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STATE SPACE IN BRST-QUANTIZATION AND KUGO-OJIMA QUARTETS

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Abstract

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The structure of the state space in the BRST-quantization is considered and the connection between different approaches to the proof of the positive definiteness of the metric on the physical state space is established. The correspondence between different expressions for the BRST-charge, quadratic in fields, is obtained. The relation between different representations of the BRST-algebra is found.

Аннотация

Рыбкин Г.Н. Пространство состояний в БРСТ-квантовании и квартеты Куго-Оджимы: Препринт ИФВЭ 89-138. - Серпухов, 1989. - 26 с., библиогр.:22.

Рассмотрена структура пространства состояний в БРСТ-квантовании и установлена связь между различными подходами к доказательству положительной определенности метрики в пространстве физических состояний. Получено соответствие между различными выражениями для квадратичного по полям БРСТ-заряда. Найдено соотношение между различными представлениями БРСТ-алгебры.

1. INTRODUCTION

The problem of the covariant quantization of gauge - invariant systems has a long history. Lately, the BRST - quantization method has been widely used to solve it. This method originates from the works by Becchi, Rouet, Stora [1] and Tyutin [2] who discovered a remarkable symmetry of the effective action for the Yang-Mills field, then called the BRST - symmetry. As was shown afterwards, the BRST - transformations could be introduced and the corresponding BRST - invariant nondegenerate action could be constructed for any gauge - invariant system [3,4]. (A more detailed presentation of these topics and references may be found in reviews [5-7].) In this, the original variables describing the states of the classical gauge-invariant system, are completed with the new ones, called ghosts and antighosts. Clearly, after the quantization of the system, described by the effective action, a too wide state space is obtained. Moreover, since in the quantum theory the BRST-symmetry is generated by the operator Q, satisfying the conditions

$$Q^2 = 0, \tag{1.1}$$

$$Q^+ = Q, \tag{1.2}$$

the total state space turns out to have an indefinite metric. Thus, the problem is to select the physical subspace with a nonnegative metric at least in the total state space. As the condition to specify the physical subspace in the BRST - quantization the equality

$$Qv = 0 \tag{1.3}$$

was taken [8,9]. The physical state space is then obtained through the factorization of the physical subspace over the vectors, which are orthogonal to all vectors from the physical subspace.

In order to have the probabilistic interpretation of the quantum theory it is necessary that the inner product on the physical state space be positive definite. In recent paper [10] we carried out a strict consideration of the structure of the state space in the BRST-quantization and proved the positive definiteness of the inner product on the physical state space. In this, we supposed that the quantization of the system described by the effective action led to a field theory model, for which the asymptotic states and asymptotic fields existed, and the BRST-charge, expressed in terms of the asymptotic fields, was quadratic in fields.

The general form of the BRST-charge, quadratic in fields, was derived in Ref. [11] on the basis of the investigation of the BRST-algebra representations. A somewhat different expression for the BRST - charge was derived in papers [12] with the help of natural assumptions on its properties. As was shown in Ref. [13], the Hamiltonian approach to the BRST - quantization of Batalin - Fradkin - Vilkovisky [4,6,7] leads to the same expression for the BRST-charge when as above the assumptions on the existence of the asymptotic states and asymptotic fields are made. In the present paper we establish the correspondence between these two different expressions for the BRST-charge.

The positive definiteness of the inner product on the physical state space was also discussed in Refs. [11-16]. However, the considerations made in these papers were formal. Indeed, as was already noted, the state space in the BRST - quantization has an indefinite metric. And to consistently study the properties of the operators in such spaces, one has to introduce an additional structure allowing to define the notion of convergency. This was not made in the works cited. In paper [10] we, in particular, got rid of this drawback sticking to the point of view that the state space in the BRST - quantization was a Krein space [17-19]. In the present paper we establish the connection between our consideration [10] and the proof of Ref. [11].

The representations of the BRST-algebra were investigated from a general standpoint in Refs. [11] and [14,15]. We obtain the relation between the representations constructed in these papers.

The present paper is organized as follows. All the necessary definitions and notations used are listed in Sect. 2. In Sect. 3 we briefly describe a concrete realization of the Krein space arising in the BRST-quantization, and present two different expressions for the BRST-charge, and establish the correspondence between them. In Sect. 4 the structure of the physical subspace is considered and the connection between two different proofs - ours and the Kugo - Ojima's - of positive definiteness of the inner product on the physical state space is shown. The relation between different representations of the BRST-algebra is obtained in Sect. 5. The results obtained are discussed in Sect. 6.

2. Z_2-GRADED KREIN SPACES

A Krein space is a Hilbert space V with the inner product (,), where an indefinite inner product < , >, related to the inner product (,) by the equality

$$\langle v,u\rangle = (v,Ju), \qquad (2.1)$$

is introduced. Here J is a selfadjoint operator satisfying the relation $J^2 = I$. A positive definite inner product (,) generates the norm

$$\|v\| = (v,v)^{1/2} \qquad (2.2)$$

for any $v \in V$. Two vectors, v and u, from a Krein space V are called J-orthogonal provided

$$\langle v,u\rangle = 0. \qquad (2.3)$$

Let A be a linear operator in a Krein space V with a dense domain. The operators, adjoint to the operator A with respect to the inner products (,) and < , >, we denote by A^* and A^+ and call adjoint and J-adjoint to the operator A, respectively. One can easily verify that they are connected by the relation

$$A^+ = JA^*J. \qquad (2.4)$$

A Krein space V is called Z_2-graded if V is a Z_2-graded Hilbert space and the operator J is even. We assume that on the homogeneous vectors of V and for the homogeneous operators in V the parity function is defined in a usual manner. The values of this function will be denoted by | |. The parity function takes its values in the ring of integers mod 2, which consists of two elements $\bar{0}$ and $\bar{1}$ and is usually denoted as Z_2. The generalized commutator [A,B] of the homogeneous operators A and B is defined as

$$[A,B] = AB - (-1)^{|A||B|}BA. \qquad (2.5)$$

In particular, if at least one of the operators A and B is even, we obtain the usual commutator $[A,B]_-$, and if both operators are odd then we obtain the anticommutator $[A,B]_+$.

With more details, the information on Z_2-graded Krein spaces is presented in our previous work [10].

3. TOTAL STATE SPACE AND BRST-CHARGE

In our previous work [10] we have shown that it is natural to treat the asymptotic state space V in the BRST-quantization as a Z_2-graded Krein space and have constructed a concrete realization of the space V. Let us recall the basic points of our construction.

First of all, let the asymptotic state space V be a Z_2-graded linear space with a nondegenerate indefinite inner product $\langle\ ,\ \rangle$.

Following Ref. [12], we consider that a set of creation and annihilation operators of physical and unphysical particles acts in V. We shall call them simply the operators of physical and unphysical particles. Denote the operators of physical particles through A_α^+ and A_α, and those of unphysical particles through a_i^+, $\bar{a}_i^+$, c_i^+, $\bar{c}_i^+$ and a_i, $\bar{a}_i$, c_i, $\bar{c}_i$. The indices α and i here can take both discrete and continuous values. The creation operators of physical and unphysical particles are adjoint to the corresponding annihilation operators with respect to the inner product $\langle\ ,\ \rangle$, which is reflected in their notation. We consider the operators of physical and unphysical particles to have a definite parity, and besides,

$$|a_i| = |\bar{a}_i|, \qquad |c_i| = |\bar{c}_i|, \tag{3.1}$$

$$|a_i| + |c_i| = \bar{1}, \tag{3.2}$$

and the parity of the creation operators coincides with that of the corresponding annihilation operators.

The commutation relations for the operators of physical and unphysical particles are of the form

$$[A_\alpha, A_\beta^+] = \delta_{\alpha\beta}, \tag{3.3}$$

$$[a_i, \bar{a}_j^+] = \delta_{ij}, \qquad [\bar{a}_i, a_j^+] = \delta_{ij}, \tag{3.4}$$

$$[c_i, \bar{c}_j^+] = \delta_{ij}, \qquad [\bar{c}_i, c_j^+] = \delta_{ij}. \tag{3.5}$$

The remaining commutators are equal to zero. Remember, that [,] denotes the generalized commutator which for brevity we shall call just a commutator.

Suppose now that in V there exists the vacuum vector ω, such that

$$A_\alpha \omega = 0, \tag{3.6}$$

$$a_i \omega = \bar{a}_i \omega = c_i \omega = \bar{c}_i \omega = 0, \tag{3.7}$$

$$\langle \omega, \omega \rangle = 1. \tag{3.8}$$

We assume that the even annihilation operators are annihilation operators for bosons, and the odd annihilation operators are annihilation operators for fermions. The vectors, generated by the action of the creation operators on the vacuum vector ω, form a basis in V. Let us assume that a vector, generated by the action of an even (odd) number of fermionic creation operators and any number of bosonic creation operators, is even (odd). In this, the vacuum vector ω is considered to be even.

Let us determine the operator J by the relations

$$J A_\alpha^+ J^{-1} = A_\alpha^+, \tag{3.9}$$

$$J a_i^+ J^{-1} = \bar{a}_i^+, \qquad J \bar{a}_i^+ J^{-1} = a_i^+, \tag{3.10}$$

$$J c_i^+ J^{-1} = \bar{c}_i^+, \qquad J \bar{c}_i^+ J^{-1} = c_i^+, \tag{3.11}$$

$$J\omega = \omega. \tag{3.12}$$

The operator J can easily be shown to satisfy here the relation $J^2 = I$. Note also, that J is an even operator. It can be easily verified that the inner product (,) related to the indefinite inner product < , > through the equality

$$(v,u) = \langle v, Ju\rangle, \tag{3.13}$$

is positive definite. The operator J is selfadjoint with respect to the inner product (,).

Thus, we have all we need to specify the total state space V as a Z_2-graded Krein space [10].

Define now in V the ghost number operator N_{gh}, which is even and J-selfadjoint. An operator A is said to have a definite ghost number, which we shall denote by gh(A), provided the following relation is fulfilled:

$$[iN_{gh}, A]_- = gh(A)A. \tag{3.14}$$

We consider the operators of physical and unphysical particles to have a definite ghost number, and besides,

$$gh(A_\alpha) = 0, \tag{3.15}$$

$$gh(a_i) = -gh(\bar{a}_i), \qquad gh(c_i) = -gh(\bar{c}_i), \tag{3.16}$$

$$gh(a_i) + gh(c_i) = 1, \tag{3.17}$$

and the ghost number of the creation operators coincides with that of the corresponding annihilation operators. Besides, we put

$$iN_{gh}\omega = 0. \tag{3.18}$$

Following again Ref. [12], we use as the BRST-charge the operator Q, given by the relation[1]

$$Q = a_i^+ c_i + c_i^+ a_i. \tag{3.19}$$

Introduce now in V other operators of unphysical particles, which will be denoted through χ_i^+, β_i^+, γ_i^+, $\bar{\gamma}_i^+$ and χ_i, β_i, γ_i, $\bar{\gamma}_i$. The annihilation operators are defined by the relations

$$\chi_i = \bar{a}_i + \frac{1}{2}\omega_{ij}a_j, \qquad \beta_i = -a_i, \tag{3.20}$$

$$\gamma_i = -ic_i, \qquad \bar{\gamma}_i = -\bar{c}_i, \tag{3.21}$$

and the creation operators as before are adjoint, with respect to the inner product < , >, to the corresponding annihilation operators. The quantities ω_{ij} in (3.20) are complex numbers. To simplify the subsequent formulae, we put

$$\omega_{ij}^+ = \omega_{ji}, \tag{3.22}$$

which means that the matrix $\|\omega_{ij}\|$ is purely Hermitian.

We shall assume that the quantities ω_{ij} form a block-diagonal matrix $\|\omega_{ij}\|$. Suppose that whenever the index i varies in the range corresponding to some block of the matrix $\|\omega_{ij}\|$, the parity in (3.1) does not depend on the values of the index i. In this case the operators of unphysical particles we have introduced by (3.20), (3.21) have a definite parity, and one can easily see with the help of Eqs. (3.1), (3.2) that

$$|\beta_i| = |\chi_i|, \qquad |\gamma_i| = |\bar{\gamma}_i|, \tag{3.23}$$

$$|\beta_i| + |\gamma_i| = \bar{1}, \tag{3.24}$$

[1]Throughout this paper, we imply the summation over the repeated indices i,j.

and the parity of the creation operators coincides with that of the corresponding annihilation operators.

The commutation relations for the operators of unphysical particles defined by (3.20), (3.21) are of the form

$$[\chi_i, \chi_j^+] = \omega_{ij}, \qquad [\chi_i, \beta_j^+] = -\delta_{ij}, \qquad [\beta_i, \chi_j^+] = -\delta_{ij}, \qquad (3.25)$$

$$[\gamma_i, \bar{\gamma}_j^+] = i\delta_{ij}, \qquad [\bar{\gamma}_i, \gamma_j^+] = -i\delta_{ij}. \qquad (3.26)$$

The remaining commutators equal zero.

Using (3.16), (3.17), we easily get convinced that, except χ_i^+, χ_i, all the operators of unphysical particles introduced by (3.20), (3.21) have a definite ghost number. Suppose that whenever the index i varies in the range corresponding to some block of the matrix $\|\omega_{ij}\|$, the ghost number in (3.16) does not depend on the values of the index i. Besides, let the following conditions be valid for each block of the matrix $\|\omega_{ij}\|$:

$$\omega_{ij} = 0, \quad \text{if} \quad gh(\beta_i) \neq 0, \qquad (3.27)$$

where every time the indices i,j vary in the range corresponding to the given block of the matrix $\|\omega_{ij}\|$. In this case the operators χ_i^+, χ_i also have a definite ghost number, and besides,

$$gh(\beta_i) = -gh(\chi_i), \qquad gh(\gamma_i) = -gh(\bar{\gamma}_i), \qquad (3.28)$$

$$gh(\beta_i) + gh(\gamma_i) = 1, \qquad (3.29)$$

and the ghost number of the creation operators coincides with that of the corresponding annihilation operators.

In terms of the operators defined by (3.20), (3.21), the BRST-charge (3.19) becomes

$$Q = i(\gamma_i^{+}\beta_i - \beta_i^{+}\gamma_i). \tag{3.30}$$

Both the operators of unphysical particles χ_i^{+}, β_i^{+}, γ_i^{+}, $\bar{\gamma}_i^{+}$ and χ_i, β_i, γ_i, $\bar{\gamma}_i$ characterized by the properties (3.23) - (3.29) and the BRST-charge of form (3.30) were introduced by the authors of Ref. [11]. In the literature these operators of unphysical particles are often referred to just as Kugo - Ojima quartets.

Thus, we have established connection between two different expressions for the BRST-charge by expressing one through the other the corresponding sets of the operators of unphysical particles which act in the total state space V.

4. PHYSICAL SUBSPACE AND KUGO-OJIMA QUARTET MECHANISM

Consider the operator Q^*, adjoint to the operator Q. Using (2.4), and the definitions of the operators J (3.9)-(3.12) and Q (3.19), one obtains the expression .

$$Q^* = \bar{a}_i^{+}\bar{c}_i + \bar{c}_i^{+}\bar{a}_i. \tag{4.1}$$

Define the operator N by the relation

$$N \equiv [Q,Q^*]_+ \equiv QQ^* + Q^*Q. \tag{4.2}$$

Using commutation relations (3.4),(3.5) and the explicit form of the operators Q (3.19) and Q^* (4.1), one can easily show that N is set by the expression

$$N = \bar{a}_i^{+}a_i + a_i^{+}\bar{a}_i + \bar{c}_i^{+}c_i + c_i^{+}\bar{c}_i \tag{4.3}$$

and that it is the unphysical particle number operator. The operator N has a purely discrete spectrum, consisting of the nonnegative integers $n=0,1,\ldots$. The space V is decomposed into the direct sum of closed mutually orthogonal and J-orthogonal subspaces V_n, such that

$$Nv_n = nv_n \tag{4.4}$$

for any $v_n \in V_n$. Any vector $v \in V$ has a unique decomposition of the form

$$v = \sum_{n=0}^{\infty} v_n, \tag{4.5}$$

where $v_n \in V_n$. In this,

$$\|v\|^2 = \sum_{n=0}^{\infty} \|v_n\|^2 < \infty. \tag{4.6}$$

The projectors P_n onto the subspaces V_n are defined by the relations

$$P_n v = v_n \tag{4.7}$$

for any vector $v \in V$. As can be easily checked, they are selfadjoint and J-selfadjoint operators satisfying the conditions of orthogonality and completeness:

$$P_n = P_n^{*} = P_n^{+}, \tag{4.8}$$

$$P_n P_m = \delta_{nm} P_n, \tag{4.9}$$

$$\sum_{n=0}^{\infty} P_n = 1. \tag{4.10}$$

In paper [10] we have strictly proved that any vector $v \in V$ can be represented in the form

$$v = P_0 v + Q\chi(v) + Q^* \chi^*(v), \qquad (4.11)$$

where it may be formally considered, that the vectors $\chi(v)$ and $\chi^*(v)$ are given by the following expressions:

$$\chi(v) = N^{-1} Q^* v, \qquad (4.12)$$

$$\chi^*(v) = N^{-1} Q v. \qquad (4.13)$$

Hence, we have the decomposition of the space V into the direct sum of three mutually orthogonal subspaces:

$$V = Ker(N) \oplus R(Q) \oplus R(Q^*). \qquad (4.14)$$

It can be shown that decomposition (4.14) entails the subspaces $Ker(N)$, $R(Q)$ and $R(Q^*)$ to be closed.

The properties (3.9), (3.12) from the definition of the operator J lead to the relation

$$\langle P_0 v, P_0 u\rangle = (P_0 v, P_0 u) \qquad (4.15)$$

for any $v,u \in V$. Therefore, with an account for the fact that $Ker(N)$ is a closed subspace of V we conclude that $Ker(N)$ is a Hilbert space with an inner product, obtained by restricting the indefinite inner product $\langle\ ,\ \rangle$ to the vectors from $Ker(N)$.

As has already been noted in the Introduction, as a physical subspace in the BRST-quantization we use the subspace, specified by condition (1.3), i.e. $Ker(Q)$. Using decomposition (4.11) together with (4.12), (4.13), one obtains

$$Ker(Q) = Ker(N) \oplus R(Q). \qquad (4.16)$$

Thus, any vector $v \in Ker(Q)$ has a unique decomposition of the form

$$v = P_0 v + Q\chi(v). \qquad (4.17)$$

Now it is clear that

$$<v,u> = <P_0 v, P_0 u> \tag{4.18}$$

for any $v,u \in Ker(Q)$. Hence, the inner product on Ker(Q), which is obtained by restricting the inner product < , > to the vectors from Ker(Q), is degenerate and, in virtue of (4.15), positive semi-definite. In this, the vectors from Ker(Q), J-orthogonal to all the vectors from Ker(Q), form a subspace R(Q). Using as the physical state space the quotient - space

$$H = Ker(Q)/R(Q), \tag{4.19}$$

we see that H can be identified with the subspace Ker(N). Hence, the physical state space H is a Hilbert space. Thus, proceeding from the physical subspace Ker(Q) one can construct the physical state space H (4.19) possessing all the properties required.

Now go back to considering the vectors from the physical subspace Ker(Q). As follows from equality (4.18), the inner products of these vectors are completely determined by the components which contain physical particles only. The components of the vectors from Ker(Q) containing unphysical particles yield no contribution into the inner products of these vectors. This phenomenon was established by the authors of Ref. [11] and named Kugo-Ojima quartet mechanism since, as was noted in the previous Section, unphysical particles form quartets. Let us present the principal points of proof by Kugo and Ojima and compare them with our results. To start with, use the commutation relations for the operators of physical particles (3.3) and obtain thus the explicit form of the projector P_0 (4.7) onto the states containing no unphysical particles

$$P_0 = \sum_{m=0}^{\infty} \frac{1}{m!} A^+_{\alpha_1} \ldots A^+_{\alpha_m} \omega < A^+_{\alpha_1} \ldots A^+_{\alpha_m} \omega, \quad >. \tag{4.20}$$

Next, with the help of the commutation relations for the Kugo-Ojima quartets (3.25), (3.26), find inductively explicit expressions for the projectors P_n (4.7) onto the states containing n unphysical particles

$$P_n = \frac{1}{n}\Big(- \beta_i^{+}P_{n-1}\chi_i - \chi_i^{+}P_{n-1}\beta_i - \beta_i^{+}\omega_{ij}P_{n-1}\beta_j + + i\gamma_i^{+}P_{n-1}\bar{\gamma}_i - i\bar{\gamma}_i^{+}P_{n-1}\gamma_i \Big), \tag{4.21}$$

where n=1,2,... . It should be noted here that the last expressions for the projectors are formal ones. Indeed, the annihilation operators are unbounded operators, that is why P_n for n=1,2,..., represented as (4.21), are defined not for all vectors of the space V. It is easy to show that

$$[Q,P_n]_- = 0 \tag{4.22}$$

for n=0,1,... . It turns out that, as Fujikawa noticed [11,20], the explicit expressions for the projectors P_n (4.21) may be rewritten in a very convenient form

$$P_n = [Q,R_n]_+ \tag{4.23}$$

for n=1,2,..., where

$$R_n = -\frac{1}{n}\Big(\bar{\gamma}_i^{+}P_{n-1}\chi_i + \chi_i^{+}P_{n-1}\bar{\gamma}_i + \beta_i^{+}\omega_{ij}P_{n-1}\bar{\gamma}_j \Big). \tag{4.24}$$

One can easily get convinced in the validity of this representation with the help of (4.22), the explicit form of the operator Q (3.30), as well as the commutation relations for the Kugo - Ojima quartets (3.25), (3.26). Using then representation (4.23) and J-selfadjointness of the operator Q (1.2), for any vectors v, u from the physical subspace Ker(Q) one immediately obtains

$$\langle v, P_n u\rangle = 0 \qquad (4.25)$$

when $n=1,2,\ldots$. Hence, unphysical particles called also Kugo - Ojima quartets yield no contribution into the inner products of the vectors from the physical subspace. As noted above, this is what the Kugo - Ojima quartet mechanism consists in. Next, using the definition of the projectors P_n (4.7), rewrite the decomposition of an arbitrary vector $v \in V$ into the components with a definite number of unphysical particles (4.5) as

$$v = P_0 v + \sum_{n=1}^{\infty} P_n v. \qquad (4.26)$$

Now, with the help of relations (4.25), (4.26), one comes again to expression (4.18) for any $v,u \in Ker(Q)$. As before, equality (4.18) allows to identify the physical state space with the subspace $Ker(N)$, the latter being a Hilbert space. The last circumstance completes the proof. As we see, Kugo and Ojima in their proof make an essential use of the explicit expressions for the projectors P_n (4.21) and, in particular, of the Fujikawa representation (4.23). Therefore, to perform a more detailed comparison with our constructions it turns out useful to rewrite these explicit expressions in terms of the operators of unphysical particles a_i^+, $\bar{a}_i^+$, c_i^+, $\bar{c}_i^+$ and a_i, $\bar{a}_i$, c_i, $\bar{c}_i$. Using relations (3.20), (3.21), for the projectors P_n instead of (4.21) we obtain the following expressions:

$$P_n = \frac{1}{n}\left(a_i^+ P_{n-1}\bar{a}_i + \bar{a}_i^+ P_{n-1} a_i + c_i^+ P_{n-1}\bar{c}_i + \bar{c}_i^+ P_{n-1} c_i \right) \qquad (4.27)$$

for $n=1,2\ldots$. The expression for the operator R_n (4.24) takes the form

$$R_n = \bar{R}_n + \Delta_{R_n} \qquad (4.28)$$

for n=1,2,... . Here the following relations are valid:

$$\bar{R}_n = \frac{1}{n}\left(\bar{c}_i^{+} P_{n-1} \bar{a}_i + \bar{a}_i^{+} P_{n-1} \bar{c}_i \right), \qquad (4.29)$$

$$\Delta_{R_n} = [Q, \bar{S}_n]_-, \qquad (4.30)$$

$$\bar{S}_n = -\frac{1}{2n}\bar{\gamma}_i^{+}\omega_{ij}P_{n-1}\bar{\gamma}_j = -\frac{1}{2n}\bar{c}_i^{+}\omega_{ij}P_{n-1}\bar{c}_j. \qquad (4.31)$$

With the help of the relations for the parity and the ghost number of the operators of unphysical particles (3.1), (3.2) and (3.16), (3.17) we see that

$$(\bar{R}_n)^{+} = \bar{R}_n, \quad |\bar{R}_n| = 1, \quad gh(\bar{R}_n) = -1. \qquad (4.32)$$

Taking into account the assumptions on the parity and the ghost number of the Kugo - Ojima quartets (3.23), (3.24) and (3.28), (3.29), and using also the properties of the matrix $\|\omega_{ij}\|$ (3.22), (3.27), one can verify that

$$(\bar{S}_n)^{+} = \bar{S}_n, \quad |\bar{S}_n| = 0, \quad gh(\bar{S}_n) = -2. \qquad (4.33)$$

In virtue of (4.30) and (4.33) we have

$$(\Delta_{R_n})^{+} = -\Delta_{R_n}, \quad |\Delta_{R_n}| = 1, \quad gh(\Delta_{R_n}) = -1. \qquad (4.34)$$

Here it should be noted that the nilpotency of the operator Q (1.1) and the representation for the operator Δ_{R_n} (4.30) ensure the validity of the relation

$$[Q, \Delta_{R_n}]_+ = 0. \qquad (4.35)$$

From here, allowing for (4.28), one obtains the representation for the projector P_n, analogous to (4.23),

$$P_n = [Q, \bar{R}_n]_+ \qquad (4.36)$$

for $n=1,2,\dots$. Thus, the usage of the operator $\bar{R}_n$ instead of R_n allows to exclude completely the quantities ω_{ij} from the theory.

Now, making use of the completeness of the set of the projectors P_n (4.10), one can perform the summation

$$\sum_{n=1}^{\infty} \bar{R}_n = N^{-1}(\bar{a}_i^+ \bar{c}_i + \bar{c}_i^+ \bar{a}_i). \qquad (4.37)$$

Taking into account the explicit expression for the operator Q^* (4.1), we conclude that

$$Q^* = N \sum_{n=1}^{\infty} \bar{R}_n = \sum_{n=1}^{\infty} n\bar{R}_n. \qquad (4.38)$$

Analogously, carrying out the summation, we find that

$$\sum_{n=1}^{\infty} \Delta_{R_n} = [Q, N^{-1}\bar{S}]_-, \qquad (4.39)$$

where

$$\bar{S} = \sum_{n=1}^{\infty} n\bar{S}_n = -\frac{1}{2}\bar{\gamma}_i^+ \omega_{ij} \bar{\gamma}_j = -\frac{1}{2}\bar{c}_i^+ \omega_{ij} \bar{c}_j. \qquad (4.40)$$

Using now the Fujikawa representation for P_n (4.23), as well as decomposition (4.26), for an arbitrary vector v from the physical subspace we obtain the decomposition

$$v = P_0 v + Q\tilde{\chi}(v), \qquad (4.41)$$

where

$$\tilde{\chi}(v) = \sum_{n=1}^{\infty} R_n v \qquad (4.42)$$

for any $v \in Ker(Q)$. Taking into account (4.28), (4.37)-(4.39), and also the expression for the vector $\chi(v)$ (4.12), we see that

$$\tilde{\chi}(v) = \chi(v) + Q\sigma(v), \qquad (4.43)$$

where

$$\sigma(v) = N^{-1}\bar{S}v \qquad (4.44)$$

for any $v \in Ker(Q)$. Obviously, when the operator $\bar{R}_n$ is used in the theory instead of R_n , there arises no additional term $Q\sigma(v)$ in (4.43). Then, with the help of relation (4.43), we conclude that

$$Q\tilde{\chi}(v) = Q\chi(v) \qquad (4.45)$$

for any $v \in Ker(Q)$. Thus, decompositions (4.17) and (4.41) for an arbitrary vector from the physical subspace, essential for our proof [10] and for the consideration by Kugo and Ojima [11], respectively, coincide.

5. REPRESENTATIONS OF BRST-ALGEBRA AND FORM OF BRST-CHARGE

Using the explicit expression for the operator Q (3.19), the definition of the ghost number (3.14) and the equalities relating ghost numbers of the operators of unphysical particles (3.16), (3.17), it is easy to calculate the following generalized commutators:

$$[Q,Q] = 2Q^2 = 0, \qquad (5.1)$$

$$[iN_{gh},Q] = Q. \qquad (5.2)$$

Hence, with respect to the operation [,] the J-selfadjoint operators Q, N_{gh} form an algebra called BRST-algebra. The operators of physical and unphysical particles we made use of in this paper, turn out to be related directly to the representations of this algebra.

It has already been noted above that unphysical particles form quartets. In fact, the operators χ_i^+, β_i^+, γ_i^+, $\bar{\gamma}_i^+$ were introduced by Kugo and Ojima as operators which, acting on the vacuum ω, generate four vectors

$$\tilde{p}_i = \chi_i^+\omega, \qquad \tilde{d}^i = -\beta_i^+\omega,$$
$$\tilde{d}_i = -i\gamma_i^+\omega, \qquad \tilde{p}^i = -\bar{\gamma}_i^+\omega, \tag{5.3}$$

forming a pair of BRST-doublets $\tilde{p}_i$, $\tilde{d}_i$ and $\tilde{p}^i$, $\tilde{d}^i$. This means the vectors $\tilde{p}_i$, $\tilde{d}_i$, $\tilde{p}^i$, $\tilde{d}^i$ are related with each other through the relations

$$\tilde{d}_i = Q\tilde{p}_i, \qquad \langle\tilde{p}^i,\tilde{d}_j\rangle = \delta_{ij},$$
$$\tilde{d}^i = Q\tilde{p}^i, \qquad \langle\tilde{d}^i,\tilde{p}_j\rangle = \delta_{ij}, \tag{5.4}$$

and besides, each of these vectors has a definite parity and a definite ghost number. Now, using relations (5.3) and the properties of the vectors $\tilde{p}_i$, $\tilde{d}_i$, $\tilde{p}^i$, $\tilde{d}^i$, one can derive commutation relations for the Kugo-Ojima quartets (3.25), (3.26), relations between their parities (3.23), (3.24) and their ghost numbers (3.28), (3.29), and also the explicit expression for the operator Q (3.30) [11].

An analogous consideration can be also performed for the other set of operators of unphysical particles used in this paper. Following relations (5.3), construct four vectors according to the formulae

$$p_i = \bar{a}_i^+\omega, \qquad d^i = a_i^+\omega,$$
$$d_i = c_i^+\omega, \qquad p^i = \bar{c}_i^+\omega. \tag{5.5}$$

Now, using the commutation relations for the operators of unphysical particles (3.4), (3.5), the relations between their parities (3.1), (3.2) and their ghost numbers (3.16), (3.17), as well as the definitions of the operators J (3.9)-(3.12) and Q (3.19), it is easy to see that the following relations are valid:

$$p_i = Q^* d_i, \qquad d^i = J p_i,$$
$$d_i = Q p_i, \qquad p^i = J d_i. \tag{5.6}$$

Moreover, each of these vectors has a definite parity and ghost number, and all their mutual inner products are equal to zero, with the only exception of

$$\langle p^i, d_j \rangle = \langle d^i, p_j \rangle = \delta_{ij}. \tag{5.7}$$

Obviously, the vectors p_i, d_i and p^i, d^i realize the doublet representations of the BRST-algebra. In this, they are all eigenstates of the operator N (4.2) belonging to the eigenvalue n=1:

$$Ne_\lambda = e_\lambda, \qquad (e_\lambda, e_\chi) = \delta_{\lambda\chi}, \tag{5.8}$$

where $e_\lambda \in \{p_i,\ d_i,\ p^i,\ d^i\}$.

Construct next the vectors

$$s_\alpha = A_\alpha^+\omega. \tag{5.9}$$

Using the properties of the operators of physical particles we see at once that

$$Qs_\alpha = Q^* s_\alpha = 0, \qquad (5.10)$$

and that the following relations are satisfied:

$$Js_\alpha = s_\alpha, \qquad \langle s_\alpha, s_\beta\rangle = \delta_{\alpha\beta}. \qquad (5.11)$$

Besides, each of these vectors has a definite parity and a zero ghost number. Clearly, the vectors s_α realize the singlet representations of the BRST-algebra.

One can also verify the validity of the relation

$$N_{gh} J = - JN_{gh}. \qquad (5.12)$$

The vectors s_α and p_i, d_i , p^i, d^i with the properties (5.6)-(5.8), (5.10), (5.11) described above were introduced in investigating the representations of the BRST-algebra in Ref. [15]. With the help of (5.3), (5.5) and (3.20), (3.21) one can easily express the vectors $\tilde{p}_i$, $\tilde{d}_i$, $\tilde{p}^i$, $\tilde{d}^i$ through p_i, d_i , p^i, d^i and vice versa, which will yield the connection between different bases of the BRST-algebra representations:

$$\tilde{p}_i = p_i + \frac{1}{2} d^j \omega_{ji}, \qquad \tilde{d}^i = d^i,$$
$$\tilde{d}_i = d_i, \qquad \tilde{p}^i = p^i. \qquad (5.13)$$

The appearance of the quantities ω_{ij} in (5.13) reflects the fact that relations (5.4) define the vectors $\tilde{p}_i$, $\tilde{d}_i$, $\tilde{p}^i$, $\tilde{d}^i$ ambiguously. These vectors coincide with the corresponding vectors p_i, d_i , p^i, d^i, specified by (5.6)-(5.8), only when $\omega_{ij} = 0$.

The investigations of the representations of the BRST - algebra performed in Refs. [14,15] make it possible to derive the form of the BRST-charge. Assume that the vectors s_α and p_i, d_i, p^i, d^i realize, respectively, the singlet and doublet representations of the BRST-algebra (5.1), (5.2) in a Z_2-graded Krein space and satisfy (5.8), and the operators of physical and unphysical particles are introduced by the formulae (5.9) and (5.5), respectively. Let relation (5.12) be also valid for the operator N_{gh}. Then, assuming in addition positive definiteness for the restriction of the inner product $< , >$ to the subspace of BRST-singlets (5.11) and using the properties of the vectors s_α and p_i, d_i, p^i, d^i and of the vacuum vector ω, we obtain the commutation relations for the operators of physical and unphysical particles (3.3)-(3.5), relations between their parities (3.1), (3.2) and their ghost numbers (3.15)-(3.17), the action of the operator J on them (3.9) - (3.11) and the explicit expression for the operator Q (3.19).

6. CONCLUSIONS

In the proofs presented in Sect. 4 an important role belongs to the unphysical particle number operator N (4.3). The introduction of the operator Q^* (4.1) made it possible to express the operator N through the BRST-charge Q (3.19) in the form (4.2). The last circumstance makes the proof of one of the basic relations (4.18) practically obvious. Indeed, using (1.1)-(1.3), (4.2), (4.4), (4.5) one obtains immediately

$$n\langle v_n, u_n\rangle = 0 \tag{6.1}$$

for $n=0,1,\ldots$ and for any $v,u\in Ker(Q)$. It follows from equality (6.1) that

$$\langle v_n, u_n\rangle = \delta_{n0}\langle v_0, u_0\rangle \tag{6.2}$$

for any $v,u\in Ker(Q)$, which, with an account of (4.5), (4.7), proves relation (4.18). It is interesting to note here that this way of proving relation (4.18) for the vectors from the physical subspace can be applied directly after the corresponding replacements in the Gupta-Bleuler quantization of Abelian gauge - invariant systems [21,22]. Let us recall that in this case there is no necessity to introduce the ghosts c_i^+, $\bar{c}_i^+$ and c_i, $\bar{c}_i$, and the only unphysical particles of the theory are a_i^+, $\bar{a}_i^+$ and a_i, $\bar{a}_i$ with the commutation relations (3.4). Correspondingly, in order to specify the physical subspace, in place of (1.3) the following conditions are used:

$$a_i v' = 0 \tag{6.3}$$

for all i, while an analog of representation (4.2) is simply the expression for the unphysical particle number operator:

$$N' = a_i^+ \bar{a}_i + \bar{a}_i^+ a_i . \tag{6.4}$$

In connection with the last proof, let us stress the following circumstance. To define correctly the physical state space and to prove the positive definiteness of its metric it would be just enough to have for any vector v from the physical subspace the decomposition

$$v = P_0 v + \zeta(v), \tag{6.5}$$

here $\zeta(v)$ satisfying the relation

$$\langle \zeta(v), u \rangle = 0 \tag{6.6}$$

for any vector u from the physical subspace [12,21]. As we saw above, in the BRST-quantization a still stronger restriction on $\zeta(v)$ from Eq. (6.5) was valid:

$$\zeta(v) = Q\chi(v). \tag{6.7}$$

Let us make now resume of our consideration. We have established the correspondence between two different expressions for the BRST-charge, quadratic in fields, obtained in Refs. [11] and [12]. Also, the connection between the BRST-algebra representations constructed in Refs. [11] and [14,15] was found. In our case, the Casimir operator Δ introduced in Ref. [15] coincides with the operator N.

We have considered the structure of the physical subspace in the BRST-quantization and shown the connection between our proof [10] of the positive definiteness of the metric on the physical state space and that proposed by Kugo and Ojima [11]. This comparison allows, in particular, to take a fresh look at the constructions of Ref. [11].

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