuative ring with a positive cone P, then P may be extended to a matrix positive cone on R by the rule* (1) *and the correspondence $P \leftrightarrow \mathcal{P}$ is a bijection between the positive cones and matrix positive cones on R.*

PROOF. We know that P can be extended uniquely to a positive cone on the field of fractions of R. Now using Theorem 2.1, this gives us a matrix positive cone whose restriction to R is the required matrix positive cone on R. Now the rest follows as before.

The scope of Theorem 2.1 may still be extended by the following almost obvious remark whose proof is clear and may be left to the reader.

PROPOSITION 2.3. *Let $f : R \to S$ be any ring homomorphism and $\mathcal{P}$ a matrix positive cone on S, then $\mathcal{P}^f$, defined by*

$$A \in \mathcal{P}^f \Longleftrightarrow A^f \in \mathcal{P}, A \in M(R),$$

is a matrix positive cone on R.

3 Positive Cones on R-fields

We now come to our main objective, which is to describe the positive cones on a field of fractions of R in terms of the matrix positive cones on R. We recall that by a *field of fractions* of R we understand a field K (possibly skew) containing R as a subring and generated, qua field, by R. More generally, an R - *field* is a field K with a homomorphism $R \to K$ and we speak of an *epic* R -field if K is generated, qua field, by the image of R. In [2] it was shown how epic R - fields (simply called "R - fields" in [2]) are determined by prime matrix ideals in R. To be precise, given an epic R - field K, the set Π of square matrices over R mapping to singular matrices over K is a prime matrix ideal, called the *singular kernel* of K, and every prime matrix ideal arises in this way as the singular kernel of an epic R - field. Moreover, given a prime matrix ideal Π, the epic R - field K with Π as singular kernel can be constructed as follows. Let Σ be the complement of Π in M(R), and form R_Σ, the "universal Σ - inverting ring". This is obtained from R by formally adjoining inverses of all matrices in Σ. Then R_Σ is a local ring whose residue class field is isomorphic to K (cf.[2],ch.7).

Let R be any ring and K an epic R - field. If P is a positive cone on K, we can form the associated matrix positive cone on K (Th.2.1) and by Proposition 2.3 use the natural homomorphism $R \to K$ to define a matrix positive cone $\mathcal{P}$ on R. We shall call $\mathcal{P}$ the matrix positive cone on R associated with P on K. Conversely, let $\mathcal{P}$ be a matrix positive cone on R, then $\Pi = M(R) \setminus \mathcal{P} \bigcup \mathcal{P}^-$ is a prime matrix ideal on R, so there is an associated epic R - field K. We recall from [2] that the elements of K are obtained as the first components of the solutions $(u_1, \ldots, u_n)^T$ of matrix equations $Au = a$, where $A = (a_1, a_2, \ldots, a_n)$ lies in the multiplicative set $\Sigma = \mathcal{P} \bigcup \mathcal{P}^-$. To define an ordering on K, put

$$(u_1; A, a) > 0 \quad \text{if} \quad A \oplus A_1 \in \mathcal{P}, \tag{2}$$

where $A_1 = (a, a_2, \ldots, a_n)$. One can easily show that (2) is independent of the choice of system (cf. [5]). Let P be the set of all positive elements of K as defined by (2). We show that P is a positive cone on K. To see this, let u_1, v_1 be the first components of the solutions of the matrix equations $Au = a, Bv = b$, respectively, where A is $n \times n$ and B is $m \times m$. We know that $u_1 + v_1$ is the first component of the solution of the matrix equation $Cz = c$, where

$$C = \begin{pmatrix} A & -a_1 0 \\ 0 & B \end{pmatrix}, c = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Now we consider the following decompositon of $C \oplus C_1$

$$\left[\begin{pmatrix} A & -a_1 0 \\ 0 & B \end{pmatrix} \oplus \begin{pmatrix} A_1 & -a_1 0 \\ 0 & B \end{pmatrix}\right] \nabla \left[\begin{pmatrix} A & -a_1 0 \\ 0 & B \end{pmatrix} \oplus \begin{pmatrix} 0 & A' & -a_1 0 \\ b & 0 & B \end{pmatrix}\right],$$

with respect to the $(n+m+1)$ - th column. Using MPC.7-12, and the fact that $B \oplus B \in \mathcal{P}, A \oplus A \in \mathcal{P}$ whenever $A, B \in \mathcal{P}$, we obtain

$$A \oplus A_1 \oplus B \oplus B \in \mathcal{P},$$

from the first matrix on the right of the decomposition of $C \oplus C_1$ and

$$A \oplus A \oplus B \oplus B_1 \in \mathcal{P},$$

from the second. Thus $C \oplus C_1 \in \mathcal{P}$, i.e. $P + P \subseteq P$. By a similar method of decomposing matrices into determinantal sums of appropriate matrices (cf. [5]), one can easily show that $PP \subseteq P$. It is straightforward to verify that $P \cap -P = \emptyset$ and $P \cup -P \cup \{0\} = K$. Thus we have proved

THEOREM 3.1. *Let R be any ring, then each matrix positive cone $\mathcal{P}$ on R determines an associated epic R - field K with a positive cone P, and conversely, every epic R - field K with a positive cone P on it arises from a matrix positive cone on R. This correspondence between matrix positive cones on R and ordered epic R - fields is bijective.*

As an illustration, take a right Ore domain R with a positive cone P defined on it, then as in the commutative case one can extend P in just one way to the field of fractions K of R (cf.[4] or [6]). Hence any positive cone on a right (or left) Ore domain arises by restriction from a unique matrix positive cone on R.

Acknowledgments. The author would like to acknowledge helpful discussions with A. Verjovsky. Professor Abdus Salam, the International Atomic Energy Agency, UNESCO, and the International Centre for Theoretical Physics, at Trieste, are acknowledged for their support.

References

[1] Artin, E., Geometric algebra, Interscience Publishers Inc., New York, 1957.

[2] Cohn, P. M., Free rings and their relations, L.M.S. Monographs No.2, Academic Press, London and New York, 1971.

[3] Cohn, P. M.,and Mahdavi-Hezavehi, M., Extension of valuations on skew fields, Springer Lecture Notes in Maths. Vol.325, 1980.

[4] Fuchs, L., Partially ordered algebraic systems, Pergamon Press LTD, London, New York, 1963.

[5] Mahdavi-Hezavehi, M., Ordering on rings and their associated skew fields, Mathematics Department, Sharif University of Technology, P. O. Box 11365-8639, Tehran, Iran, Preprint 1979.

[6] Mahdavi-Hezavehi, M., Totally ordered rings and fields, Bull. of Iranian Mathematical Soc.,1980.

[7] Mahdavi-Hezavehi, M., Matrix valuations and their associated skew fields, Resultate der Mathematik, Vol.5,1982.

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