IC/90/223

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

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INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1990 MIRAMARE-TRIESTE

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and
United Nations Educational Scientific and Cultural Organization
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ABSTRACT

The self–organized critical state exhibited by a sandpile model is shown to correspond to motion on an attractor characterized by an invariant distribution of the height variable. The largest Lyapunov exponent is equal to zero. The model nonetheless diplays intermittent chaos, with a multifractal distribution of local expansion coefficients in history space. Laminar spatio-temporal regions are interrupted by chaotic bursts caused by avalanches. We introduce the concept of local histories in configuration space and show that their expansion parameters also exhibit a multifractal distribution in time and space.

MIRAMARE – TRIESTE
August 1990

* Submitted for publication.

The ubiquity in nature of phenomena exhibiting self-similarity over a wide range of spatial and temporal scales suggests that mechanisms giving rise to such behaviour must be robust with respect to variations of system parameters or initial conditions, as well as being independent of the detailed physics in question. This has led Bak, Tang and Wiesenfeld [1,2] to propose a very general scenario for "self organized criticality" where an extended dissipative system spontaneously evolves into a "critical" state possessing spatial and temporal scale invariance.

The sandpile [1-5] is an explicit, discrete realization of a continuum model with nonlinear diffusive dynamics in the presence of noise, as has already been discussed by several authors [2,5,7]. Nevertheless, a systematic investigation of the underlying dynamics and the properties of the attractor associated with the self organized critical (SOC) state is still lacking. The crucial observation to be made here is that the SOC state is reached and maintained under a finite [6,8] density of grains of sand being added, or in other words, that the addition of grains of sand is an integral part of the dynamics. Then it follows that the "critical state" itself corresponds to motion on the attractor for the dynamics of this system. It is the purpose of this letter to clarify some aspects of the SOC state by investigating the scaling properties of the attractor in phase space to which motion is confined over long times, and to see how these are interrelated with spatial and temporal intermittency.

We first show that the attractor may be characterized by an invariant probability distribution for the height variable, the conserved quantity in the model. A computation of the relevant correlation integrals shows that in history space the model has a multifractal distribution of local expansion parameters, although the largest Lyapunov exponent is found to be zero. We show that multifractality in history space is mirrored by a multifractal distribution of expansion coefficients defined in configuration space. Further, we find that at any given time, the active sites form a fractal set with dimension $D_f = 1.59 \pm 0.01$.

We begin by recalling the definition of the model. In two dimensions, to which we will confine our attention, the "sandpile" is characterized by integer z_k at all sites k. The external driving force corresponds to adding a particle at a random site k such that $z_k(n+1) = z_k(n) + 1$, where n is the discrete time variable. If at any time $z_k \geq 4$, then

$$\begin{aligned} z_k(n+1) &= z_k(n) - 4 \\ z_{k+\delta}(n+1) &= z_{k+\delta}(n) + 1 \end{aligned} \qquad (1)$$

where δ signifies the unit vector to nearest neighbor sites. For simplicity we have taken "closed" boundary conditions [1] on two contiguous edges and "open" boundary conditions on the other two. In what follows, n will

signify time measured in number of particles added.

The stationarity condition is given by the requirement that the rate of flow into a particular microscopic state - a particular z value - should equal the rate of flow out of that state [9]. This implies that the steady state is characterized by an invariant distribution $\rho(z)$. It should be remarked that the deterministic dynamics of relaxation from super-critical configurations is not sufficient to reach this invariant distribution, but that the annealing effect of the presence of noise is necessary [10]. For linear system sizes of L=30,40, we have obtained, $\rho(0) = .07 \pm 4\%, \rho(1) = .17 \pm 7\%, \rho(2) = .31 \pm 9\%, \rho(3) = .45 \pm 3\%$, where the percent errors are estimated from fluctuations over runs with $n = 10^4$. The size of the relative fluctuations decreases with system size.[11]

A more detailed description of the attractor is afforded by the hierarchy of exponents that have been introduced [12-15] to characterize the multifractal distribution of phase points on the attractor. We have computed the correlation integral [12,13]

$$C(l) = \lim_{M\to\infty} \frac{1}{M^2}\{\text{number of pairs } (\{z\}_i, \{z\}_j) \text{ with } d(\{z\}_i, \{z\}_j) < l\} \quad (2)$$

where $\{z\}_i$ is the configuration of the system at time step i=*1,...* M and $d(.,.)$ the distance [15] in phase space between the two configurations i and j. We find that $C(l) \sim l^\nu$ with $\nu = 0.37 \pm .01 << F = L^2$, which is the signature of a nontrivial attractor. Making use of [13] $D_{LB} \leq \nu \leq D \leq D_{KY}$, where D_{LB} is the number of positive Lyapunov exponents, D is the fractal dimension of the attractor and D_{KY} is the Kaplan-Yorke dimension [16], we immediately conclude from the leftmost inequality that all the Lyapunov exponents λ_i are equal to or less than zero. This implies $D_{KY} = j$ where j is the last integer such that $\sum_i^j \lambda_i \geq 0$ (assuming $\lambda_1 \geq \lambda_2 \geq ...\lambda_F$), and from $D_{KY} > \nu$ it follows that $j \geq 1$. Thus the largest Lyapunov exponent(s) is(are) zero, whose eigendirection(s) correspond to conserved quantities, the existence of which is claimed [5] to underlie spatio-temporal scale invariance.

To picture the motion in phase space with $\lambda_1 = 0$, it is instructive to consider the behaviour of the Hamming distance $H(n)$,

$$H(n) \equiv \sum_{all\ sites\ k} (z_k(n) - z_k'(n))^2 \tag{3}$$

between two slightly different configurations as a function of time. Although over long times $n >> L^2$, $H(n)$ is not diverging it displays intermittent [17] behaviour. In Fig.1, one clearly sees the laminar regions, where the two trajectories evolve in parallel, interrupted by abrupt rises caused by the avalanches, with eventual collapses to nearby configurations.

Clearly, the Lyapunov exponents, which are in some sense average quantities, do not suffice to describe this situation and one should instead probe the distribution of local expansion parameters in history space [18]. To do this, we have computed the generalized Renyi entropies [18,19] $K(q)$, which

play an analogous role to the generalized dimensions characterizing the scaling properties of the mass distribution on the attractor. We define NF dimensional vectors, or "histories" [18] via

$$\mathbf{X}_i^{(N)} = (\{z\}_i, \{z\}_{i+1}, ...\{z\}_{i+N+1}) \tag{4}$$

starting at the phase point $\{z\}_i$ at time $i = 1,..M$. The probability of encountering another history in a neighborhood of size l around $\mathbf{X}_0^{(N)}$ in history space, is

$$P_l^{(N)}(\mathbf{X}_0^{(N)}) = \frac{1}{M}\sum_{j=1}^{M} \Theta[l - \Delta(\mathbf{X}_0^{(N)} - \mathbf{X}_j^{(N)})] \tag{5}$$

where Θ is the step function and Δ is the distance between the pair of histories as defined in Ref.18. The local expansion parameters [18] λ_I are defined via $P_l^{(N)}(I) \sim e^{-\lambda_I N}$ for fixed l, and the q^{th} moments of the probability distribution $P_l^{(N)}(I)$

$$\Gamma_N(q) \equiv \sum_I P_l^{(N)}(I)^q \sim e^{-N(q-1)K(q)} \tag{6}$$

where $K(q)$, are the generalized Renyi entropies. We find that they depend nonlinearly on q (see Fig. 2) signalling intermittency in the sense of multifractality in history space. The topological entropy is $K_0 = 1.1 \pm 0.1$ and the Kolmogorov entropy $K_1 = 0.6 \pm 0.1$.

Armed with this insight, we would now like to examine how the "turbulent" and laminar regions are distributed in the 2 space + 1 time–dimensions. For ease of visualization, we take a one dimensional cut through the sandpile and plot, as a function of n, all points along this cut for which $z_k(n-1) - z_k(n) \neq 0$ (see Fig.3). It should be noted that this is a different approach from the one usually taken [1,2,20] in depicting avalanche clusters. What we get is a series of snapshots of the active regions falling on this particular intersection. Seen in this way, the active regions are non-compact and on the intersection sets we compute $D_\perp = 0.59 \pm 0.01$. The usual argument of additivity of co-dimensions leads to $D_f = 1.59 \pm 0.01$ for the fractal dimension of the active regions at any moment in time, and $D_f + 1 = 2.6 > 2$ for the dimension of the same set embedded in 2+1 dimensions. Projecting this on to two dimensions, as is done in the usual treatments[1,2,20] where one implicitly integrates the mass of the avalanche clusters over time, one would indeed find them to be compact.

Finally, we would like to introduce a new quantity, which we shall call a "local history", as a measure of how differences in local configurations scale with time and spatial seperation. We define

$$h_k(r,n) = \sum_{m=1}^{n} (z_k(n) - z_{k+r}(n))^2 \tag{7}$$

and

$$R(q;n,r) = \sum_{all\ sites\ k} h_k(r,n)^q. \tag{8}$$

In analogy with the multifractal scaling in history space, we now have, for the "local" histories, for fixed r, $R(q;n,1) \sim e^{n\tau(q)}$ and for fixed n, $R(q;r) \sim r^{\varsigma(q)}$, over length scales $r < L$ and $n \sim (O(L))$. We have

plotted $\tau(q)$ and $\varsigma(q)$ in Figs.4 and 5, where one clearly observes the nonlinear dependence on q. For the case of $\varsigma(q)$, the close similarity with multifractal scaling in turbulent media [15,21,22] is obvious.

ACKNOWLEDGEMENTS

We are grateful for interesting discussions we have had with D. Dhar, and for the hospitality of the Condensed Matter group at the International Center of Theoretical Physics. The authors would also like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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FIGURE CAPTIONS

1. Hamming distance in arbitrary units between two nearby configurations, as a function of time. In eq.(5) we set $z'_i(0) = z_i(0)$, except at four randomly chosen sites where $z'_i(0) = z_i(0) + 1$.

2. Generalized Renyi entropies $K(q)$ vs q.

3. Evolution in time of active sites (shown in black) on a 1–dimensional cut through the sandpile. The vertical axis corresponds to the position along the cut and the horizontal axis to time.(L=30).

4. Intermittency reflected as multifractality in the distribution of expansion parameters of local histories, for L=30, r=1. (see Eq. (10) in text).

5. Nonlinear q dependence of the q^{th} moments of the local histories as a function of spatial seperation.

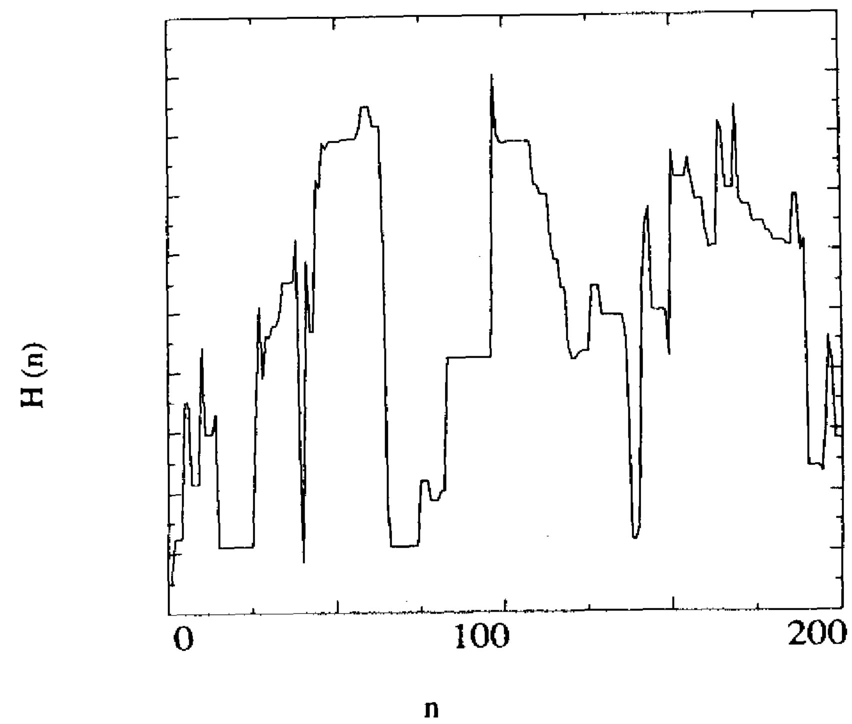

Fig.1

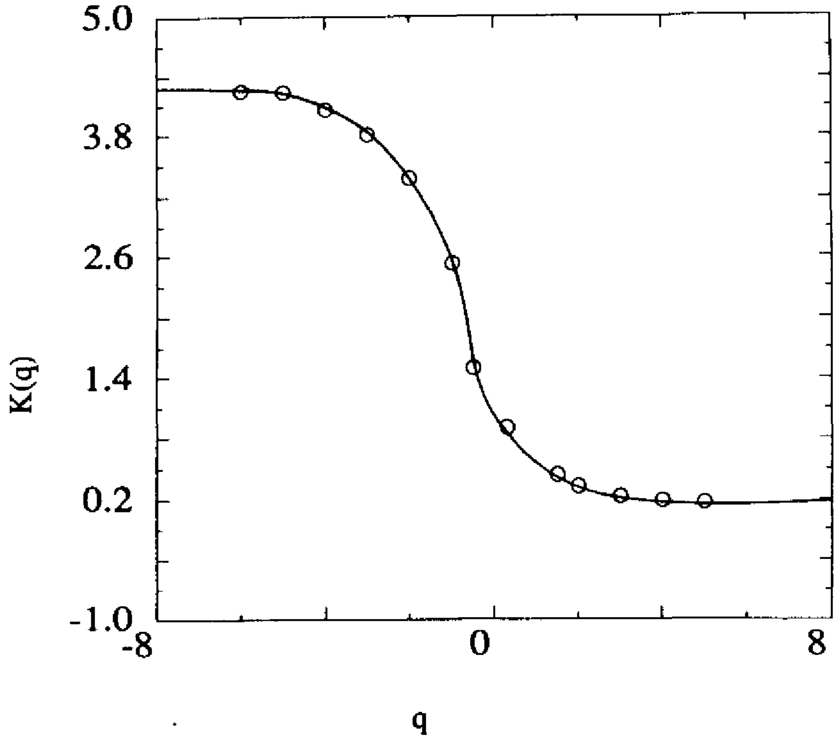

Fig.2

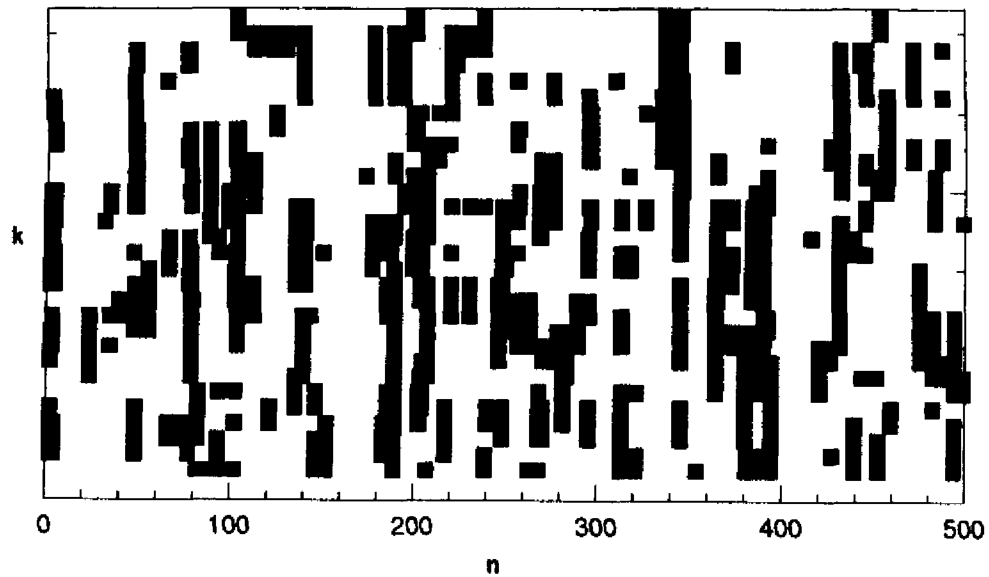

Fig.3

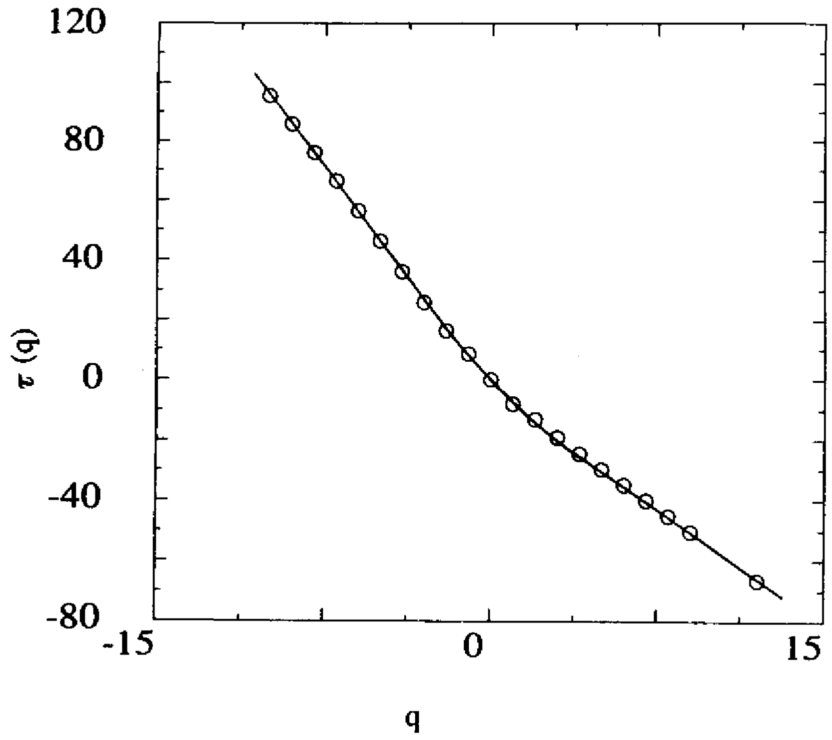

Fig.4

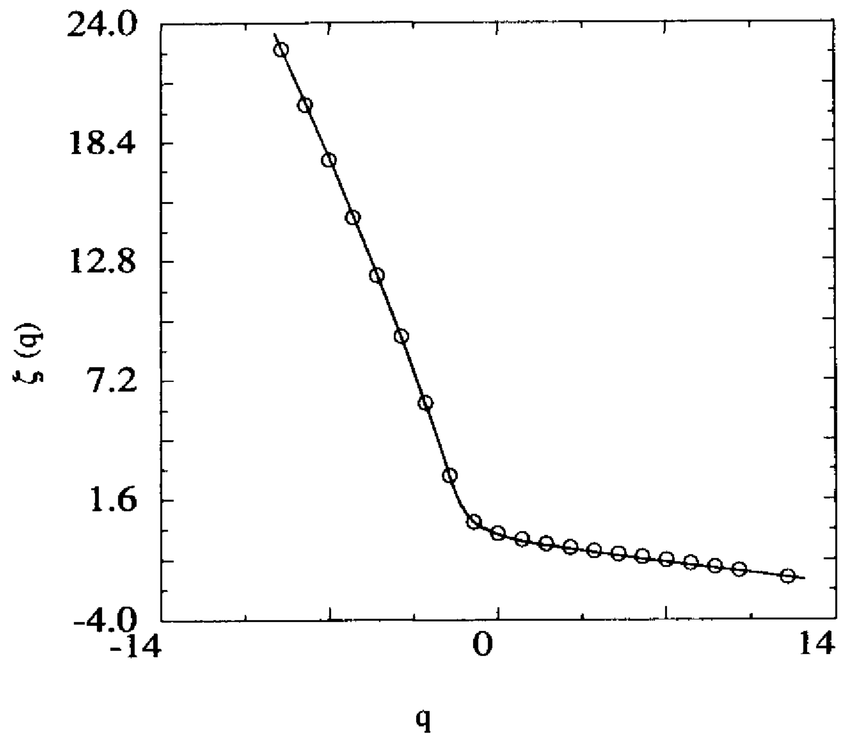

Fig.5

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