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LIOUVILLE ACTION IN CONE GAUGE

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The effective action of the conformally invariant field theory in the curved background space is considered in the light cone gauge. The effective potential in the classical background stress is defined as the Legendre transform of the Liouville action. This potential is tightly connected with the sl(2) current algebra. The series of the covariant differential operators is constructed and the anomalies of their determinants are reduced to this effective potential.

Fig. - , ref. - 7

1. Introduction

Recent progress in two-dimensional gravity (Liouville theory) is due to the cone gauge parametrization of a 2D surface, introduced by Polyakov [1]. This gauge is defined by the following form of the space-time distance (x^+ and x^- are coordinates on 2D surface, which are complex in the euclidean case and not necessarily related by the complex conjugation)

$$ds^2 = dx^+(dx^- + h(x^+, x^-)dx^+). \tag{1.1}$$

Contrary to the commonly used conformal gauge, where the space-time distance has the form

$$ds^2 = e^{\varphi}|dz|^2 \tag{1.2}$$

the cone gauge leads to well-defined perturbation theory in the inverse central charge of the theory [1]. Moreover, in this gauge new unexpected symmetry of the theory under sl(2) current algebra shows up [1] which permits one an exact calculation of scaling parameters for $c < 1$ [2]. The connection with the sl(2) current algebra was recently recovered through the path integral quantization of the coadjoint orbits of the Virasoro algebra [3].

Two-dimensional Liouville gravity is essentially an induced gravity and Liouville action arises as an effective action of some massless conformally invariant field theory in curved background space-time. It is characterized by single parameter i.e. central charge c of the inducing field theory, which essentially defines the effective number of massless degrees of freedom. Hence the theory is an effective infrared theory and all fields with finite correlation length for Liouville theory point of view contribute to ultraviolet cutoff only.

In two-dimensional conformal field theory the trace of the stress tensor T_{ab}, which defines the reaction of the effective action on an infinitesimal variation of background metric

$$\delta \log Z = \frac{1}{4\pi} \int T_{ab} \delta g^{ab} \sqrt{g}\, d^2x \qquad (1.3)$$

is simply related to the space-time geometry

$$T_{ab} g^{ab} = -\frac{c}{12} R \qquad (1.4)$$

where R is scalar curvature of the background metric. Combined with the stress tensor conservation this relation defines unambiguously (at least in the space with trivial topology) the vacuum expectation value of the stress tensor components and therefore fixes the effective action which is clearly determined by the value of the central charge c only and does not depend on other details of the conformal field theory.

In the conformal gauge the system (1.3), (1.4) is easily integrated and leads to the Liouville action in the familiar form

$$Z \sim \exp \frac{c}{96\pi} \int (\partial_a \varphi)^2 d^2x . \qquad (1.5)$$

Here this effective action S_{Liouv} will be considered in the cone gauge, where it appears to be of more complicated form. Now it is functional of h and according to (1.3) generates connected correlations of "left" stress components $T = T_{--}$. Related to the Liouville action is another interesting functional that we shall denote as the effective potential W(u). It may be obtained as the Legendre transform of Liouville action and classically may be considered as an effective action in some background stress u. Moreover this correspondence persists on quantum level, i.e. after suitable renormalization of constants $W(u)$ is also the Fourier transform of the cone gauge Liouville action. Interesting observation by Polyakov [1] is that this potential $W(u)$ is tightly related to sl(2) current algebra; in fact it generates connected correlation functions of sl(2) currents. In sects. 2 and 3 we consider these two functionals in more detail. Similar approach of the Liouville action was used in [4].

The Liouville action is well known to arise when calculating determinants of conformally covariant Laplace operator acting in the space of differentials of weight λ. It turns out that the potential $W(u)$ also can be recovered as an anomaly in determinants of a family of covariant differential operators of order $2j+1$ (for all positive integer or half-integer j). Simplest nontrivial example of this type (second order) is the operator $\partial_-^2+\frac{u}{2}$. In sect.4 we consider this family of operators, analyse their determinants and calculate corresponding anomaly (i.e. the central charge k_j in the sl(2) current algebra).

Consideration of these determinants was motivated by one-loop calculation in cone quantum gravity, performed by Polyakov. This beautiful calculation uses the determinant of the third-order operator of the family. It is reproduced in sect.5 of this paper.

2. Liouville action

In the cone gauge we consider metric tensor of the following form

$$g_{ab}=\begin{pmatrix} h & 1/2 \\ 1/2 & 0 \end{pmatrix};\quad g^{ab}=\begin{pmatrix} 0 & 2 \\ 2 & -4h \end{pmatrix};\quad \sqrt{g}=1/2i. \tag{2.1}$$

The corresponding metric connection is as follows

$$\begin{aligned} \Gamma^{+}_{++} &= -\partial_- h \\ \Gamma^{-}_{++} &= \partial_+ h + 2h\partial_- h \\ \Gamma^{-}_{+-} &= \partial_- h \end{aligned} \tag{2.2}$$

all other components are zero. The scalar curvature is

$$R = 4\partial_-^2 h. \tag{2.3}$$

Covariant conservation of the stress tensor in the cone coordinates acquires the form

$$(\partial_+ - 2h\partial_- - 3\partial_- h)T_{--} + \partial_- T_{+-} = 0 \tag{2.4a}$$

$$\partial_- T_{++} + (\partial_+ - 2h\partial_- - 2\partial_- h)T_{+-} - \partial_+ h T_{--} = 0 \tag{2.4b}$$

and eq.(1.4) is rewritten as

$$T_{+-} - hT_{--} = -\frac{c}{12}\partial_-^2 h. \tag{2.5}$$

With (2.5) the first of eqs.(2.4) turns to the following closed equation for $T = T_{--}$

$$(\partial_+ - h\partial_- - 2\partial_- h)T_{--} = \frac{c}{12}\partial_-^3 h. \tag{2.6}$$

In the cone gauge relation (1.3) acquires the following form

$$\delta \log Z = \delta S_{Liouv}(h) = -\frac{1}{\pi}\int \langle T_{--}\rangle \delta h \, d^2x \tag{2.7}$$

where by d^2x we denote $\frac{1}{2i}\, dx^+ \wedge dx^-$. This equation combined with eq.(2.6) gives closed system of functional relations for $S_{Liouv}(h)$.

Note that the second of eqs.(2.4) after substitution of (2.5) may be written in the form

$$\partial_-(T_{++} - h^2 T_{--} + \frac{c}{6} h\partial_-^2 h) = \frac{c}{12}(\partial_+\partial_-^2 h - h\partial_-^3 h). \tag{2.8}$$

This equation leads to the following expression for the component T_{++}

$$T_{++} = h^2 T_{--} + \frac{c}{12}\left[\partial_+\partial_- h - 3h\partial_-^2 h + \frac{1}{2}(\partial_- h)^2\right] \tag{2.9}$$

which holds up to zero modes of the operator ∂_-.

Considering h as a small perturbation one could solve eq.(2.6) iteratively and then integrate (2.7) to find $S_{Liouv}(h)$ as a series in h. Alternatively this series may be obtained with the use of formal expansion

$$Z(g+\delta g) = Z(g)\sum_{n=0}^{\infty} \frac{1}{(4\pi)^n n!}$$

$$\int \langle T_{a_1b_1}(x_1)\ldots T_{a_nb_n}(x_n)\rangle \delta g^{a_1b_1}\ldots\delta g^{a_nb_n}\sqrt{g_1}d^2x_1\ldots\sqrt{g_n}d^2x_n . \qquad (2.10)$$

In the cone gauge this means that Z(h) generates the stress tensor $T = T_{--}$ correlation functions on the infinite plane

$$Z(h) = \sum_{n=0}^{\infty} \frac{(-)^n}{\pi^n n!} \int \langle T(x_1)\ldots T(x_n)\rangle h(x_1)d^2x_1\ldots h(x_n)d^2x_n . \qquad (2.11)$$

All these infinite plane correlations can be recovered from the following Ward identities, which reflect the commutation relations for conformal symmetry generators

$$\langle T(x^-)T(x_1^-)\ldots T(x_n^-)\rangle = \sum_{i=1}^{n} \frac{c}{2(x^- - x_i^-)^4} \langle T(x_1^-)\ldots \not{T}(x_i^-)\ldots T(x_n^-)\rangle +$$

$$\sum_{i=1}^{n}\left[\frac{2}{(x^- - x_i^-)^2} + \frac{1}{x^- - x_i^-}\,\partial/\partial x_i^-\right]\langle T(x_1^-)\ldots T(x_n^-)\rangle . \qquad (2.12)$$

Hence the Liouville action $S_{Liouv}(h) = \log Z(h)$ generates connected correlation functions $\langle T(x_1^-)\ldots T(x_n^-)\rangle_c$. Connected correlations are produced by the same Ward identities (2.12) with the central term omitted

$$\langle T(x_1^-)T(x_2^-)\rangle_c = \frac{c}{2(x_{12}^-)^4} \tag{2.13}$$

$$\langle T(x^-)T(x_1^-)\dots T(x_n^-)\rangle_c = \sum_{i=1}^{n}\left(\frac{2}{(x^- - x_i^-)^2} + \frac{1}{x^- - x_i^-}\,\partial/\partial x_i^-\right)\langle T(x_1^-)\dots T(x_n^-)\rangle_c .$$

Note that besides recurrence relations (2.13) these connected correlation functions permit more explicit representation. As was mentioned above the Liouville action in particular results from the anomaly in the determinant of the covariant operator $\bar{\partial}_\lambda$, acting from the space of differentials of weight λ to the space of differential forms of weight $(1,\lambda)$. Corresponding central charge is

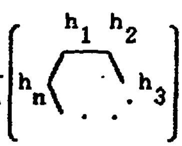
$$c_\lambda = -2(1+6\lambda-6\lambda^2). \tag{2.14}$$

In the cone gauge corresponding operators are $\partial_+ - h\partial_- - \lambda\partial_- h$ and one finds

$$\det(\partial_+ - h\partial_- - \lambda\partial_- h) = Z_{c_\lambda}(h). \tag{2.15}$$

Determinant representation leads to the following diagram expansion of the Liouville action

$$S_{Liouv}(h)\Big|_{c=c_\lambda} = -\sum_{n=1}^{\infty}\frac{1}{n\pi^n}\left[\;\cdots\;\right] d^2x_1\dots d^2x_n \tag{2.16}$$



where propagators are $1/x_{ij}^-$ and in each vertex the operator

$$\rangle\cdots = \lambda\overrightarrow{\partial}_- - (1-\lambda)\overleftarrow{\partial}_- \tag{2.17}$$

acts. This is a λ-dependent family of representations for the connected stress tensor correlations. The expansion with $\lambda = 0$ is the most convenient one. Performing vertex differentiation one finds the same diagram expansion as (2.16) with propagator $1/(x^-_{ij})^2$ and unity in the vertex.

Since central charge enters S_{Liouv} simply as a multiplier it is convenient to get rid of it and introduce the standard functional $S_L(h)$, independent on c:

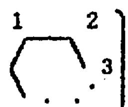
$$S_{Liouv}(h) = \frac{c}{2} S_L(h), \tag{2.18}$$

S_L is determined by the following functional relations

$$\begin{aligned} \delta S_L(h) &= -\frac{1}{6\pi}\int u(h)\delta h d^2x \\ (\partial_+ - h\partial_- - 2\partial_- h)u &= \partial_-^3 h \end{aligned} \tag{2.19}$$

and expands in h as follows

$$\delta S_L(h) = \sum_{\iota=1}^{\infty} \frac{1}{n\pi^n}\left[n \quad \right] h(x_1)\ldots h(x_n)d^2x_1\ldots d^2x_n \tag{2.20}$$



with propagator $1/(x^-_{ij})^2$.

In particular the following closed expression for connected stress correlations in conformal field theory holds

$$\langle T_1 \ldots T_n \rangle_c = c\sum_1^{(n-1)!/2} \frac{1}{(x^-_{C_1C_2} x^-_{C_2C_3} \ldots x^-_{C_nC_1})^2} \tag{2.21}$$

where the summation is over all cycles of length n in permutations of n elements (1,2,...,n).

3. Potential W(u)

On attempting to quantize gravity in the cone gauge one is brought to perform something like functional integral over h weighted with the Liouville action $S_{Liouv}(h)$. Define a generating functional for h-correlations through the following formal expression

$$e^{W(u)} = \int \mathcal{D}h \; e^{S_{Liouv}(h)+\frac{c}{12\pi}\int u(x)h(x)d^2x} \qquad (3.1)$$

where u is a classical background stress. In the quasiclassical limit $c \to -\infty$ one can use the classical equation of motion for h_{cl}

$$\frac{c}{12\pi} u + \frac{c}{2} \frac{\delta S_L}{\delta h} (h_{cl}) = 0. \qquad (3.2)$$

Comparing this equation with the first of eqs.(2.19) one finds that it reduces to the second one of eqs.(2.19), which should be read now "from right to left", i.e. as the equation for h with u given. Classical action is determined by the relation

$$\delta W_{cl}(u) = \frac{c}{12\pi}\int h(u)\delta u(x)d^2x. \qquad (3.3)$$

To get rid of the central charge define standard functional $W_L(u)$

$$W_{cl}(u) = \frac{-c}{12} W_L(u) \qquad (3.4)$$

which is fixed by the c-independent system of relations

$$\begin{aligned} &\delta W_L(u) = \frac{1}{\pi}\int h\delta u \; d^2x \\ &(\partial_-^3 + 2u\partial_- + \partial_- u)h = \partial_+ u \; . \end{aligned} \qquad (3.5)$$

Now we are arrived at new functional: potential $W_L(u)$. To get insight to its origin consider the sl(2) current algebra. It contains three currents j^0 and $j^{\pm}$ holomorphically dependent on x^+. Commutation relations are defined by the following operator product expansions

$$j^{0}(x_1^{+})j^{0}(x_2^{+}) = \frac{k}{2(x_{12}^{+})^{2}} + \text{reg.}$$

$$j^{+}(x_1^{+})j^{+}(x_2^{+}) = \text{reg.}$$

$$j^{-}(x_1^{+})j^{-}(x_2^{+}) = \text{reg.} \qquad (3.6)$$

$$j^{0}(x_1^{+})j^{\pm}(x_2^{+}) = \pm \frac{1}{x_{12}^{+}}\, j^{\pm}(x_2^{+}) + \text{reg.}$$

$$j^{0}(x_1^{+})j^{0}(x_2^{+}) = \frac{k}{(x_{12}^{+})^{2}} + \frac{2}{x_{12}^{+}}\, j^{0}(x_2^{+}) + \text{reg.}$$

where k is the central charge of the current algebra. Define the following x^{+} and x^{-} dependent field [1,5]:

$$2H(x) = j^{-}(x^{+}) + 2x^{-}j^{0}(x^{+}) - (x^{-})^{2}j^{+}(x^{+}) \ . \qquad (3.7)$$

This field enjoys the following operator product expansion, which encodes all the commutation relations (3.6)

$$H(x)H(0) = -\frac{k}{4}\left(\frac{x^{-}}{x^{+}}\right)^{2} - \frac{x^{-}}{x^{+}}H(0) - \frac{(x^{-})^{2}}{2x^{+}}\partial_{-}H(0) + \text{reg.} \qquad (3.8)$$

This expansion can be utilized in the form of Ward identities for H-correlations

$$\langle H(x)H(x_1)\ldots H(x_n)\rangle = \sum_{i=1}^{n} -\frac{k}{4}\left(\frac{x^{-}-x_i^{-}}{x^{+}-x_i^{+}}\right)^{2}\langle H(x_1)\ldots H(\not{x}_i)\ldots H(x_n)\rangle -$$

$$\sum_{i=1}^{n}\left[\frac{x^{-}-x_i^{-}}{x^{+}-x_i^{+}} + \frac{1}{2}\frac{(x^{-}-x_i^{-})^{2}}{x^{+}-x_i^{+}}\frac{\partial}{\partial x_i^{-}}\right]\langle H(x_1)\ldots H(x_n)\rangle . \qquad (3.9)$$

Using these Ward identities one successively recovers all the multipoint correlation functions of fields H on the infinite plane x^{+}, x^{-}. These are nothing but condensed form of the sl(2)-currents correlation functions.

Define the generating functional for these multipoint correlations

$$U(u) = \sum_{n=0}^{\infty} \frac{1}{n!\pi^n} \int \langle H(x_1)\ldots H(x_n)\rangle u(x_1)\ldots u(x_n) d^2x_1\ldots d^2x_n . \qquad (3.10)$$

It is clear that the functional $W(u) = \log U(u)$ generates the "connected" H-correlations $\langle H(x_1)\ldots H(x_n)\rangle_c$, which are recovered by Ward identities (3.9) without central term

$$\langle H(x_1)H(x_2)\rangle_c = -\frac{k}{4}\left(\frac{x^-_{12}}{x^+_{12}}\right)^2 , \qquad (3.11)$$

$$\langle H(x)H(x_1)\ldots H(x_n)\rangle_c = -\sum_{i=1}^{n}\left(\frac{x^- - x^-_i}{x^+ - x^+_i} + \frac{1}{2}\frac{(x^- - x^-_i)^2}{x^+ - x^+_i}\frac{\partial}{\partial x^-_i}\right)\langle H(x_1)\ldots H(x_n)\rangle_c .$$

Now it remains to check that the functional obtained this way

$$W(u) = \sum_{n=2}^{\infty} \frac{1}{n!\pi^n} \int \langle H(x_1)\ldots H(x_n)\rangle_c u(x_1)\ldots u(x_n) d^2x_1\ldots d^2x_n \qquad (3.12)$$

satisfies the same defining relations (3.5) as $W_L(u)$. Perturbative derivation proceeds as follows: consider variation

$$\delta W(u) = \frac{k}{2\pi}\int h\delta u \, d^2x . \qquad (3.13)$$

For h the following diagram expansion holds

$$\frac{k}{2} h(x) = \sum_{n=1}^{\infty} \frac{1}{n!\pi^n} \int \langle H(x)H(x_1)\ldots H(x_n)\rangle u_1\ldots u_n d^2x_1\ldots d^2x_n . \qquad (3.14)$$

Differentiating one has

$$\frac{k}{2}\partial_-^3 h(x) = \sum_{n=1}^{\infty} \frac{1}{n!\pi^n} \int \langle \partial_-^3 H(x)H_1\ldots H_n\rangle u_1\ldots u_n d^2x_1\ldots d^2x_n . \qquad (3.15)$$

Using Ward identities for connected correlation functions one finds

$$\langle \partial_-^3 H(x) H(x') \rangle_c = \pi \frac{k}{2}\, \partial_+ \delta^2(x-x'),$$

$$\langle \partial_-^3 H(x) H(x_1) \ldots H(x_n) \rangle_c =$$

$$- \sum_{i=1}^{n} \left(\pi \partial_- \delta^2(x-x_i) + \pi \delta^2(x-x_i) \partial / \partial x_i^- \right) \langle H(x_1) \ldots H(x_n) \rangle_c . \qquad (3.16)$$

Finally one arrives at

$$\frac{k}{2} \partial_-^3 h = \frac{k}{2} \partial_+ u - \frac{k}{2} \left(u \partial_- h + \partial_-(uh) \right).$$

This is exactly the equation (3.5) and therefore

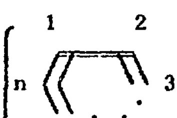
$$W(u) = \frac{k}{2} W_L(u). \qquad (3.17)$$

To add to the analogy with the Liouville action it is interesting to construct a determinant representation for the potential $W_L(u)$. It turns out that in this case there is a family of covariant differential operators with the anomaly being reducible to the functional $W_L(u)$. The simplest example is the operator

$$\partial_-^2 + \frac{u}{2} . \qquad (3.18)$$

This observation leads in particular to the following diagram expansion for $W_L(u)$

$$W_L(u) = 2 \sum_{n=2}^{\infty} \frac{(-)^{n+1}}{n(2\pi)^n} \left[\quad \right] u_1 \ldots u_n d^2x_1 \ldots d^2x_n \qquad (3.19)$$



where the double line corresponds to propagator x_{ij}^- / x_{ij}^+ and the vertex is equal to 1. In the next section several more complicated determinant representations are suggested.

To close the section it is worth to note that recent calculations by Knizhnik et.al. [2] suggest that the abovementioned relation between functionals $S_L(h)$ and $W_L(u)$ holds also in the quantum case. Namely after suitable renormalization of constants the generating functional for quantum correlations of field h reduces to the potential $W_L(u)$. Quantum relation has the form

$$\langle e^{\frac{1}{\pi}\int Th\, d^2x} \rangle_c = e^{\frac{k}{2} W_L(zT)} \qquad (3.20)$$

where $\langle ... \rangle_c$ means the average over the fluctuations of the induced gravity with the central charge c while k and z are in the following relation to c

$$c-13 = 6(k+2 + \frac{1}{k+2}) \qquad (3.21)$$

$$z = \frac{2}{k+2} \quad , \qquad (3.22)$$

4. Determinants of covariant operators

Consider the symmetric operator (3.18) in the two-dimensional space with coordinates x^+, x^-. To study the local properties of its determinant introduce commutative (bosonic) field $\varphi(x)$. Then

$$\det^{-1/2}(\partial_-^2 + \frac{u}{2}) = \int \mathcal{D}\varphi\, e^{-\frac{1}{2\pi}\int \varphi(\partial_-^2 + \frac{u}{2})\varphi\, d^2x} \quad . \qquad (4.1)$$

In the free field theory defined by (4.1) the unperturbed propagator (for u = 0) is

$$\langle \varphi(x)\varphi(0) \rangle = - \frac{x^-}{x^+} \quad . \qquad (4.2)$$

while the interaction with the external field $u(x)$ is through the current $H(x)$ defined as $\delta\mathrm{Action} = \frac{1}{\pi}\int H(x)\delta u(x) d^2x$, where

$$H(x) = \frac{1}{4}\varphi^2. \tag{4.3}$$

In unperturbed theory the two-current operator product expansion is

$$H(x)H(0) = \frac{1}{8}\left(\frac{x^-}{x^+}\right)^2 - \frac{x^-}{x^+}H(0) - \frac{(x^-)^2}{2x^+}\partial_- H(0) + \text{reg.} \tag{4.4}$$

Comparing this with (3.8) we conclude that the theory under consideration exhibits the current algebra with $k = -1/2$ and the effective action in external field u is generated by the connected current correlations. Therefore

$$\det^{-1/2}(\partial_-^2 + \frac{u}{2}) = e^{-\frac{1}{4}W_L(u)}. \tag{4.5}$$

In particular the standard determinant expansion lead to diagram representation (3.18) for the potential $W_L(u)$.

The nonperturbative derivation of this relation is also possible. In nonzero external u(x) the field φ satisfy the linear equation of motion

$$(\partial_-^2 + \frac{u}{2})\varphi(x) = 0. \tag{4.6}$$

Classically this equation lead to the following third order current conservation equation

$$(\partial_-^3 + 2u\partial_- + \partial_- u)H(x) = 0. \tag{4.7}$$

We shall see now that in the quantum case there is an anomalous term in the r.h.s. of (4.7) and calculate it. To make expression (4.3) sensible in quantum case define current through the operator product expansion

$$H(x) = \frac{1}{4}\overline{\lim_{\varepsilon \to 0}}\left[\varphi(x+\varepsilon)\varphi(x) + \varepsilon^-/\varepsilon^+\right]. \tag{4.8}$$

Consider the two-point expectation value $G(x,x') = \langle\varphi(x)\varphi(x')\rangle$. This symmetric in x and x' function satisfy the equation

$$(\partial_-^2 + \frac{1}{2}u(x))G(x,x') = \pi\delta^2(x-x'). \tag{4.9}$$

Let $\phi_1(x^-,x^+)$ and $\phi_2(x^-,x^+)$ by a pair of linear independent zero modes of the operator (3.18), non-singularly parameterized by the coordinate x^+ and normalized by the condition $\phi_1(x)\partial_-\phi_2(x) - \partial_-\phi_1(x)\phi_2(x) = 1$. Then

$$G(x,x') = \frac{\phi_1(x)\phi_2(x')-\phi_2(x)\phi_1(x')}{(x-x')^+} + \Omega(x,x') \tag{4.10}$$

where $\Omega(x,x')$ is zero mode of (3.18) in each variable x^- and $(x')^-$. By definition (4.8)

$$4\langle H(x)\rangle = \partial_+\phi_1(x)\phi_2(x) - \phi_1(x)\partial_+\phi_2(x) + \Omega(x,x). \tag{4.11}$$

With this representation one finds

$$(\partial_-^3 + 2u\partial_- + \partial_-u)\langle H(x)\rangle = -\frac{1}{4}\partial_+u. \tag{4.12}$$

More careful analysis shows that (4.12) holds not only for vacuum expectation but in the operator sense, i.e. for every correlation function.

Note that the operator (3.18) enjoys the following covariance property. Consider the group of coordinate x^- diffeomorphisms: $x^- \to f(x^-)$ (they may depend on x^+ as a parameter; presently this is not important). If u transforms as the projective connection under these substitutions, i.e.

$$\delta u = \omega\,\partial_-u + 2\partial_-\omega\, u + \partial_-^3\omega \tag{4.13}$$

for infinitesimal $f = x^- + \omega$, then this operator acts covariantly from the space of differentials of weight $j = -1/2$, i.e. quantities that transform as

$$\delta\psi_j = \omega\, \partial_- \psi_j + j \partial_- \omega\, \psi_j \tag{4.14}$$

to the space of differentials of weight 3/2. In this sense the action in (4.1) is invariant under these diffeomorphisms. It is also invariant under substitutions $x^+ \to x^+ + \epsilon(x^+)$, *if one considers* φ as a differential of weight 1/2 under transformations of this type.

Classically the current $H(x)$ *defined* by (4.3) transforms as a differential of weight -1 under x^- coordinate substitutions. Like stress tensor in conformal field *theory* the quantum current (4.8) exhibits an inhomogeneous term under these transformations

$$\delta H = \omega\, \partial_- H - \partial_- \omega\, H + \frac{k}{2}\, \partial_-^3 \omega. \tag{4.15}$$

For *finite* substitutions it has the *form*

$$H(f) = \frac{\partial f}{\partial x^-}\, H(x) - \frac{k}{2}\, \frac{\partial f}{\partial x^+}\,. \tag{4.16}$$

The antisymmetric third order differential operator $\partial_-^3 + 2u\partial_- + \partial_- u$, which we have already met in (3.5), enjoys the similar covariance property: it acts from the space of differentials of weight -1 to the space of weight 2 differentials. Anomaly of its determinant also reduces to functional $W_L(u)$. Introduce anticommuting fields ϕ and ψ

$$\det(\partial_-^3 + 2u\partial_- + \partial_- u) = \int \mathcal{D}\phi \mathcal{D}\psi\, e^{\frac{1}{\pi}\int \phi(\partial_-^3 + 2u\partial_- + \partial_- u)\psi\, d^2x}\,. \tag{4.17}$$

The action here is also invariant under x^- *diffeomorphisms*. Now the unperturbed two point function is

$$\phi(x)\psi(0) = -\frac{1}{2}\frac{(x^-)^2}{x^+} \qquad (4.18)$$

and the current has the form

$$H(x) = \phi\,\partial_-\psi - \partial_-\phi\,\psi . \qquad (4.19)$$

The operator product expansion of two currents is again of the form (3.8) with k = 4. Therefore

$$\det(\partial_-^3 + 2u\partial_- + \partial_- u) = e^{2W_L(u)} . \qquad (4.20)$$

There is a remarkable family of differential operators in ∂_- with the abovestated covariance property. Consider the space of differential operators of order 2j+1 with j non-negative integer or half-integer number and with coefficients containing homogeneously the only function $u(x^-)$ and its x^--derivatives. They are homogeneous in the following sense: operator $\partial/\partial x_-$ weights 1 and the weight of the function u is 2 (then $\partial_- u$ weights 3, etc.). Then there is a unique operator D_j (up to an overall constant) of this type which act covariantly from the space of differentials of weight -j to the space of weight j+1 differentials.

Requiring this property one finds few simple examples

$$\begin{aligned} D_{1/2} &= \partial_-^2 + \frac{1}{2}u \\ D_1 &= \partial_-^3 + 2u\partial_- + \partial_- u \\ D_{3/2} &= \partial_-^4 + 5u\partial_-^2 + 5\partial_- u\partial_- + \frac{3}{2}\partial_-^2 u + \frac{9}{4}u^2 \\ D_2 &= \partial_-^5 + 10u\partial_-^3 + 15\partial_- u\partial_-^2 + 9\partial_-^2 u\partial_- + 16u^2\partial_- + 2\partial_-^3 u + 16u\partial_- u . \end{aligned} \qquad (4.21)$$

Calculations become rather involved for higher order operators. They can be simplified considerably by the use of the following observation. The differential operators under consideration are related to the null-vectors in the Verma modules of the Virasoro algebra [6] in the so called classical limit [7]. Consider the Virasoro Verma module and tend the value of the central charge $c \to -\infty$, keeping the highest weight Δ and the level under consideration to be finite. The series of the Kac curves [6] $\Delta_{(1,n)}(c)$ in this limit tend to finite weights $\Delta_{(1,n)}(c \to -\infty) = -(n-1)/2$ and the corresponding null-vectors can be associated with the operators described above.

In this sense the space of differential operators of the type described above, acting on the differential of weight j may be associated with the highest weight representation of the "classical" Virasoro algebra. To reach this classical limit (i.e. the limit $c \to -\infty$) one should substitute in the commutation relations of the Virasoro algebra

$$\left[L_m, L_n\right] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n} \qquad (4.22)$$

as follows

$$L_m = \frac{c}{12}\ell_m \qquad \text{for } m \le 2\ ,$$

$$L_m = \ell_m \qquad \text{for } m > 2$$

and in the resulting structure constants set $c \to \infty$. The limiting commutation relations are free of parameter c and one results in the following Lie algebra

$$\begin{aligned}
\left[\ell_m, \ell_n\right] &= (m-n)\ell_{m+n} \quad \text{if } (m > -2 \text{ and } (n > -2 \text{ or } m+n \le -2)) \text{ or} \\
&\qquad\qquad\qquad (n > -2 \text{ and } m+n \le -2) \\
\left[\ell_m, \ell_{-m}\right] &= m^3-m \qquad \text{if } |m| > 1 \qquad\qquad (4.23) \\
\left[\ell_m, \ell_n\right] &= 0 \qquad \text{rest cases.}
\end{aligned}$$

We shall not analyse in detail the structure of this algebra, but simply note, that all the operators ℓ_m with $m \le -2$ are commutative and may be represented as multiplication by functions, for instance $\ell_{-2} \to u(x)$. With the representation of ℓ_{-1} as $\partial/\partial x$ the commutation relations lead to

$$\ell_{-n-2} \to \frac{1}{n!}\partial^n u \qquad n \geq 0$$

and the arbitrary vector in the Verma module becomes the differential operator with the coefficients constructed of u and its derivatives, acting on a differential of the highest weight. The action of ℓ_n's with positive n in the module is defined completely by the following rules

$$\begin{aligned} \ell_0\phi_j &= (j+n)\partial^n\phi_j \\ \ell_1\partial^n\phi_j &= n(2j+n-1)\partial^{n-1}\phi_j \\ \ell_2\partial^n\phi_j &= n(n-1)(3j+n-2)\partial^{n-2}\phi_j \end{aligned} \tag{4.24}$$

where ϕ_j is the highest vector of weight j, and the commutation relations

$$\begin{aligned} &[\ell_0,\partial^n u] = (n+2)\partial^n u, && \text{for } n \geq 0 \\ &[\ell_1,\partial^n u] = n(n+3)\partial^{n-1}u, && \text{for } n \geq 0 \\ &[\ell_2,u] = 6, \quad [\ell_2,\partial^n u] = n(n-1)(n+4)\partial^{n-2}u, && \text{for } n \geq 1\,. \end{aligned} \tag{4.25}$$

The covariant differential operator of the order 2j+1 is the null-vector in the module of highest weight -j (it is annihilated by all ℓ_n's with positive n.

This approach can be used to calculate the covariant differential operators. In particular one can find in general the linear in u term of the operator D_j

$$D_j = \partial^{2j+1} + \frac{j(j+1)(2j+1)}{3}\sum_{k=0}^{2j-1} \frac{6C_k^{2j-1}}{(k+2)(k+3)}\,\partial^k u\partial^{2j-1-k} + O(u^2). \tag{4.26}$$

This leads to the following expression for the zero field (u = 0) current

$$H_j = \sum_{k=0}^{2j-1} \frac{(2j-k)(k+1)}{2} (-)^k \partial^k \phi \, \partial^{2j-1-k} \psi \qquad (4.27)$$

with the anticommuting fields ψ and ϕ. In this normalization the two-point function is

$$\langle \phi(x) \psi(0) \rangle = -\frac{1}{(2j)!} \frac{(x^-)^{2j}}{x^+} \qquad (4.28)$$

and one finds that the two-current operator product expansion is of the form (3.8) with

$$k_j = \frac{2j(2j+1)(2j+2)}{6} . \qquad (4.29)$$

This suggests that

$$\det D_j = \exp \frac{k_j}{2} W_L(u). \qquad (4.30)$$

One question naturally arises after the last calculation. It looks like the answer (4.30) holds for the operator (2.26) with the nonlinear in u terms truncated, i.e.when the expression (4.27) is exact. Now, what is the role of the nonlinear terms? The perturbative analyses of few first operators suggests the following answer. For $j > 1$ the two-current operator product expansion (3.8) is not correct as the relation between the distributions and holds only up the functions with local support. The δ-functions and their derivatives actually emerge in the r.h.s of (3.8). They result in the contact terms in the diagram expansion of the effective potential. The nonlinear in u terms are designed in just the way to cancel all the contact terms contributions. To support this picture one can perform the nonperturbative derivation of the current anomaly, as it was done for the second order operator, and find that the correct form of the anomaly is characteristic exactly for the currents of the covariant operators.

5. Discussion

It was shown by A.M.Polyakov (unpublished) that the expression (4.20) for the third order determinant can be used to calculate quickly the one-loop quantum renormalization of the constants (3.21) and (3.22). Up to this order they are

$$k = \frac{c-25}{6} - \ldots$$
$$z = \frac{2}{k+2} = 12/c + 156/c^2 + \ldots \tag{5.1}$$

This is to be compared with the result of the quasiclassical evaluation of the functional integral (3.20)

$$\langle e^{\frac{1}{\pi}\int Th\, d^2x} \rangle_c = \int \mathcal{D}h\, e^{S_{Liouv}(h)+\frac{1}{\pi}\int T(x)h(x)d^2x} . \tag{5.2}$$

In the zero-loop order this is estimated by the classical solution h_{cl}, i.e. the solution of the equation

$$(\partial_-^3 + 2u\partial_- + \partial_- u)h_{cl} = \partial_+ u \tag{5.3}$$

with the identification $T = \frac{c}{12} u$

$$e^{S_{Liouv}(h_{cl})+\frac{1}{\pi}\int T(x)h_{cl}(x)d^2x} = e^{\frac{c}{12} W(\frac{12}{c} T)} . \tag{5.4}$$

Expanding $h = h_{cl} + f$ one has to the one-loop order the gaussian integral

$$\int \mathcal{D}f\, e^{\int \frac{\delta^2 S_{Liouv}(h_{cl})}{\delta h(x)\delta h(x')} f(x)f(x')d^2x\, d^2x'} =$$

$$\det{}^{-1/2}\left[\frac{\delta^2 S_{Liouv}(h_{cl})}{\delta h(x)\delta h(x')}\right]. \tag{5.5}$$

On the other hand $\delta S_L/\delta h(x) = -\frac{1}{6\pi}\, u(x)$, where u is implicitly determined by h_{cl} through eq.(5.3). Therefore, the one-loop determinant is $\det^{-1/2}\left[\delta u(x)/\delta h(x')\right]$. Differentiating eq. (5.3) one finds this to be

$$\det{}^{-1/2}\left[(\partial_+ - h\partial_- - 2\partial_- h)^{-1}(\partial_-^3 + 2u\partial_- + \partial_- u)\right]. \tag{5.6}$$

Determinants for both operators entering (5.6) are ready:

$$\begin{aligned} \det(\partial_+ - h\partial_- - 2\partial_- h) &= e^{-\frac{c_2}{2} S_L(h)} = e^{-13 S_L(h)} \\ \det(\partial_-^3 + 2u\partial_- + \partial_- u) &= e^{-\frac{k_1}{2} W_L(u)} = e^{2W_L(u)} \end{aligned} \tag{5.7}$$

and for one-loop contribution we have

$$\det{}^{-1/2}(\ldots) = e^{-\frac{13}{2} S_L(h_{cl}) - W_L(\frac{12}{c} T)}. \tag{5.8}$$

Combining with the classical contribution one has

$$e^{\frac{c-25}{12} W_L(\frac{12}{c} T) + \frac{13}{\pi c}\int T h_{cl} d^2x}. \tag{5.9}$$

Recall now that $\delta W/\delta u$ is $\frac{1}{\pi}$ h. So, to this order (5.9) reduces to

$$e^{\frac{c-25}{12} W_L\left[\frac{12}{c}(1+\frac{13}{c})T\right]} \tag{5.10}$$

in agreement with the expansion (5.1).

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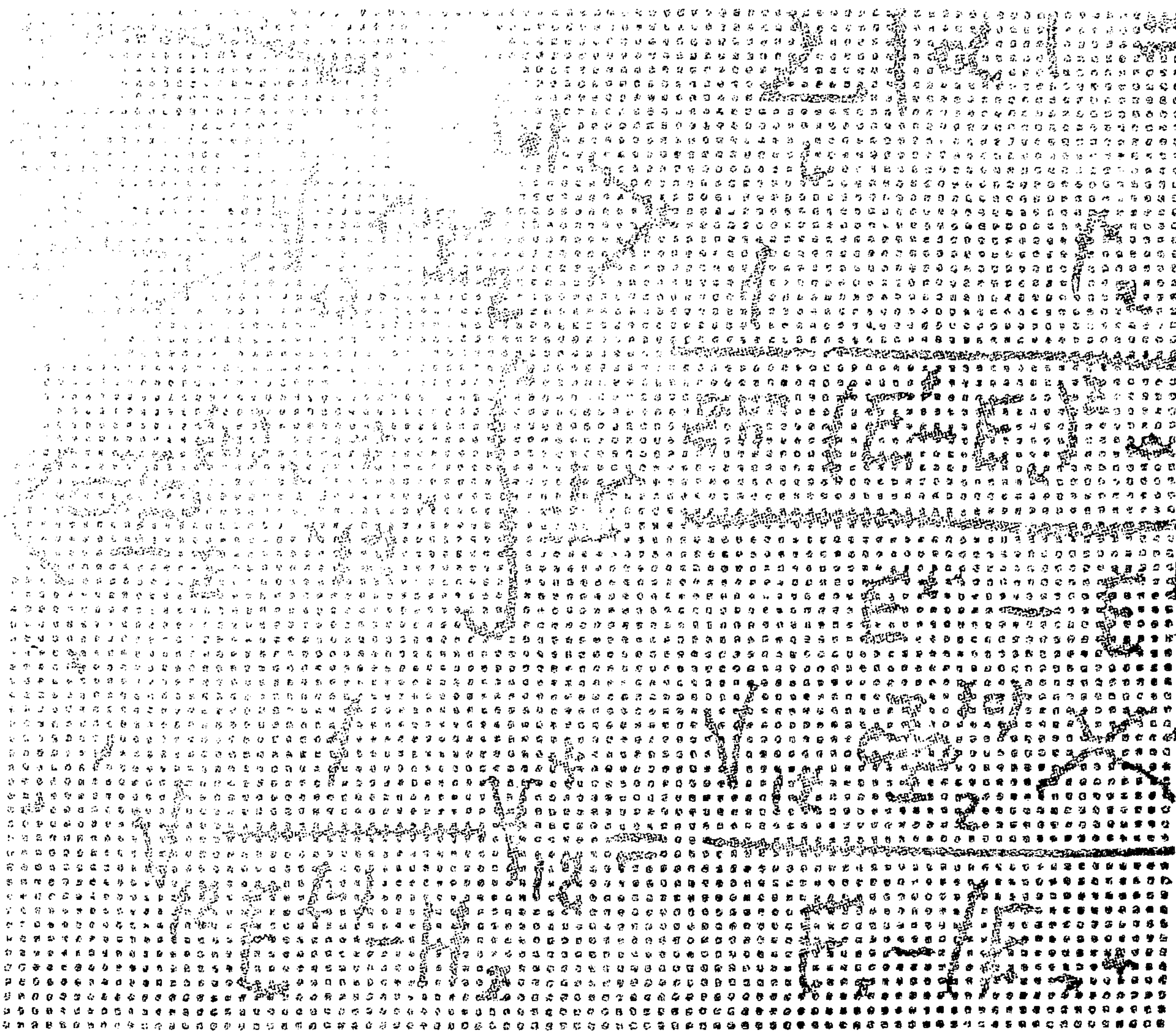

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