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Dynamical Entropy for Infinite Quantum Systems[1]

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Abstract

We review the recent physical applications of the so-called Connes-Narnhofer-Thirring entropy, which is the successful *quantum mechanical* generalization of the classical Kolmogorov-Sinai entropy and, by its very conception, is a dynamical entropy for *infinite* quantum systems. We thus comparingly review also the physical applications of the *classical* dynamical entropy for *infinite* classical systems.

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1 Introduction

The concept of *dynamical* (or Kolmogorov-Sinai) *entropy* plays a central rôle in classical "information dynamics" (cf. Atmanspacher and Scheingraber [1]), and thus presumably also in some of the other contributions collected in this proceedings volume. In particular, the two final pages (before the conclusion) of the included review by Wehrl [2] could even be read as a *mathematical* introduction to the subject of this present contribution: Here, we want to review the recent *physical* applications of the so-called Connes–Narnhofer–Thirring (CNT) entropy (mathematically defined in full generality at the end of Wehrl's text), which is the successful *quantum mechanical* generalization of the classical Kolmogorov–Sinai (KS) entropy.

As briefly indicated by Wehrl [2], here the "generalization" has to be understood in a mathematical sense: Classically, (bounded) *observables* of a physical system are represented by (bounded and possibly continuous, but in any case "measurable") real–valued functions on the phase (or "configuration"–, see section 2) space of the system, endowed with a natural algebraic structure given by point–wise addition and (commutative) multiplication of these functions; and the classical (measure theoretic) KS theory can be equivalently "translated" into this (function–) algebraic language (cf. Hudetz [3, 4, 5]). The (bounded) observables of a *quantum* system, however, are represented by self–adjoint (and bounded, but not necessarily *all* of the self–adjoint bounded) operators on a suitable Hilbert space, with the natural algebraic structure of operator addition and (*non*–commutative) multiplication; and in this sense *classical* systems are just "quantum" systems with a *commutative* (or Abelian) algebra of observables (technically speaking, an Abelian von Neumann– or general C^*–algebra; cf. Thirring [6, 7]).

From this far–sighted point of view, the quantum mechanical generalization of the classical dynamical entropy amounts to extending the algebraic formulation of the KS entropy theory from commutative to *non*–Abelian operator algebras; and this generalization was performed *successfully* (in a "physical" sense which should become clear from the following), partly by Connes and Størmer [8] and "completely" by Connes, Narnhofer and Thirring [9], as also cited by Wehrl [2]. The success of this algebraic approach to quantum dynamical entropy, within the general framework of *algebraic* quantum mechanics, was made possible by the fact that the latter applies not only to microscopic quantum systems (with finitely many degrees of freedom, such as single atoms or small molecules; cf. Thirring [6]) but also to macroscopic systems (i.e. "very large" many–particle systems which can be idealized as *infinite*, with infinitely many degrees of freedom; cf. Thirring [7]). An easily readable exposition of this generalized (algebraic) quantum mechanical framework can be found in part I of the book by Sewell [10], where an introductory statement explicitely says that "the quantum theory of infinite systems (...) contains physically relevant structures, that do not occur in the quantum theory of finite systems"; and this fully applies also to the physically *relevant* concept of quantum dynamical entropy, as developed by Connes, Narnhofer and Thirring [9]. We thus claim that, by its very conception, this generalized KS entropy is a dynamical entropy for *infinite* quantum systems; and our aim in this review paper is to convince the reader thereof, by "physical" arguments and comparing with the physical applications of the *classical* dynamical entropy to classical statistical mechanics of *infinite* systems (in the same sense as above).

The comparative reading of section 2 (classical dynamical entropy) and section 3 (quantum dynamical entropy, both for infinite systems respectively) is enabled by a simultaneous subdivision of the two sections; and the reader is advised to leaf back and forth but to keep the historical succession of classical before quantum theory. The review list of references, as far as they concern the development and applications of the CNT entropy, is tentatively complete; but again the historical order is preferable to the alphabetic one (which in turn would be preferable here to the order of citation as chosen by the editors): Distinguishing *two* historical "phases", the "mathematical phase" was set about by Connes and Størmer [8] with their definition (cited by Wehrl [2]) of the dynamical entropy w. r. t. a normalized trace on a von Neumann algebra (which can be reinterpreted "physically" as a quantum state at *infinite* temperature; cf. Thirring [7]), temporarily rejected again by Connes and Størmer [11] themselves. This definition has been applied "mathematically" by Shukhov [12], Besson [13], Pimsner and Popa [14], Quasthoff [15] and Choda [16]. But already in 1985, a "phase transition" to the "mathematical physics" (and *finite* temperature) phase manifested itself, announced by Connes [17] with distant correlation to Narnhofer and Thirring [18], and fully developed by Connes, Narnhofer and Thirring [9] together. Immediately after the transition, in Narnhofer and Thirring [19] there occurred a certain "theory–dynamical" discontinuity w. r. t. the mathematical phase (cf. Størmer and Voiculescu, [20]); and since then the "mathematical physics" phase has coexisted in Narnhofer and Thirring [21, 22], Benatti and Narnhofer [23, 24], Narnhofer [25, 26, 27] and Hudetz [28, 3, 4, 5]. But only recently, the work of Gaspard [29] seems to announce a possible second "phase transition" to a really "physical" phase (see (2.7) resp. (3.7) below).

2 Classical Dynamical Entropy

(2.1) The KS entropy was introduced by Kolmogorov and refined by Sinai as a conjugacy invariant of measure preserving transformations on probability spaces (cf. Wehrl [2] and the references there). In physical applications, it refers to a transformation (e.g. unit time evolution) $T : \Omega \to \Omega$ of the phase (or "configuration"–, see (2.4) below) space Ω of a classical system and assigns a characteristic real number $h_\mu(T) \geq 0$, still depending on the invariant probability measure μ on Ω (e.g. an "equilibrium state" for the time evolution); and we emphasize this dependence by our notation in contrast to the notation $s(T)$ used by Wehrl [2]. As is well–known, this entropy $h_\mu(T)$ has been applied successfully to (spatially) *finite* classical systems with *finitely* many degrees of freedom.

(2.2) But $h_\mu(T) = 0$ *identically* (i.e. for all possible T) for spatially finite classical systems with a phase space Ω which has a *finite* "grain–size" (mathematically, for an *atomic* measure space Ω), with any "physically reasonable" measure μ. This is so because then the *total* entropy $S(\mu)$ of the partition of Ω into its "atoms" is *finite*, and the entropy of *any* partition of Ω (as defined e.g. in Wehrl, [2]) is bounded above by $S(\mu) < \infty$. From the definition of $h_\mu(T)$ as in Wehrl [2] it then follows that $h_\mu(T) \leq \lim_{n\to\infty} S(\mu)/n = 0$.

(2.3) Put differently: There is *no intrinsic* (minimal) "scale of size" in Ω for $h_\mu(T) > 0$ (cf. also the remarks on classical vs. quantum chaos by Lindblad [30]). But careful: Ω need not be the *phase* (or even position) space of the physical system; and in particular for classical systems with *infinitely* many degrees of freedom and *infinite* spatial extension (i.e. in classical statistical mechanics) we can have $h_\mu(T) > 0$ *in spite* of an intrinsic (minimal) length scale in physical *position* space!

(2.4) In fact, the KS entropy was first applied to classical statistical mechanics (of infinite systems) by Robinson and Ruelle [31] for the *classical spin models*: For simplicity, we first consider a *one*-dimensional "classical spin" *chain*, where $\Omega_0 = \{s_0, \ldots, s_N\}$ is a set of $(N+1)$ different "spin values", and Ω is given by the spin *configuration space* $\Omega = \times_{k\in\mathbf{Z}}\Omega_k$ with $\Omega_k \equiv \Omega_0$ $\forall k \in \mathbf{Z}$ (i.e. the space of two-sided sequences $x = (x_k)_{k\in\mathbf{Z}}$ with elements $x_k \in \Omega_0$, and with the measurable sets generated by "cylinder sets", as is well-known). Let $T : \Omega \to \Omega$ be induced by the *lattice shift* along the chain, i.e. for $x = (x_k)_{k\in\mathbf{Z}}$ let $Tx = (x_{k+1})_{k\in\mathbf{Z}}$. Then, for a T-invariant probability measure ("state") μ on Ω: $h_\mu(T) = s(\mu)$, where $s(\mu) \leq \log(N+1)$ denotes the *entropy density* of the state μ on the chain (in the sense of classical statistical mechanics; cf. Ruelle [32]).

(2.5) Robinson and Ruelle [31] proved this equality already for n-dimensional (cubic) spin lattice systems (physically $n = 2, 3$), implicitely using the definition and properties of the *joint* dynamical entropy $h_\mu(S_1, \ldots, S_n)$ of n *commuting* transformations $(S_1, \ldots, S_n) = \vec{S}$ (symbolic notation) of Ω ("exactified" e.g. by Conze, [33]); where for the lattice translations $S_1, \ldots, S_n$ on $\Omega = \times_{\mathbf{k}\in\mathbf{Z}^n}\Omega_{\mathbf{k}}$ with $\Omega_{\mathbf{k}} \equiv \Omega_0$ $\forall \mathbf{k} \in \mathbf{Z}^n$ and with $\vec{S}$-invariant state μ again: $h_\mu(\vec{S}) = s(\mu)$, with the entropy density of μ on the lattice (cf. Ruelle, [32]).

(2.6) But conversely: For infinite classical systems of infinitely many particles moving in a *continuous* physical position space ($\mathbf{Z}^n$ in (2.5) replaced by $\mathbf{R}^n$), "typically" both $h_\mu(\vec{S}) = \infty$ for the unit space translations $\vec{S} = (S_1, \ldots, S_n)$ in the coordinate directions, and $h_\mu(T) = \infty$ for the unit time evolution, *independently* of the particles' interaction (mutual or with an external field)! Goldstein [34] thus considered the so-called *space-time entropy* $h_\mu(\vec{S}, T)$ (defined as in Conze [33]) of such systems with translationally invariant equilibrium state μ for the time evolution T (generated by a translationally invariant interaction or external field): He showed that $h_\mu(\vec{S}, T) = 0$ for the infinite classical *ideal gas*, whereas $h_\mu(\vec{S}, T) > 0$ for the infinite *periodic Lorentz gas* (where $\vec{S}$ is given by the periodicity translations, acting on the space Ω of particle "configurations" in the one-particle phase space X), for a Maxwellian momentum distribution as well (inducing a probability measure μ on Ω given by a "Poisson" distribution of *independent* particles in X; cf. Cornfeld, Fomin and Sinai [35]).

(2.7) Only recently, Gaspard [29] announced an explicit *estimate* for the space resp. time entropy of the (here *one*–dimensional) infinite *ideal gas*: For the unit space translation S and for a partition ξ of the configuration space Ω (as in (2.6) above), induced by a fine-graining of the one-particle phase space X with position (resp. velocity) grain size Δx (resp. Δv), he gets "asymptotically" (as $\Delta x \Delta v \to 0$) for the entropy of S w. r. t. ξ (cf. Wehrl, [2]):

$$h_\mu(S,\xi) \cong \rho \int_{-\infty}^{\infty} dv\, f(v) \log \frac{e}{\rho\, f(v) \Delta x \Delta v},$$

where ρ (resp. $f(v)$) denotes the constant spatial particle number density (resp. the velocity distribution density w. r. t. the measure dv). A detailed and rigorous derivation of this "heuristic" formula will be part of a forthcoming joint publication by Gaspard and Hudetz. Similarly, then, we get for the unit time evolution T of the infinite ideal gas:

$$h_\mu(T,\xi) \cong \rho \int_{-\infty}^{\infty} dv\, |v|\, f(v) \log \frac{e}{\rho\, f(v) \Delta x \Delta v}.$$

For a *Maxwellian* velocity distribution $f(v)$ at inverse temperature β (times Boltzmann's constant), inducing the equilibrium state μ_β of the ideal gas, one then gets:

$$h_{\mu_\beta}(S,\xi) \cong \rho \log \frac{(2\pi m)^{1/2}}{\rho \beta^{1/2} \Delta x \Delta p} + \frac{3\rho}{2},$$

where $\Delta p \equiv m \Delta v$ with particle mass m, resp.

$$h_{\mu_\beta}(T,\xi) \cong \frac{2\rho}{(2\pi m \beta)^{1/2}} \log \frac{(2\pi m)^{1/2}}{\rho \beta^{1/2} \Delta x \Delta p} + \frac{4\rho}{(2\pi m \beta)^{1/2}}.$$

Note that $h_{\mu_\beta}(S) \geq \lim_{\Delta x \Delta p \to 0} h_{\mu_\beta}(S,\xi) = \infty$, resp. $h_{\mu_\beta}(T) \geq \lim_{\Delta x \Delta p \to 0} h_{\mu_\beta}(T,\xi) = \infty$, as generally stated in (2.5) above. But remember that in this case $h_{\mu_\beta}(S,T) = 0$, as shown by Goldstein [34].

(2.8) On the other hand, Goldstein [34] quite generally showed for infinite, spatially *continuous* and *periodic* systems of mutually non-interacting particles (e.g. the infinite periodic Lorentz gas, as already cited in (2.6) above) the following: Let T resp. $\vec{S}$ be the unit time evolution resp. the periodicity translations (acting on the configuration space Ω) with invariant state μ on Ω. For a minimal periodicity volume Λ, we denote by T_Λ the restriction of T to Λ by periodic boundary conditions and ignoring all particles not in Λ. Then, under some rather mild technical assumptions: $h_\mu(\vec{S},T) = h_\mu(T_\Lambda)/|\Lambda|$, where $|\Lambda|$ denotes the volume of Λ. Goldstein [34] further conjectured that for general infinite (periodic) systems of *interacting* particles: $h_\mu(\vec{S},T) = \lim_{\Lambda \to \infty} h_\mu(T_\Lambda)/|\Lambda|$, where $\Lambda \to \infty$ in the sense that the smallest sides of the increasing (periodicity) volumes Λ approach ∞.

In fact, Sinai and Chernov [36] showed this for the infinite system of hard spheres with elastic collisions ("infinite billiard") with unit time evolution T and unit coordinate translations $\vec{S}$ and with (microcanonical) invariant state μ (with *reflecting* boundary conditions for T_Λ and with "suitable" volumes $\Lambda \to \infty$). And Chernov [37] considered the infinite system of infinitely many particles interacting via a pair –, finite range and hard core potential, with unit time evolution T and unit space translations $\vec{S}$, and with (grand canonical) equilibrium state $\mu_{\beta,\rho}$ at inverse temperature β and for (sufficiently small) particle density ρ. He showed that $h_{\mu_{\beta,\rho}}(\vec{S}, T)$ is *finite* and has an upper bound $\rho \cdot c(\beta)$.

3 Quantum Dynamical Entropy

(3.1) As already mentioned in the introduction, the CNT entropy was developed as a conjugacy invariant of automorphisms acting on a C^*–algebra with an invariant "state" (cf. Wehrl [2]). In physical applications, it refers to a transformation (e.g. unit time evolution) θ of a quantum system, acting (in the *Heisenberg representation*; cf. Sewell, [10]) on the (bounded) observables $A \to \theta(A)$ (which are self–adjoint bounded operators on a suitable Hilbert space $\mathcal{H}$); and it assigns a characteristic real number $h_\omega(\theta) \geq 0$, still depending on the invariant quantum state ω of the system (which, e.g. for *finite* systems, is given by a density matrix ρ_ω on $\mathcal{H}$ with $\theta(\rho_\omega) = \rho_\omega$, where $\omega(A) = \mathrm{Tr}(\rho_\omega A)$ with the trace Tr in $\mathcal{H}$). Many of the properties of this entropy $h_\omega(\theta)$ directly generalize (by construction) the properties of the classical KS entropy.

(3.2) But $h_\omega(\theta) = 0$ *identically* (i.e. for all possible θ) for quantum systems with *finitely* many particles (or degrees of freedom), with any "physically reasonable" state ω. This is so because then the *total* entropy $S(\omega)$ of the system is *finite*; e.g. $S(\omega) = -\mathrm{Tr}\rho_\omega \log \rho_\omega < \infty$ for states ω with finite *energy*, cf. Wehrl [2], Thirring [7]. The H_ω–functional, as cited and used in definition (i) at the end of Wehrl's text, is always bounded above by $S(\omega)$; and from this definition (i) and (ii) of $h_\omega(\theta)$ it then follows that $h_\omega(\theta) \leq \lim_{n\to\infty} S(\omega)/n = 0$; cf. Narnhofer [26].

(3.3) Roughly speaking, the "quantum mechanical phase space" has grain size h^N (with Planck's constant h and the number N of degrees of freedom of the system; cf. Lindblad [30]); and we can get $h_\omega(\theta) > 0$ only for $N \to \infty$, either in *finite* spatial volume V (i.e. in quantum *field* theory) or together with the "thermodynamic limit" $V \to \infty$ (and $N/V \to \rho$, i.e. in quantum *statistical* mechanics). Put differently, a "quantization" of a *finite* (but continuous) *chaotic* classical system (here in the sense that $h_\mu(T) > 0$), if it is performed "chaos–preserving" (e.g. in the sense that $h_\mu(T) = h_\omega(\theta)$ with obvious notations), has to lead to an *infinite* (macroscopic or "mesoscopic", cf. e.g. Dittrich and Graham, [38]) quantum system, but *not* necessarily continuous in space.

(3.4) For the well–known classical toy–model of the "baker's transformation", this "quantization" is performed explicitely (e.g.) in Narnhofer, Pflug and Thirring [39]; leading first (by conjugacy and equivalent algebraic "translation", cf. the introduction) to a one–dimensional classical spin model (2.4), and then to a *quantum* spin chain (cf. also Benatti, Narnhofer and Sewell [40] for a "quantization" of Arnold's "cat map"). In fact, the CNT entropy was first applied to quantum statistical mechanics by Connes [17] for these *one-dimensional quantum spin models*, where the Hilbert spaces $\mathcal{H}_0 = \mathbf{C}^{(N+1)}$, $\mathcal{H}_k \equiv \mathcal{H}_0\ \forall k \in \mathbf{Z}$, and $\mathcal{H} = \bigotimes_{k\in\mathbf{Z}} \mathcal{H}_k$ mean a "spin $N/2$-particle" at each lattice site, with θ given by the lattice shift $\theta : \mathcal{B}(\mathcal{H}_k) \to \mathcal{B}(\mathcal{H}_{k+1})\ \forall k \in \mathbf{Z}$. Then, for a θ-invariant quantum state ω of the spin chain: $h_\omega(\theta) \leq s(\omega)$ at first, where $s(\omega) \leq \log(N+1)$ denotes the *entropy density* of the state ω (in the sense of quantum statistical mechanics, cf. Bratteli and Robinson [41]).

This general inequality was sharpened to $h_\omega(\theta) = s(\omega)$ (as also classically, cf. (2.4)) by Narnhofer and Thirring [21] at least for a θ-invariant *short range* interaction between near-by spins, with an *equilibrium* state ω for the generated time evolution. And also for a one-dimensional *Fermi lattice gas* with "quasi-free" state ω invariant w. r. t. the lattice shift θ (cf. e.g. Narnhofer, Pflug and Thirring [39]) $h_\omega(\theta) = s(\omega)$, as shown by Connes, Narnhofer and Thirring [9].

(3.5) The definition and properties of the *joint* quantum dynamical entropy $h_\omega(\sigma_1, \ldots, \sigma_n)$ of n *commuting* quantum transformations $(\sigma_1, \ldots, \sigma_n) = \vec{\sigma}$ (symbolic notation) were extended from (2.5) by Hudetz [28] (following the classical route of Conze, [33]); and the author generalized the quantum *inequality* of (3.4) to n-dimensional (cubic) quantum spin lattice systems with the lattice translations $\sigma_1, \ldots, \sigma_n$ and $\vec{\sigma}$-invariant state ω: $h_\omega(\vec{\sigma}) \leq s(\omega)$, with the quantum entropy density of ω on the lattice (cf. Bratteli and Robinson [41]). Again, $h_\omega(\vec{\sigma}) = s(\omega)$ for an n-dimensional Fermi lattice gas with $\vec{\sigma}$-invariant "quasi-free" state ω (cf. Hudetz [3]).

(3.6) Narnhofer and Thirring [19] performed the *continuum limit* for the one-dimensional Fermi lattice gas as in (3.4) above, leading to the one-dimensional "ideal" Fermi gas with translationally invariant, quasi-free equilibrium state ω, which is determined by the (two-point function) number density distribution in momentum space: $n_\omega(k) = (1 + \exp[\beta(\epsilon(k) - \mu)])^{-1}$, where β (times Boltzmann's constant) is the inverse temperature, k is momentum ("wave number" times Planck's reduced constant $\hbar$), $\epsilon(k)$ is the one-particle energy-momentum spectrum, and μ is a "chemical potential" of the grand canonical equilibrium state ω (cf. Bratteli and Robinson, [41]). With the "fermionic" entropy functional $s_F(n) = -n\log n - (1-n)\log(1-n)$ of $n \in [0,1]$, Narnhofer and Thirring [19] expressed the entropy of the unit space translation σ as:

$$h_\omega(\sigma) = \int_{-\infty}^{\infty} \frac{dk}{2\pi\hbar}\, s_F(n_\omega(k)).$$

As mentioned in the introduction, the derivation of this (physically *correct*) formula involved a certain "physical" discontinuity w. r. t. the "mathematical" phase before, which is still present by now (cf. Størmer and Voiculescu, [20]) but will presumably disappear in the future. And for the one-particle implemented ("quasi-free") unit time evolution τ given by $\epsilon(k)$ as one-particle Hamiltonian in momentum representation, the *dynamical* entropy of the "ideal" Fermi gas was deduced (by conjugacy invariance) from $h_\omega(\sigma)$:

$$h_\omega(\tau) = \int_{-\infty}^{\infty} \frac{dk}{2\pi\hbar} \left|\frac{d\epsilon(k)}{dk}\right| s_F(n_\omega(k)).$$

(3.7) Gaspard [29] performed for both $h_\omega(\sigma)$ and $h_\omega(\tau)$ the *high temperature – dilute gas* limit, where the mean *particle density* $\rho = (2\pi\hbar)^{-1}\int n_\omega(k)dk$ tends to zero such that the "fugacity" is approximately $\exp(\beta\mu) \approx 2\pi\hbar\rho(\beta/2\pi m)^{1/2}$, cf. Bratteli and Robinson [41]. Then, for the "classical" free one–particle time evolution given by $\epsilon(k) = k^2/2m$, he gets approximately (to first order in ρ):

$$\begin{aligned} h_\omega(\sigma) &\approx \rho\log\frac{(2\pi m)^{1/2}}{2\pi\hbar\rho\beta^{1/2}} + \frac{3\rho}{2}, \\ h_\omega(\tau) &\approx \frac{2\rho}{(2\pi m\beta)^{1/2}}\log\frac{(2\pi m)^{1/2}}{2\pi\hbar\rho\beta^{1/2}} + \frac{4\rho}{(2\pi m\beta)^{1/2}}. \end{aligned}$$

Note that, with these approximations, exact "correspondence" to (2.7) is established by $\Delta x\Delta p = 2\pi\hbar \equiv h$, illustrating the general "heuristic" statement in (3.3) above. But in contradistinction to the classical situation (2.7), $h_\omega(\sigma) < \infty$ resp. $h_\omega(\tau) < \infty$ cannot be exceeded any more in the quantum case. More generally, it can be shown for the n–dimensional continuous Fermi gas with the unit space translations $(\sigma_1, \ldots, \sigma_n) = \vec{\sigma}$ in the coordinate directions and with *any* (not necessarily quasi–free) $\vec{\sigma}$–invariant state ω with *finite* particle density ρ, that $h_\omega(\vec{\sigma}) < \infty$ (using the results of (3.5) above; cf. Hudetz [3, 28]).

(3.8) But whereas for the Fermi gas in $n = 1$ dimension (as in (3.6) above) also $h_\omega(\tau) < \infty$ for any physically reasonable quasi–energy $\epsilon(k)$, we get $h_\omega(\tau) = \infty$ for $n > 1$, as also classically in (2.6), at least with one–particle implemented unit time evolution τ and quasi–free equilibrium state ω. This (at first sight surprising) result follows (see below) from the *scaling* of the multi–entropy $h_\omega(\theta_1, \ldots, \theta_n)$ as cited in (3.5), which implies the following (also surprising) property: If $h_\omega(\theta_{i(1)}, \ldots, \theta_{i(m)}) < \infty$ for $0 < m < n$ (with $i(j) \neq i(k)$), then $h_\omega(\theta_1, \ldots, \theta_n) = 0$ (Hudetz [28]).

From this we first have to draw the following "no–go" conclusion (Hudetz [3, 28]): For an n–dimensional continuous Fermi gas with *any* translationally invariant (mutual) interaction such that the (unit) time evolution τ exists, with the unit coordinate translations $(\sigma_1, \ldots, \sigma_n) = \vec{\sigma}$ and $\vec{\sigma}$–invariant equilibrium state ω (with finite particle density), because of $h_\omega(\vec{\sigma}) < \infty$ from (3.7) together with the scaling property, the space–time entropy *vanishes*: $h_\omega(\vec{\sigma}, \tau) = 0$, even if $h_\omega(\tau) = \infty$ for $n > 1$ as announced above. For these infinite quantum systems of *interacting* particles in $n > 1$ dimensions (physically $n = 3$), the space–time entropy thus can *not* distinguish between different time evolutions τ (as it *does* in the classical case (2.8)): $h_\omega(\tau) \equiv \infty$ but $h_\omega(\tau, \vec{\sigma}) \equiv 0$!

But the above scaling property can be "circumvented" by omitting one of the coordinate translations $\vec{\sigma}$ (without loss of generality, σ_1) such that the "time–space" entropy $h_\omega(\tau, \sigma_2, \ldots, \sigma_n)$ can become *positive* and non–trivial: Again for the one–particle implemented (*non*–interacting) unit time evolution τ of the "ideal" Fermi gas (for quasi–energy $\epsilon(\vec{k})$) with quasi–free equilibrium state ω determined by a density–momentum distribution

$n_\omega(\vec{k})$ as for $n = 1$ in (3.6), the following expression was derived by Hudetz [3] as for $n = 1$ by Narnhofer and Thirring [19]:

$$h_\omega(\tau, \sigma_2, \ldots, \sigma_n) = \int_{\mathbb{R}^n} \frac{d^n k}{(2\pi\hbar)^n} \left| \frac{\partial \epsilon(\vec{k})}{\partial k_1} \right| s_F(n_\omega(\vec{k})).$$

This "time–space" entropy is *finite* (for reasonable $\epsilon(\vec{k})$) and of course *positive*, from which by the scaling property it follows that $h_\omega(\tau) = \infty$ as announced above.

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