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THE JORDAN STRUCTURE OF LIE AND KAC-MOODY ALGEBRAS*

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ABSTRACT

In this paper we establish a precise relation between the structures of Lie and Jordan algebras by presenting a method of constructing one type of algebra from the other. Our method differs in some aspects of the Tits construction and Jordan pairs. The examples of the Lie algebras associated to simple Jordan algebras $M_m^{(n)}$ and Clifford algebras are discussed in detail. We believe our approach will shed light on the role of the realizations of Jordan algebras through some types of Fermi fields used in the construction of Kac-Moody and Virasoro algebras as well as its relevance in the study of some aspects of conformal fields theories.

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1. INTRODUCTION

It has become clear in the past few years that a semi direct product of Kac-Moody (K-M) and Virasoro algebras plays a crucial role in underlying the algebraic structure of physical systems possessing conformal invariance in two dimensions[1]. In fact, the K-M algebras can be thought of having a more fundamental structure since from the Sugawara construction[2], the Virasoro algebra can be constructed out of K-M generators. The Virasoro generators obtained in this manner depend upon the representation of the K-M algebra appearing in the Sugawara form, and hence upon the value of its central element K (see ref. 3 for a review).

It is a well known fact that the representations of K-M algebras can be constructed in terms of quantum fields[3]. In particular, the quark model construction provides K-M currents bilinear in Fermi fields, i.e.

$$T^a(z) = (i/2)H^\alpha(z)\, M^a{}_{\alpha\beta}\, H^\beta(z) \qquad (1.1)$$

where M^a are real antisymmetric matrices constituting a finite dimensional representation of a compact Lie algebra g. $H^\alpha(z)$, α = 1, 2 ...dim M are real independent Fermi fields defined on a two dimensional space-time parametrized by the complex variable z. The Laurent coefficients of (1.1) are the generators of the affine Kac-Moody algebra $\hat{g}$ with central element given by the Dynkin index of the finite dimensional representation[4] X_M as

$$K\delta^{ab} = (1/2)X_H \psi^2 \delta^{ab} = -(1/2)\mathrm{Tr}(H^a H^b) \qquad (1.2)$$

where ψ is the highest root of g.

We have a particular interest in studying representations of K-M algebras for which the quantity $X = 2K/\psi^2$, called the level, takes unit values. These are the representations admiting conformal embeddings[5]. It has been shown that for g classical, i.e. SO(n), SU(n) and SP(n), level 1 representations can be obtained with H^a in the defining representation of g [6]. For the exceptional Lie algebras, F_4, E_6, E_7 and E_8 however, there is no matrix representation yielding $X = 1$.

Although there are no such matrix representations for the exceptional algebras (apart from G_2), level 1 representations still can be constructed bilinearly in Fermi fields using vertex operators[7,8]. These fields however, are no longer independent in the sense that their operator product expansion is given by [9]

$$\psi^\alpha(Z)\,\psi^\beta(\xi) = \frac{Z^{1/2}}{(Z-\xi)^{1/2}}\,\psi^\gamma(Z) + \text{reg. terms} \qquad (1.3.a)$$

$|Z| > |\xi|$ and $\alpha \neq \beta \neq \gamma$. That contrasts with the operator product expansion for independent fields, i.e.

$$H^\alpha(Z)\,H^\beta(\xi) = \frac{Z}{Z-\xi}\,\delta^{\alpha\beta} + \text{reg. terms} \qquad (1.3.b)$$

$|Z| > |\xi|$.

The object of this paper is to establish the existence of a more fundamental algebraic structure underlying the construction of Lie (Kac-Moody) algebras. This structure is based upon Jordan algebras which unifies (1.3.a) and (1.3.b) as a field theoretic version of a Jordan product as we now explain.

A Jordan algebra [10,11] $\mathfrak{J}$ is an algebra endowed with a symmetric (commutative) product

$$a \circ b = b \circ a, \qquad a,b \in \mathfrak{J} \tag{1.4}$$

satisfying the Jordan identity or power associativity law

$$(a^2 \circ b) \circ a = a^2 \circ (b \circ a) \tag{1.5}$$

A simple example can be seen by the Clifford algebra with generators γ^α, $\alpha = 1,2,\ldots N$ and Jordan product given by the anticommutator,

$$\gamma^\alpha \circ \gamma^\beta \equiv (1/2)\{\gamma^\alpha, \gamma^\beta\} = \delta^{\alpha\beta} 1 \tag{1.6}$$

Notice the resemblance between eq. (1.6) and the operator product expansion (1.3.b).

Further examples of Jordan algebras are provided by nxn hermitean matrices over the real ($\mathbb{R}$), complex ($\mathbb{C}$) and quaternionic numbers ($\mathbb{Q}$) denoted by $M_n^{(1)}$, $M_n^{(2)}$ and $M_n^{(4)}$ respectively. For n = 3 there is an exceptional Jordan algebra $M_3^{(8)}$ consisting of 3x3 hermitean matrices over the octonions ($\mathbb{O}$).

In all these cases the Jordan product is given by one half of the anticommutator. When restricted to off diagonal matrices, the Jordan product resembles the operator product expansion (1.3.a)[9,12,13,14].

The main point of this paper is to establish a precise relation between elements of a Jordan algebra and the generators of a Lie algebra. Relations of this kind have already been discussed in the mathematical literature[15,16] and for some particular cases they can be connected to the results of this paper. However our approach is more fruitfull in the sense that the relation is established in both directions, i.e. one can construct a Jordan algebra from a Lie algebra and vice-versa. In addition, it provides an intimate connection among the properties of the two different algebraic structures. We believe our approach will be useful in understanding the role of Jordan algebras in the construction of representations of Kac-Moody and Virasoro algebras.

This paper is organized as follows. In section 2 we review a few properties of a general Jordan algebra and its algebra of automorphisms. The construction of Lie algebras in terms of Jordan generators is discussed in section 3. We propose a realization of the Jordan product in terms of three subspaces, each isomorphic to the Jordan algebra and related to each other by an automorphism of order 3 (triality). It is shown that these subspaces are Lie algebra valued leading to the construction of a sequence of Lie algebras.

In section 4 we discuss the necessary conditions a Lie algebra L is required in order to possess a Jordan structure. This establishes the construction of Jordan algebras from a given Lie algebra.

The first and most familiar example is given in section 5 where we start from a Clifford algebra generated by gamma matrices and Jordan product given by (1.6). We show that there is a sequence of orthogonal Lie algebras that can be constructed out of these gamma matrices. Conversely, we show that the orthogonal Lie algebras B_n and D_n may be decomposed according to section 4 exhibiting its Jordan structure.

Next in section 6 we discuss the construction of the Freudenthal magic square out of the Jordan algebras $M_3^{(1)}$, $M_3^{(2)}$, $M_3^{(4)}$ and $M_3^{(8)}$ and its relation with dependent fermions. In section 7 we illustrate our construction with the Poincaré algebra which is related to a non semi simple Jordan algebra. Finally in section 8 we conclude and discuss the possible extensions of our arguments to Kac- Moody algebras.

2. PRELIMINARIES

Since the Jordan algebra is a commutative algebra, it follows that

$$(a,b,c) + (c,b,a) = 0 \qquad (2.1)$$

$$(a,b,c) + (c,a,b) + (b,c,a) = 0 \tag{2.2}$$

where

$$(a,b,c) = (a\circ b)\circ c - a\circ(b\circ c) \tag{2.3}$$

is called the associator. When replacing a by $a + \lambda b$ in the Jordan identity (1.5) and collecting the linear terms in the real parameter λ we get $(a^2,b,c) + 2(a\circ c,b,a) = 0$. Now replace a by $a + \lambda d$ to conclude that the Jordan identity is equivalent to

$$(a\circ c,b,d) + (c\circ d,b,a) + (d\circ a,b,c) = 0. \tag{2.4}$$

Let us now define linear operators R_a by

$$R_a(b) = a\circ b = b\circ a \tag{2.5}$$

generating an associative algebra under map composition. As a consequence, the vector space $\mathfrak{L} = R + [R,R]$ is a Lie algebra under the commutator[11]. From (2.5) we then get

$$\mathcal{D}_{a,b}(c) \equiv [R_a,R_b](c) = -(a,c,b) \tag{2.6}$$

The linear operators $\mathcal{D}_{a,b}$ generate a subalgebra $\mathcal{D}$ of $\mathfrak{L}$ which is the Lie algebra of inner derivations of the Jordan algebra $\mathfrak{J}$. In fact it follows from (2.1),(2.2) and (2.4) that $(a,c\circ d,b) = c\circ(a,d,b) + (a,c,b)\circ d$ and therefore

$$\mathcal{D}_{a,b}(c\circ d) = \mathcal{D}_{a,b}(c)\circ d + c\circ\mathcal{D}_{a,b}(d). \tag{2.7}$$

Consequently transformations of the form

$$c \longrightarrow c - \epsilon(a,c,b) + (\epsilon^2/2)(a,(a,c,b),b) + \dots \qquad (2.8)$$

are automorphisms of $\mathfrak{J}$. It has been shown that for a finite dimensional semi-simple Jordan algebra, all derivations have the form (2.6) [17].

From the fact that $\mathcal{D}$ is a derivation yields

$$[\mathcal{D}, R_a] = R_{\mathcal{D}(a)}, \qquad (2.9)$$

we therefore obtain from (2.6) and (2.9) the commutation relations for the Lie algebra $\mathcal{D}$,

$$[\mathcal{D}_{a,b}, \mathcal{D}_{c,d}] = \mathcal{D}_{\mathcal{D}_{a,b}(c),d} + \mathcal{D}_{c,\mathcal{D}_{a,b}(d)} \qquad (2.10)$$

Notice that in terms of the Lie algebra valued generators $\mathcal{D}_{a,b}$ the Jordan identity (2.4) reads.

$$\mathcal{D}_{a\cdot b,c} + \mathcal{D}_{c\cdot a,b} + \mathcal{D}_{b\cdot c,a} = 0. \qquad (2.11)$$

3. CONSTRUCTION OF LIE ALGEBRAS FROM JORDAN ALGEBRAS

Let $\mathfrak{J}$ be an arbitrary Jordan algebra and L a vector space admiting a decomposition into

$$L = T^1 + T^2 + T^3 + D \qquad (3.1)$$

where T^i, $i = 1,2,3$ are three vector spaces isomorphic to $\mathfrak{J}$, i.e. there exist a one to one map between $\mathfrak{J}$ and T^i such that for any $a \in \mathfrak{J}$ there is a corresponding $T^i(a)$, $i = 1,2,3$ satisfying

$$T^i(a + \lambda b) = T^i(a) + \lambda T^i(b) \qquad (3.2)$$

where λ is a c-number.

Let us introduce a bilinear, antisymmetric operation defined in L through the relations

$$[T^i(a), T^j(b)] = i\epsilon^{ijk} T^k(a \cdot b) + \delta^{ij} D_{a,b} \qquad (3.3.a)$$

$$[D_{a,b}, T^i(c)] = T^i(\mathcal{D}_{ab}(c)) = -T^i((a,c,b)) \qquad (3.3.b)$$

$$[D_{a,b}, D_{c,d}] = D_{\mathcal{D}_{a,b}(c),d} + D_{c,\mathcal{D}_{a,b}(d)} \qquad (3.3.c)$$

where $a \cdot b$ denotes the Jordan product between a and b and $D_{a,b}$ are elements of the subspace D constituting a realization of the derivation algebra $\mathcal{D}$ (see (2.10)). In fact, relation (3.3.a) defines $D_{a,b}$ and we assume it spans D. As a consequence of (3.2) and the fact that the bracket operation is antisymmetric yields that $D_{a,b}$ is bilinear and antisymmetric in elements of the Jordan algebra, i.e.

$$D_{a+\lambda b,c} = D_{a,c} + \lambda D_{b,c} \qquad (3.4.a)$$

$$D_{a,b} = -D_{b,a}. \qquad (3.4.b)$$

The algebra given by the relations (3.3) have already been constructed in connection with the Jordan algebras $H_n^{(1)}$, $H_n^{(2)}$, $H_n^{(4)}$ and $H_n^{(8)}$ and was shown to generate the first three lines of the Freudenthal magic square [14]. We shall now show that the vector space L endowed with the operation (3.3) is a Lie algebra for any given Jordan algebra $\mathfrak{J}$. For any three elements of L, consider the jacobian

$$j(\ell_1,\ell_2,\ell_3) = [[\ell_1,\ell_2],\ell_3] + [[\ell_3,\ell_1],\ell_2] + [[\ell_2,\ell_3],\ell_1] \qquad (3.5)$$

Using (3.2), (3.3) and (3.4) we get

$$j(T^i(a),T^j(b),T^k(c)) = i\varepsilon^{ijk}(D_{a\bullet b,c} + D_{c\bullet a,b} + D_{b\bullet c,a}) \qquad (3.6.a)$$

$$j(T^i(a),T^j(b),D_{c,d}) = -i\varepsilon^{ijk}\,T^k(\,\mathcal{D}_{c,d}(a\bullet b) - \mathcal{D}_{c,d}(a)\bullet b - a\bullet\mathcal{D}_{c,d}(b)\,) \qquad (3.6.b)$$

$$j(T^i(a),D_{b,c},D_{d,e}) = -T^i\Big([\mathcal{D}_{b,c}\,,\mathcal{D}_{d,e}] - \mathcal{D}_{\mathcal{D}_{b,c}(d),e}(a) - \mathcal{D}_{d,\mathcal{D}_{b,c}(e)}(a)\Big) \qquad (3.6.c)$$

$$J(D_{a,b}, D_{c,d}, D_{e,f}) =$$

$$= -D\Big([\mathcal{D}_{a,b}, \mathcal{D}_{c,d}](c) - \mathcal{D}_{\mathcal{D}_{a,b}(c),d}(e) - \mathcal{D}_{c,\mathcal{D}_{a,b}(d)}(e)\Big),f -$$

$$- D_{e,}\Big([\mathcal{D}_{a,b}, \mathcal{D}_{c,d}](f) - \mathcal{D}_{\mathcal{D}_{a,b}(c),d}(f) - \mathcal{D}_{c,\mathcal{D}_{a,b}(d)}(f)\Big)$$

(3.6.d)

From eqs.(3.2) and (3.4) follows that $\dot{T}(o) = D_{o,a} = 0$, hence the jacobians (3.6.b), (3.6.c) and (3.6.d) vanish as a consequence of (2.7) and (2.10). Although the derivation algebra $\mathcal{D}$ satisfy (2.11), the elements $D_{a,b} \in D$ does not necessarily do. However $\Delta(a,b,c) = D_{a\circ b,c} + D_{c\circ a,b} + D_{b\circ c,a}$ generate a central element in the sense that

$$[\Delta(a,b,c) , \dot{T}(d)] = 0 \qquad (3.7.a)$$

$$[\Delta(a,b,c), D_{d,e}] = 0. \qquad (3.7.b)$$

Due to the antisymmetry of the Levi-Civita symbol on the r.h.s. of (3.6.a), the subspace $D + T^{i}$ (fixed i) is a Lie algebra even when the quantity $\Delta(a,b,c)$ does not vanish. The subalgebra D is then homomorphic to the derivation algebra $\mathcal{D}$ and $\Delta(a,b,c)$ lies in the kernel of such homomorphism. The vector space L is a Lie algebra only if D and $\mathcal{D}$ are actually isomorphic. Since the subspace Δ generated by $\Delta(a,b,c)$ is an abelian ideal of D, the factor algebra $\bar{D} = D/\Delta$ is isomorphic to $\mathcal{D}$. Consequently the vector space $\bar{L} = T^1 + T^2 + T^3 + \bar{D}$ is always a Lie algebra. The quantity $\Delta(a,b,c)$ is in fact a 3-cocycle.

Notice that the commutation relations given by (3.3) implies the existence of an automorphism τ of order 3 ($\tau^3 = 1$) permuting the three subspaces T^i and leaving D invariant i.e.

$$\tau(T^i(a)) = T^{i+1 \bmod 3}(a) , \qquad \tau(D_{a,b}) = D_{a,b}. \tag{3.8}$$

In the cases where the Jordan algebra has an identity element such automorphism is realized by

$$\tau(\ell) = g_1 g_2 \ell \, g_2^{-1} g_1^{-1}, \quad \ell \in L \tag{3.9}$$

where $g_i = \exp(i\pi T^i(1)/2)$. In addition, the representatives of the identity generate an SU(2) subalgebra commuting with D i.e.

$$[T^i(1), T^j(1)] = i\varepsilon^{ijk} T^k(1) \tag{3.10}$$

$$[D_{a,b}, T^i(1)] = 0 \tag{3.11}$$

since $(a,1,b) = 0$ for any $a,b \in \mathfrak{J}$. Notice that this resembles the Tits construction[15] in the sense that the derivation algebra for the quaternions, namely SU(2), commutes with the Jordan derivation algebra. However, as it will become clear in the examples, the SU(2) subalgebra (3.10) is not necessarily related to quaternions.

The structures of the two algebras $\mathfrak{J}$ and L are intimately related through eqs.(3.3). Suppose the Jordan algebra $\mathfrak{J}$ possess an ideal I with elements denoted by x,y,z,... . The subspace K $\subset$ L spanned by $T^i(x)$, $i=1,2,3$ and $D_{x,a}$ with $x \in I$, $a \in \mathfrak{J}$ is then an ideal of L as can be easily verified from eq. (3.3). Therefore, L is simple only if $\mathfrak{J}$ is so. If I is an abelian ideal (i.e. $x^2 = 0$ for any $x \in I$), then K is abelian only if $D_{x,y}$ vanish for all $x,y \in I$. However, since we have shown that if L is a Lie algebra, D and $\mathcal{D}$ are isomorphic, we get using $(x,a,y) = 0$, $a \in \mathfrak{J}$, that $\mathcal{D}_{x,y} = 0$ and so is $D_{x,y}$. Therefore, if L is a Lie algebra, K is an abelian ideal of L whenever I is so of $\mathfrak{J}$. Hence, the Lie algebra L is semi simple only if the underlying Jordan algebra is.

The eqs. (3.3) also allow us to relate invariant bilinear forms defined on $\mathfrak{J}$ and on L. Let $Tr(\ell\ \ell')$ denote a bilinear invariant form on the algebra L, i.e.

$$Tr([\ell,\ell']\ \ell'') + Tr(\ell'[\ell,\ell'']) = 0 \qquad (3.12)$$

As a consequence of (3.3) and (3.12), $Tr\ (T^i(a)T^i(b)) = Tr(\tau(T^i(a))\tau(T^i(b)))$ where τ is as in eq. (3.8). We can therefore define a bilinear form on $\mathfrak{J}$ by restricting Tr to the subspace T^i, i.e.

$$\langle a,b\rangle = Tr(T^i(a)\ T^i(b)) \qquad (3.13)$$

which is independent of i. Using (3.12) for $T^i(a)$, $T^j(b)$ and $T^k(c)$ with $i\neq j\neq k$ we get

$$\langle a\circ c, b\rangle - \langle a, b\circ c\rangle = 0 \qquad (3.14)$$

So $\langle a,b\rangle$ is an invariant bilinear form in the Jordan algebra $\mathfrak{J}$[17].

4. CONSTRUCTION OF JORDAN ALGEBRAS FROM LIE ALGEBRAS

In this section we discuss the requirements a Lie algebra has to fulfill in order to possess a Jordan structure of the type (3.3). The construction presented here has some points in common with the usual construction of Jordan algebras using Jordan pairs[15,16,18,19]. However those constructions are based on triple commutators so that the Jordan and Lie structures are not so closely related as ours.

Let L be a Lie algebra graded as

$$L = L_{-1} + L_0 + L_1 \qquad (4.1)$$

with

$$[L_m, L_n] \subset L_{m+n} \quad , \quad m,n = 0,\pm1 \qquad (4.2)$$

It then follows that L_1 and L_{-1} are abelian. In general such decomposition is obtained by a U_1 generator Q

$$[Q, L_m] = mL_m \qquad (4.3)$$

which does not necessarily belong to L. Suppose that L possess an involutive automorphism σ, ($\sigma^2 = 1$) providing a one to one map between the abelian subspaces L_1 and L_{-1}, i.e.

$$\sigma(L_1) = L_{-1} \qquad (4.4)$$

This involution also decomposes the subalgebra L_0 into even and odd subspaces, i.e.

$$L_0 = D + T^3, \quad \sigma(D) = D, \quad \sigma(T^3) = -T^3 \qquad (4.5)$$

In the cases where $Q \in L$, eqs. (4.3) and (4.4) imply

$$\sigma(Q) = -Q \qquad (4.6)$$

and therefore $Q \in T^3$.

We denote by T^1 and T^2 the odd and even combinations according to σ of L_1 and L_{-1} respectively, i.e.

$$T^1 = L_1 - \sigma(L_1), \qquad T^2 = L_1 + \sigma(L_1) \qquad (4.7)$$

Hence the Lie algebra L is decomposed into

$$L = T^1 + T^2 + T^3 + D \qquad (4.8)$$

Using (4.2) and the fact that σ is an automorphism one gets

$$[T^i, T^j] \subset T^k \quad (i \neq j \neq k) \qquad (4.9.a)$$

$$[T^i, T^i] \subset D \tag{4.9.b}$$

$$[D, T^i] \subset T^i \tag{4.9.c}$$

$$[D, D] \subset D. \tag{4.9.d}$$

The graded structure (4.1) defines another automorphism in L, namely σ' defined by

$$\sigma'(L_m) = (-1)^m L_m. \tag{4.10}$$

When the integers m are eigenvalues of Q, σ' is realized by $\sigma'(L) = \exp(i\pi Q) L \exp(-i\pi Q)$.

The involutions σ and σ' commute, hence define a third involution, $\sigma'' = \sigma\sigma' = \sigma'\sigma$. The identity map together with the three involutions in fact constitute a $Z_2 \times Z_2$ group. Associated with these involutions there are three symmetric spaces[20]. The algebra of the group of isometries of these spaces are $H = D + T^2$, $H' = D + T^3$ and $H'' = D + T^1$ which are invariant under σ, σ' and σ'' respectively. Notice that T^2 is odd under σ' or σ'', T^3 is odd under σ or σ'' and finally T^1 is odd under σ or σ'. Therefore when dividing those isometry groups by the compact subgroup whose algebra is D, we end up with three new symmetric spaces, namely H/D, H′/D and H′′/D. We should point out that this algebraic structure is direct consequence of the grading (4.1) and the involution (4.4).

Given the grading (4.1) the involution σ satisfying (4.4) is not unique. By varying σ, we vary the subalgebra D and the subspace T^3 defined in (4.5). We are interested in those cases where the involution is such that there exist an automorphism of order 3 , τ ($\tau^3 = 1$) permuting the subspaces T^i, $i = 1,2,3$ and leaving D invariant as in (3.8). The vector spaces T^1 and T^2 are already isomorphic due to (4.7) and the existence of τ implies an isomorphism among T^1, T^2 and T^3. At this point we introduce a vector space $\mathfrak{J}$ with elements denoted by $a, b, c, \ldots$, which is isomorphic to T^i, $i = 1,2,3$. We shall denote a representative of an element $a \in \mathfrak{J}$ in T^i by $T^i(a)$, $i=1,2,3$ which are related by τ i.e., $\tau(T^i(a)) = T^{i+1 \bmod 3}(a)$. Let us introduce a product law in $\mathfrak{J}$, denoted by $a{\bullet}b$, as follows: take the Lie bracket between the representative of a in T^1 and the representative of b in T^2. As a consequence of (4.9), the result is an element in T^3 which we define to be the representative of the element $a{\bullet}b$, i.e.

$$[T^1(a),\ T^2(b)] \equiv i T^3(a{\bullet}b) \tag{4.11}$$

where the factor i was introduced for convinience. Due to τ, this product can as well be defined as $[T^2(a),\ T^3(b)]$ or $[T^3(a),\ T^1(b)]$. The fact that L_1 and L_{-1} are abelian guarantees the symmetry of the product . Indeed, using (4.7) yields

$$[T^1(a),\ T^2(b)] = [T^1(b),\ T^2(a)]. \tag{4.12}$$

We shall now prove that our product satisfy the Jordan identity and hence defines a Jordan algebra. From (4.11),

$$T^3((a^2 \circ b)\circ a) \;=\; i[[[T^3(a),\ T^1(a)],\ T^3(b)],T^2(a)] \qquad (4.13.a)$$

and

$$T^3((a \circ b)\circ a^2) \;=\; i[[T^2(a),\ T^3(b)],\ [T^3(a),\ T^1(a)]] \qquad (4.13.b)$$

When calculating (4.13), we have to keep in mind that, by the definition (4.11), the result of an operation must be brought (using τ) to one of the subspaces T^1 or T^2 before operating with it again. Using the Jacobi identity, we find

$$T^3((a^2,b,a)) \;=\; i[[[T^3(a),\ T^1(a)],\ T^2(a)],\ T^3(b)] \qquad (4.14)$$

The quantity $M_{312}(a) \;=\; [[T^3(a), T^1(a)],\ T^2(a)]$ is an element of D and being so is invariant under τ, i.e.

$$M_{123} \;=\; \tau(\ M_{312}) \;=\; M_{312} \qquad (4.15.a)$$

and

$$M_{231} \;=\; \tau^2(M_{312}) \;=\; M_{312} \qquad (4.15.b)$$

In addition, from the Jacobi identity,

$$M_{123}(a) \;+\; M_{231}(a) \;+\; M_{312}(a) \;=\; 0. \qquad (4.16)$$

Therefore $M_{123}(a) \;=\; M_{231}(a) \;=\; M_{312}(a) \;=\; 0$ and hence

$$T^3((a^3,b,a)) \;=\; 0 \qquad (4.17)$$

The vector space $\mathfrak{J}$ endowed with the product defined by (4.11) is then a Jordan algebra.

According to (4.9) the Lie bracket between two elements of the same subspace T^i is an element of D. Due to the triality τ, this is independent of i. We then define

$$D_{a,b} \equiv [T^i(a), T^i(b)], \quad i = 1,2,3. \qquad (4.18)$$

Therefore using (4.11) and the triality τ we establish (3.3.a). Now using the definition (4.18) , (3.3.a) and the Jacobi identity we find that $D_{a,b}$ satisfy (3.3b) and (3.3.c) (when choosing a convinient value for i).

Except for the possibility of the existence of elements of D which are not of the form (4.18) these are all the commutation relations for L. In any case, the elements $D_{a,b}$ constitute a subalgebra of D which realizes the derivation algebra of the Jordan algebra $\mathfrak{J}$. We have then shown that any Lie algebra possessing the grading (4.1-4.2) and automorphisms σ and τ defined in (4.4) and (3.8) respectively, has a Jordan structure in the sense of eq. (3.3).

5. CLIFFORD ALGEBRAS AND ORTHOGONAL LIE ALGEBRAS

Having explained in sections 3 and 4 how Lie algebras can be constructed out of Jordan algebras and vice versa, we now consider some examples. The simplest and most familiar example of a simple

Jordan algebra consist of generators γ^α, $\alpha = 1,\ldots,N$ of a Clifford algebra with product given by the anticommutator (1.6). In this case the relations (3.3) read

$$[T^i(\gamma^\alpha),\ T^j(\gamma^\beta)] \ = \ i\epsilon^{ijk}\,\delta^{\alpha\beta}\,T^k(1) \ + \ \delta^{ij}D_{\alpha,\beta} \qquad (5.1.a)$$

$$[T^i(1),\ T^j(\gamma^\alpha)] \ = \ i\epsilon^{ijk}\,T^k(\gamma^\alpha) \qquad (5.1.b)$$

$$[D_{\alpha,\beta},\ T^i(\gamma^\rho)] \ = \ \delta^{\beta,\rho}\,T^i(\gamma^\alpha) \ - \ \delta^{\alpha,\rho}\,T^i(\gamma^\beta) \qquad (5.1.c)$$

$$[D_{\alpha,\beta},\ T^i(1)] \ = \ 0 \qquad (5.1.d)$$

$$[D_{\alpha,\beta},\ D_{\gamma,\delta}] \ = \ \delta^{\beta\gamma}D_{\alpha,\delta} \ - \ \delta^{\alpha\gamma}D_{\beta,\delta} \ + \ \delta^{\beta\delta}D_{\gamma,\alpha} \ - \ \delta^{\alpha\delta}D_{\gamma,\beta} \ . \qquad (5.1.e)$$

where we have used $(\gamma^\alpha,\gamma^\rho,\gamma^\beta) \ = \ \delta^{\alpha\rho}\gamma^\beta - \delta^{\rho\beta}\gamma^\alpha$, and $\mathcal{D}_{\alpha\beta}(1) \ = \ 0$. The element $D_{1,\alpha}$ does not appear on the r.h.s. of (5.1.b) since we have assumed the ideal Δ has been divided off from D (see comments following eq. (3.7)). According to the arguments of section 3, the vector space $L = T^1 + T^2 + T^3 + D$ endowed with the operations (5.1) is a Lie algebra.

The Lie algebra D is easily recognizable since (5.1.e) defines the special orthogonal algebra SO(N). Notice that eq. (5.1.a) for $i = j$ resembles the construction of SO(N) generators bilinearly in gamma matrices.

Denoting $D_{N+1,\alpha} = iT^1(\gamma^\alpha)$ we see that eqs. (5.1.a) and (5.1.c) for $i = j = 1$ are written in the same form as (5.1.e). Therefore the subspace $D + T^1(\gamma^\alpha)$, $\alpha = 1,\ldots,N$ is then the Lie algebra SO(N+1). Adding the U_1 generator $T^1(1)$ we get the non semisimple Lie algebra O(N+1). Analogously denoting $D_{N+2,\alpha} = iT^2(\gamma^\alpha)$ and $D_{N+1,N+2} = iT^3(1)$ we can easily check that the subspace $D + T^1(\gamma^\alpha) + T^2(\gamma^\alpha) + T^3(.)$ constitute the Lie algebra SO(N+2). Finally denoting $D_{N+3,\alpha} = iT^3(\gamma^\alpha)$, $D_{N+2,N+3} = iT^1(1)$ and $D_{N+1,N+3} = -iT^2(1)$ we find that the L = SO(N+3). Therefore the construction explained in section 3 provide, for the case of the Clifford algebra a series of four orthogonal Lie algebras.

We now want to apply the construction of section 4 to the orthogonal Lie algebras and obtain an underlying Clifford algebra (1.6). A pratical procedure in decomposing a simple Lie algebra L into the form (4.1) is by taking the U_1 generator Q in (4.3) to have the form

$$Q = 2\lambda_a.H/\alpha_a^2 \tag{5.2}$$

where α_a ($a = 1,\ldots,$rank L) is a simple root of L and λ_a its corresponding fundamental weight. They satisfy $2\alpha_a.\lambda_b/\alpha_a^2 = \delta^{ab}$. The generators H_i ($i = 1,\ldots,$rank L) constitute the Weyl-Cartan basis for the Cartan subalgebra of L (they are related to the Chevalley basis by $H_a = 2\alpha_a.H/\alpha_a^2$ (see appendix)). Obviously they have zero grading w.r.t. Q,

$$[Q, H_i] = 0, \tag{5.3.a}$$

$i=1,\ldots,\text{rank } L$. The step operators satisfy

$$[Q, E_\alpha] = (2\lambda_a.\alpha / \alpha_a^2)\, E_\alpha = n_a E_\alpha \qquad (5.3.b)$$

where n_a are the integers in the expansion $\alpha = n_a \alpha_a$. The highest grade of Q is given by the coefficient m_a of α_a in the expansion of the highest root ψ ($\psi = m_a \alpha_a$). Therefore, Q has integer eigenvalues varying from $-m_a$ to m_a and hence it decomposes L into $(2m_a+1)$ subspaces. If we now want to decompose L as in (4.1), we have to choose λ_a such that $m_a = 1$. In table 1 we give the integers m_a for the orthogonal algebras SO(2r) and SO(2r+1). In order to construct the Clifford algebra (1.6), for both cases we have to choose Q in the direction of λ_1. The other two possibilities arising when L = SO(2r) provide the Jordan algebras $M_N^{(4)}$, for r = 2N. This exemple will be discussed in the next section.

The subalgebra L_o is therefore generated by all Cartan subalgebra generators H_i together with the step operators E_α for roots α not containing α_1 in their expansion in terms of simple roots. The Dynkin diagram for L_o is then obtained by deleting the point corresponding to α_1 in the Dynkin diagram of SO(N) (N=2r+1 or 2r, see fig. 1). Consequently the orthogonal algebras are decomposed as

$$SO(N) = L_1 + L_{-1} + SO(N-2) + U(1)_a \qquad (5.4)$$

where the U(1) factor is generated by Q. The subspace L_1 (L_{-1}) is generated by all positive (negative) step operators for the roots containing α_1 once in its expansion. In terms of an orthonormal basis e_i ($i= 1,2,\ldots,r$) the roots of the orthogonal algebras are[20]

$$\text{roots of } SO(2r+1) = \{\pm e_i \pm e_j,\ \pm e_i,\ i,j = 1,2,\ldots,r\} \qquad (5.5a)$$

$$\text{roots of } SO(2r) = \{\pm e_i \pm e_j,\ \ i,j = 1,2,\ldots,r\} \qquad (5.5b)$$

In both cases the fundamental weight λ_1 is equal to e_1, and since $\alpha_1^2=2$, we have[21]

$$Q = e_1 . H \qquad (5.6)$$

Therefore the subspace L_1 is given by[(*)1]

$$L_1^{so(2r+1)} = \{E_{e_1}\ ,\ E_{e_1 \pm e_l}\ ,\ l= 2,3,\ldots,r\ \} \qquad (5.7a)$$

$$L_1^{so(2r)} = \{E_{e_1 \pm e_l}\ ,\ l= 2,3,\ldots,r\ \}. \qquad (5.7b)$$

In both cases the subspace L_{-1} is generated by their corresponding

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(*) Formally, these are generators of the complex algebras B_r and D_r and not of the compact real forms SO(2r + 1) and SO(2r). The decomposition (5.4) is made, in fact, in the complexification of the orthogonal Lie algebras.

negative root step operators.

The involution (4.4) in the case of the orthogonal algebras is an inner automorphism associated to the product of the Weyl reflections through the roots e_s+e_r and e_s-e_r. It is given by

$$\sigma(T) = e^{i\pi(S_2(e_s+e_r) + S_2(e_s-e_r))}\, T\, e^{-i\pi(S_2(e_s+e_r) + S_2(e_s-e_r))} \tag{5.8}$$

where $S_2(\alpha)$ is defined in (5.15).

One can easily check that

$$\sigma(Q) = -Q. \tag{5.9}$$

It then follows from (4.3) and the fact that σ is an automorphism that $\sigma(L_1)$ has grade -1. Since σ is of order two, it satisfies (4.4).

Using relations (1.3-5) and (1.8) in the appendix one gets from (5.8)

$$\sigma(e_r.H) = -e_r.H \tag{5.10a}$$

$$\sigma(E_{e_s+e_r}) = -E_{-(e_s+e_r)} \tag{5.10b}$$

$$\sigma(E_{e_s-e_r}) = -E_{-(e_s-e_r)} \tag{5.10c}$$

$$\sigma(E_{e_r}) = \epsilon(e_1+e_r,-e_1)\epsilon(e_1-e_r,e_r)E_{-e_r} \quad (5.10e)$$

$$\sigma(E_{e_1\pm e_l}) = \epsilon(e_1\pm e_l,-e_1-e_r)\epsilon(-e_r\pm e_l,-e_1+e_r)E_{-e_1\pm e_l} \quad (5.10f)$$

$$\sigma(E_{e_r\pm e_l}) = \epsilon(e_r\pm e_l,-e_1-e_r)\epsilon(e_1-e_r,-e_1\pm e_l)E_{-e_r\pm e_l} \quad (5.10g)$$

where $l=2,3,\ldots r-1$. The generators $e_l.H$, $E_{\pm e_l}$ and $E_{\pm e_l \pm e_m}$, $l,m=2,3,\ldots,r-1$, are invariant under σ.

Therefore, according to (4.5), the subalgebra $L_o = SO(N-2)\times U(1)$ split into $D + T_s$. For $N = 2r$, the subspaces are giving by

$$D^{so(2r)} = \{e_l.H;\ E_{\pm e_l \pm e_m};\ E_{e_r\pm e_l} + \sigma(E_{e_r\pm e_l});\ l,m=2,3,\ldots,r-1\} \quad (5.11a)$$

$$T_s^{so(2r)}=\{Q,\ e_r.H,\ E_{e_r\pm e_l} - \sigma(E_{e_r\pm e_l});\ l=2,3,\ldots,r-1\} \quad (5.11b)$$

and for the case $N = 2r+1$ the subspaces are

$$D^{so(2r+1)}= \{D^{so(2r)},\ E_{e_r}+\sigma(E_{e_r}),\ E_{\pm e_l}, l=2,3,\ldots,r-1\} \quad (5.12a)$$

$$T_s^{so(2r+1)}= \{T_s^{so(2r)},\ E_{e_r}-\sigma(E_{e_r})\} \quad (5.12b)$$

Notice $D^{so(2r)}$ is the algebra $SO(2r-3)$ since the generators $E_{e_r \pm e_l} + \sigma(E_{e_r \pm e_l})$ are the step operators for the short roots $\pm e_l$. On the other hand $D^{so(2r+1)}$ is the algebra SO(2r-2) since $E_{e_r} + \sigma(E_{e_r})$ together with $e_l . H$ generate the Cartan subalgebra, and suitable linear combinations of the generators $E_{\pm e_l}$ and $E_{e_r \pm e_l} + \sigma(E_{e_r \pm e_l})$ are the step operators for roots $\pm e_r \pm e_l$. In both cases the subspace T_3 has the same dimension as L_1 in (5.7).

Consider the inner automorphism

$$\tau(T) = g(e_1+e_r)g(e_1-e_r)\; T\; g^{-1}(e_1-e_r)g^{-1}(e_1+e_r) \tag{5.13}$$

where

$$g(\alpha) = \exp(-i\frac{\pi}{2} S_1(\alpha)) \exp(-i\frac{\pi}{2} S_2(\alpha)) \tag{5.14}$$

$$S_1(\alpha) = \frac{E_\alpha + E_{-\alpha}}{2} , \quad S_2(\alpha) = \frac{E_\alpha - E_{-\alpha}}{2i} . \tag{5.15}$$

Using the relations of the appendix, one can check that τ ciclically permutes the subspaces T_i ($i=1,2,3$), i.e.

$$\tau[(1 - \sigma)E_{e_1}] = -i(1 + \sigma)E_{e_1} \tag{5.16a}$$

$$\tau[(1 + \sigma)E_{e_1}] = i\varepsilon(e_r, e_1-e_r)\; (1 - \sigma)E_{e_r} \tag{5.16b}$$

$$\tau[(1 - \sigma)E_{e_r}] = \varepsilon(e_r, e_1- e_r)\; (1 - \sigma)E_{e_1} \tag{5.16c}$$

$$\tau[(1-\sigma)E_{e_1\pm e_r}] = -i(1+\sigma)E_{e_1\pm e_r} \qquad (5.17a)$$

$$\tau[(1+\sigma)E_{e_1\pm e_r}] = i(e_1\pm e_r).H \qquad (5.17b)$$

$$\tau((e_1\pm e_r).H) = (1-\sigma)E_{e_1\pm e_r} \qquad (5.17c)$$

$$\tau[(1-\sigma)E_{e_1\pm e_l}] = -i(1+\sigma)E_{e_1\pm e_l} \qquad (5.18a)$$

$$\tau[(1+\sigma)E_{e_1\pm e_l}] = i\,\epsilon(e_1\pm e_l,-e_1+e_r)\ (1-\sigma)E_{e_r\pm e_l} \qquad (5.18b)$$

$$\tau[(1-\sigma)E_{e_r\pm e_l}] = \epsilon(e_1\pm e_l,-e_1+e_r)\ (1-\sigma)E_{e_1\pm e_l} \qquad (5.18c)$$

where $l=2,3,...,r-1$ and τ leaves the elements of $D^{so(2r)}$ and $D^{so(2r+1)}$ invariant. Therefore this automorphism satisfies the requirements of section 4 and :cording to the arguments given there the Lie algebras SO(2r) and SO(2r+1) possess a Jordan structure in the sense of (3.3). The Jordan algebra obtained, from the product law (4.11) is indeed the Clifford algebra (1.6). The representatives of the identity and the γ-matrices in the subspace T_1 are given by

$$T^1(1) = \frac{1}{2}\ (1-\sigma)\ (E_{e_1+e_r} + E_{e_1-e_r}) \qquad (5.19a)$$

$$T^1(\gamma_{2r-3}) = \frac{1}{2}\,(1-\sigma)\,(E_{e_1+e_r} - E_{e_1-e_r}) \qquad (5.19b)$$

$$T^1(\gamma_{2l-3}) = \frac{1}{2}\,(1-\sigma)\,(E_{e_1+e_l} - \eta_l E_{e_1-e_l}) \qquad (5.19c)$$

$$T^1(\gamma_{2l-2}) = \frac{-i}{2}\,(1-\sigma)\,(E_{e_1+e_l} + \eta_l E_{e_1-e_l}) \qquad (5.19d)$$

where $l=2,3,\ldots,r-1$, and η_l is the product of cocycles appearing in (5.10f), i.e.

$$\eta_l = \varepsilon(e_1+e_l,-e_1-e_r)\varepsilon(-e_r+e_l,-e_1+e_r) =$$
$$= \varepsilon(e_1-e_l,-e_1-e_r)\varepsilon(-e_r-e_l,-e_1+e_r). \qquad (5.20)$$

The equality between the product of those cocycles can be easily checked using the relations of the appendix.

For the case of SO(2r+1) there is an additional γ-matrix representated in T^1 by

$$T^1(\gamma_{2r-2}) = \Big[\varepsilon(e_1+e_r,-e_1)\varepsilon(e_r,e_1-e_r)\Big]^{1/2}(E_{e_1} - \sigma(E_{e_1}))/2 \qquad (5.21)$$

The representatives in the subspaces T^2 and T^3 are obtained by applying τ and τ^2 respectively on the generators above. That this choice of representatives does provide the algebra (1.6) can be easily verified by checking directly into (5.1) and using the relations (1.3-5) and (1.8).

The step operators satisfy the hermiticity condition $E_\alpha^+ = E_{-\alpha}$. Therefore, from (5.10d) one observes that the square root of the product of cocycles in (5.21) guarantees the hermiticity of $T^1(\gamma_{2r-2})$. The generators introduced in (5.19) are also hermitian. Since the triality τ, defined in (5.13), is a unitary transformation it follows that the representatives in the subspace T^2 and T^3 are also hermitian operators. Consequently the relations (3.3) for these examples are the commutation relations for the compact simple Lie algebra SO(2r) and SO(2r+1). This fact accounts for the Euclidean metric of the Clifford algebras obtained here (see (1.6)). If one wants a Clifford algebra with a non Euclidean metric one has to work with non compact real forms of the Lie algebras B_r and D_r.

6. JORDAN ALGEBRAS AND THE FREUDENTHAL MAGIC SQUARE.

In ref. (8) Goddard et. al. proposed a construction of the Freudenthal magic square based on their work on vertex operator representations for non simply laced Kac-Moody algebras. An important feature of the construction is the concept of Z_2 lattice pairs[8,9]. Let Λ and Λ_0 be two lattices such that $\sqrt{2}\Lambda$ and Λ_0 are even integral lattices (but Λ not necessarily integral). If $\Lambda \subset \Lambda_0^*$ (the dual lattice of Λ_0) and Λ/Λ_0 is isomorphic to the group $(Z_2)^m$ for some m, then (Λ, Λ_0) is a Z_2 lattice pair The root lattices of Lie algebras are prototypes of such pairs. Indeed two examples are given by:

i)Λ being the root lattice of a non simply laced Lie algebra g (excluding G_2, i.e. B_r , C_r and F_4) and Λ_o the root lattice of its simply laced subalgebra g_L generated by the Cartan subalgebra and the step operators for the long roots of g.

ii)Λ being the root lattice of a simply laced algebra, i.e. A_r, D_r, E_6, E_7, and E_8 (with roots taken to be short, i.e. of square length one) and $\Lambda_o = 2\Lambda$.

The effect of this structure on the root system Φ of a Lie algebra is to split the short roots into disjoint orbits Ω_i ($i=1,\dots,2^m-1$) which are defined to be the set of unit length points in the cosets $\Lambda_i \subset \Lambda/\Lambda_o$. In case i), Ω_i are the orbits of the Weyl group of g_L. The set of long roots $\Phi_L \subset \Phi$ is made of the points of square length 2 in Λ_o. In case ii) Φ_L obviously vanish. The short roots in a given orbit are either orthogonal or antiparallel and therefore the sum of two of them in the same orbit either vanish or is a long root. On the other hand two short roots e_i and e_j in distinct orbits (but in cosets belonging to the same $(Z_2)^2$ subgroup of Λ/Λ_o), satisfy $e_i.e_j = \pm 1/2$ (when normalizing $e_i^2=1$) and consequently their sum can only be a root if they make an angle $2\pi/3$. In such case the resulting root is short and belongs to a third orbit distinct to those of e_i and e_j (but inside the same $(Z_2)^2$ subgroup).

Another important concept is that of matching systems. Two Z_2 lattice pairs are said to match if they share a common value of m. In such cases the cosets Λ_i and hence the orbits Ω_i can be put into a one to one correspondence respecting the $(Z_2)^m$ structure. Given two Lie algebras g_1 and g_2 such that their corresponding Z_2

lattice pairs match, we can construct a simply laced Lie algebra g_{12} in the following way: The root space of g_{12} is defined to be the direct sum of the root spaces of g_1 and g_2. The roots of g_{12} consist of the long roots of g_1 and g_2 together with compositions of the form $e+e'$ where e and e' are short roots of g_1 and g_2 respectively, belonging to corresponding orbits (see ref. 8 and 9 for more details).

The algebras SU(n) (with roots taken to be short) and SP(n) match since both have n(n-1)/2 orbits. For the case n=3, the algebra F_4 also match with SU(3) and SP(3) since they all have three orbits. In ref. (8) and (9), it was shown that the matching of these algebras provide the simply laced algebras appearing in the third line of the Freudenthal magic square (see table 2). For the algebras SU(3), SP(3) and F_4 their orbits contain 2, 4 and 8 unit length vectors each respectively and they are related to division algebras of the complex numbers ($\mathbb{C}$), quaternions ($\mathbb{Q}$) and octonions ($\mathbb{O}$). Indeed by associating fermionic vertex operators to the points in such orbits one can realize those division algebras using a suitable definition of the product for the vertex operators (see ref. (9)).

In this section we shall apply the construction of section 4 to the Lie algebras appearing in the third line of the magic square, namely SP(3), SU(6), SO(12) and E_7 and show that they are related, via eq.(3.3) to the simple Jordan algebras $M_3^{(1)}$, $M_3^{(2)}$, $M_3^{(4)}$ and $M_3^{(8)}$ respectively. These are the Jordan algebra of 3x3 hermitean matrices over the real, complex, quaternionic and octonionic numbers with product given by

$$a \circ b = b \circ a = (ab + ba)/2 \qquad a,b \in \mathfrak{J} \qquad (6.1)$$

We will do that exploring the fact that SU(6), SO(12) and E_7 are obtained by matching g_1 = SP(3) with g_2 = SU(3), SP(3) and F_4 respectively. It is straightfoward to generalize our results to show that the algebras SP(n), SU(2n) and SO(4n) are related to the Jordan algebras $M_n^{(1)}$, $M_n^{(2)}$ and $M_n^{(4)}$ respectively. In fact some of these results have already been discussed in a previous paper[14].

The roots of the Lie algebras L on the third line of the magic square can be written , using the matching and the notation of ref. (8) as the long roots of g_1 = SP(3) i.e., $\pm\sqrt{2}e_i$, i=1,2 and 3, the long roots Φ_L of g_2 together with $(\pm e_1 \pm e_2)/\sqrt{2}$ + Ω_1, $(\pm e_2 \pm e_3)/\sqrt{2}$ + Ω_2 and $(\pm e_1 \pm e_3)/\sqrt{2}$ + Ω_3 where Ω_i i=1,2 and 3 vanish for L=SP(3) and correspond to the orbits of SU(3),SP(3) and F_4 for L= SU(6), SO(12) and E_7 respectively. The set of long roots Φ_L and the unit length vectors in Ω_i are given according to the notation of ref.(8) by

a)for g_2=SU(3), Φ_L vanish and

$$\Omega_1=(\pm(e_4-e_5)/\sqrt{2}), \quad \Omega_2= (\pm(e_5-e_6)/\sqrt{2}) \quad \text{and} \quad \Omega_3= (\pm(e_4-e_6)/\sqrt{2}) \qquad (6.2a)$$

b)for g_2=SP(3), Φ_L= $(\pm\sqrt{2}e_i$, i=4,5 and 6) and

$$\Omega_1= ((\pm e_4 \pm e_5)/\sqrt{2}), \quad \Omega_2= ((\pm e_5 \pm e_6)/\sqrt{2}) \quad \text{and} \quad \Omega_3= ((\pm e_4 \pm e_6)/\sqrt{2}) \qquad (6.2b)$$

c)for g_2=F_4, Φ_L = $(\pm e_i \pm e_j$, i,j=4,5,6 and 7) and

$$\Omega_1= (\pm e_i, i=4,5,6,7), \quad \Omega_2= ((\pm e_4 \pm e_5 \pm e_6 \pm e_7)/2)_{even} \quad \text{and}$$

$$\Omega_3 = \{(\pm e_4 \pm e_5 \pm e_6 \pm e_7)/2\}_{odd} \tag{6.2c}$$

where Ω_2 and Ω_3 contain an even and odd number of minus signs respectively.

In a previous paper[14] we have discussed the relation of Jordan and division algebras with the construction of Lie algebras appearing in the Freudenthal Magic Square[15,16] (Tits construction). We shall now show that those algebras namely, SP(n), SU(2n), SO(4n) and E_7 can be decomposed according to the structure of section 4. As in section 5, the U_1 generator Q has the form

$$Q = 2\frac{\lambda_a . H}{\alpha_a^2} \tag{6.3}$$

where α_a is a simple root and the fundamental weights λ_a are choosen as follows: λ_n for Sp(n) and SU(2n), λ_{2n-1} or λ_{2n} for SO(4n) and λ_7 for E_7 according to the convention adopted in the expansion of the highest root given in table 1. Again the subalgebra L_0 is obtained by deleting the point in the Dynkin diagram of g corresponding to α_a (see fig. 1). They are L_0 = SU(n)xU_1, SU(n)xSU(n)xU_1, SU(2n)xU_1, E_6xU_1 according to g = SP(n), SU(2n), SO(4n) and E_7 respectively. In fact the subalgebras L_0 above are those appearing in the second line in the Freudenthal magic square. For simplicity, from now on we shall only consider n=3. The second and third lines are constructed by matching the root system of $SU(3)/\sqrt{2}$ and $SP(3)$ respectively with $SU(3)/\sqrt{2}$, $SP(3)$ and F_4. The simple roots of $SP(3)$ are those of $SU(3)/\sqrt{2}$ together with $\sqrt{2}e_3$. Then the fundamental weight λ_a in (6.2) (which is orthogonal to all roots of L_0 and $\sqrt{2}e_3.\lambda_a = 1$) for all entries in the third line is $\lambda_a = (e_1+e_2+e_3)/\sqrt{2}$. We now define, following the notation of ref.(8) and (9), the abelian subspaces L_1 and L_{-1} (see (4.3))

$$L_1 = (E^{\frac{e_1+e_2}{\sqrt{2}}+\Omega_1}, E^{\frac{e_2+e_3}{\sqrt{2}}+\Omega_2}, E^{\frac{e_1+e_3}{\sqrt{2}}+\Omega_3}, E^{\sqrt{2}e_i}, i=1,2,3\ )$$

with $L_{-1} = (\ L_1)^+$ where Ω_1, Ω_2 and Ω_3 are defined in (6.2).

The vectors $\frac{e_1+e_2}{\sqrt{2}}$, $\frac{e_2+e_3}{\sqrt{2}}$ and $\frac{e_1+e_3}{\sqrt{2}}$ are in fact short roots of $SP(3)$. The automorphism (4.4) mapping L_1 into L_{-1} can be constructed out of three Weyl reflections generated by the long roots of $SP(3)$ namely: $\sqrt{2}e_1$, $\sqrt{2}e_2$ and $\sqrt{2}e_3$. It is defined as

$$\sigma(L) = \sigma_{\sqrt{2}e_1}(\sigma_{\sqrt{2}e_2}(\sigma_{\sqrt{2}e_1}(L))) \tag{6.4}$$

where $\sigma_x(L) = e^{i\pi S_2(x)}\ L\ e^{-i\pi S_2(x)}$ and $S_2(x) = \frac{E^x - E^{-x}}{2i}$. It therefore follows, using the relations of the appendix

$$\sigma(E^{\sqrt{2}e_i}) = -E^{-\sqrt{2}e_i} \tag{6.5a}$$

$$\sigma(e_i.H) = -e_i.H \tag{6.5b}$$

$$\sigma\left[E^{\frac{\xi e_i+\eta e_j}{\sqrt{2}}+\Omega_{ij}}\right] = \rho_{ij}^{(\xi,\eta)}\ E^{\frac{-\xi e_i-\eta e_j}{\sqrt{2}}+\Omega_{ij}} \tag{6.5c}$$

where $\rho_{ij}^{(\xi,\eta)} = \eta\xi\varepsilon(-\xi\sqrt{2}e_i, \frac{\xi e_i + \eta e_j}{\sqrt{2}} + \Omega_{ij})\varepsilon(-\eta\sqrt{2}e_j, \frac{-\xi e_i + \eta e_j}{\sqrt{2}} + \Omega_{ij})$

and $\xi, \eta = \pm 1$, $i,j,k = 1,2,3$. We had denoted the orbits given in (6.2) as $\Omega_1 = \Omega_{12}$, $\Omega_2 = \Omega_{23}$ and $\Omega_3 = \Omega_{13}$ (see paragraph preciding (6.2)). Since the 3 Weyl reflections in (6.4) commute among themselves,

$$\sigma^2(L) = \sigma^2_{\sqrt{2}e_1}(\sigma^2_{\sqrt{2}e_2}(\sigma^2_{\sqrt{2}e_1}(L))) = 1$$

Apart from mapping L_1 into L_{-1} the automorphism σ defined in (6.4) also split L_0 into $D+T_3$ such that $\sigma(D)=D$ and $\sigma(T_3)=-T_3$. We get

$$T^3 = \left\{(1-\sigma)E^{\frac{e_i - e_j}{\sqrt{2}} + \Omega_{ij}},\quad \sqrt{2}e_i.H,\quad i,j=1,2 \text{ and } 3\right\}.$$

so that $\dim T_3 = \dim L_1$. Before constructing the automorphism of order 3 (triality) satisfying (3.8), let us first define the identity elements. According to section 4 the identity elements belong to the centralizer of D. In fact there is a SU(2) subalgebra commuting with D generated by

$$T^1(1) = \left(\frac{E^{\sqrt{2}e_1} + E^{-\sqrt{2}e_1}}{2}\right) + \left(\frac{E^{\sqrt{2}e_2} + E^{-\sqrt{2}e_2}}{2}\right) + \left(\frac{E^{\sqrt{2}e_3} + E^{-\sqrt{2}e_3}}{2}\right) \tag{6.6a}$$

$$T^2(1) = \left(\frac{E^{\sqrt{2}e_1} - E^{-\sqrt{2}e_1}}{2i}\right) + \left(\frac{E^{\sqrt{2}e_2} - E^{-\sqrt{2}e_2}}{2i}\right) + \left(\frac{E^{\sqrt{2}e_3} - E^{-\sqrt{2}e_3}}{2i}\right) \tag{6.6b}$$

and

$$T^{3}(1) = \left(\frac{e_1+e_2+e_3}{\sqrt{2}}\right).H \qquad (6.6c)$$

satisfying (3.10).

Let us now define τ by

$$\tau(T)=\exp\left(\frac{-i\pi}{2}\,T^{1}(1)\right)\exp\left(\frac{-i\pi}{2}\,T^{2}(1)\right)\;T\;\exp\left(\frac{i\pi}{2}\,T^{2}(1)\right)\exp\left(\frac{i\pi}{2}\,T^{1}(1)\right). \qquad (6.7)$$

It is then straightforward to obtain

$$\tau\left[(1-\sigma)E^{\sqrt{2}e_i}\right] = -i(1+\sigma)E^{\sqrt{2}e_i} \qquad (6.8a)$$

$$\tau\left[(1+\sigma)E^{\sqrt{2}e_i}\right] = i\sqrt{2}e_i.H \qquad (6.8b)$$

$$\tau(\sqrt{2}e_i.H) = (1-\sigma)E^{\sqrt{2}e_i} \qquad (6.8c)$$

Also using the relations among the cocycles, given in the appendix, and the fact that σ is an automorphism of L, it is tedious but straightfoward to show that

$$\tau\left[(1-\sigma)E^{\frac{e_i+e_j}{\sqrt{2}}+\Omega_{ij}}\right] = -i(1+\sigma)E^{\frac{e_i+e_j}{\sqrt{2}}+\Omega_{ij}} \qquad (6.9a)$$

$$\tau\left[(1+\sigma)E^{\frac{e_i+e_j}{\sqrt{2}}+\Omega_{ij}}\right] = -i\epsilon\left(-\sqrt{2}e_j,\frac{e_i+e_j}{\sqrt{2}}+\Omega_{ij}\right)(1-\sigma)E^{\frac{e_i-e_j}{\sqrt{2}}+\Omega_{ij}} \qquad (6.9b)$$

$$\tau\left[(1-\sigma)E^{\frac{e_i-e_j}{\sqrt{2}}+\Omega_{ij}}\right] = -\epsilon(-\sqrt{2}e_j,\frac{e_i+e_j}{\sqrt{2}}+\Omega_{ij})(1-\sigma)E^{\frac{e_i+e_j}{\sqrt{2}}+\Omega_{ij}} \tag{6.9c}$$

We denote the elements of the Jordan algebra $M_3^{(n)}$ by

$$a_1 = \begin{bmatrix}1 & & \\ & 0 & \\ & & 0\end{bmatrix}, \quad a_2 = \begin{bmatrix}0 & & \\ & 1 & \\ & & 0\end{bmatrix} \quad \text{and} \quad a_3 = \begin{bmatrix}0 & & \\ & 0 & \\ & & 1\end{bmatrix}$$

$$a_{12}=\begin{bmatrix}0 & w & 0\\ \bar{w} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad a_{23}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & w\\ 0 & \bar{w} & 0\end{bmatrix}, \quad a_{13}=\begin{bmatrix}0 & 0 & w\\ 0 & 0 & 0\\ \bar{w} & 0 & 0\end{bmatrix} \tag{6.10}$$

where w and its conjugate $\bar{w}$ represent real, complex, quaternionic and octonionic numbers according to n=1,2,4 and 8 respectively. We can check that the following choice of representatives

$$T^1(a_i) = (1-\sigma)E^{\sqrt{2}e_i}, \quad T^2(a_i) = \tau(T^1(a_i)) \text{ and } T^3(a_i) = \tau^2(T^1(a_i))$$

and

$$T^1(a_{ij}) = (1-\sigma)E^{\frac{e_i+e_j}{\sqrt{2}}+\Omega_{ij}},$$

$$T^2(a_{ij}) = \tau(T^1(a_{ij})) \quad \text{and} \quad T^3(a_{ij}) = \tau^2(T^1(a_{ij})),$$

reproduces the multiplication table of the Jordan algebras (6.10) via eq. (3.3). It is then clear that the Lie algebras SP(3), SU(6), SO(12) and E_7 in fact generate the Jordan algebras $M_3^{(n)}$, n=1,2,4 and 8 respectively.

Notice that the Jordan subalgebra generated by the elements a_i (i=1,2,3) is related via (3.3) to the simply laced subalgebra g_L = SU(2)xSU(2)xSU(2) common to SP(3), SU(6), SO(12) and E_7. In all these cases g_L is generated by the step operators $E^{\pm\sqrt{2}e_i}$ together with the Cartan subalgebra generators $\sqrt{2}e_i$.H, i=1,2 and 3. In addition, it becomes clear from (3.3) that the sequence of subalgebras SP(3) ⊂ SU(6) ⊂ SO(12) ⊂ E_7 is a consequence of the sequence of Jordan subalgebras $M_3^{(1)} \subset M_3^{(2)} \subset M_3^{(4)} \subset M_3^{(8)}$.

7. THE POINCARE ALGEBRA

The Lorentz algebra in 3+1 dimensions is generated by the rotations J_i, and the boost operatores K_i(i=1,2,3). The Poincaré algebra contains, in addition, the translations P_μ (μ=0,1,2,3). The commutations relations are:

$$
\begin{aligned}
&[J_i, J_j] = i\epsilon_{ijk} J_k , \qquad && [P_i, K_j] = i\delta_{ij}P_o \\
&[J_i, K_j] = i\epsilon_{ijk} K_k , && [P_o, K_i] = iP_i \\
&[K_i, K_j] = -i\epsilon_{ijk} J_k , && [J_i, P_o] = 0 \\
&[J_i, P_j] = i\epsilon_{ijk} P_k , && [P_\mu, P_\nu] = 0
\end{aligned}
\qquad (7.1)
$$

Since we are using the Lorentz metric, the boost and rotation operators generate the non compact algebra SO(3,1) (Lorentz algebra). The translations generate an abelian ideal. These operators are all hermitian.

Taking the generator Q in (4.3) to be J_3, we decompose the Poincaré algebra as in (4.1) with

$$L_0 = \{J_3,\ K_3,\ P_3,\ P_0\} \tag{7.2a}$$

$$L_1 = \{J_1+iJ_2,\ K_1+iK_2,\ P_1+iP_2\} \tag{7.2b}$$

$$L_{-1} = L_1^{+} \tag{7.2c}$$

We introduce the inner automorphisms

$$\sigma(T) \equiv e^{i\pi J_2}\, T\, e^{-i\pi J_2} \tag{7.3a}$$

$$\tau(T) \equiv e^{-i\frac{\pi}{2}J_1}\, e^{-i\frac{\pi}{2}J_2}\, T\, e^{i\frac{\pi}{2}J_2}\, e^{i\frac{\pi}{2}J_1} \tag{7.3b}$$

which are of order two and three respectively. One can easily check that these automorphisms fulfill the requirements of the construction explained in section 4. That is, σ obeys (4.4) and τ permutes the subspaces T_i (i=1,2,3) defined in (4.5) and (4.7), leaving the subalgebra D invariant. Therefore according to the arguments given there the Poincaré algebra is related to a Jordan algebra via equations (3.3). Such Jordan algebra, has an identity element 1 and is given by

$$\gamma_r \circ \gamma_s = g_{rs} 1 \qquad r,s = 1,2 \tag{7.4}$$

where $$g_{rs} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \qquad (7.5)$$

So, it lives inside a Clifford algebra of the degenerated metric g_{rs}.

One can check that this is indeed true by writing the relations (7.1) in the form (3.3) using the following choice of representatives of the Jordan elements

$$T_i(1) \equiv J_i, \quad T_i(\gamma_1) \equiv K_i, \qquad T_i(\gamma_2) \equiv P_i, \qquad (7.6a)$$

and

$$D_{\gamma_1,\gamma_2} \equiv -iP_0 \;, \qquad D_{1,\gamma_r} \equiv 0, \quad r=1,2 \qquad (7.6b)$$

The relations (3.3a) and (3.3b) can be checked, in this case, using the fact that all associators vanish, except

$$(\gamma_2,\gamma_1,\gamma_1) = \gamma_2 \qquad (7.7)$$

The algebra (7.4) is not semisimple since γ_2 generates a solvable ideal. Acording to the discussion at the end of sec.3 this is a consequence of the fact the Poincaré algebra is not semisimple. One easily sees that the Jordan subalgebra generated by 1 and γ_1 is semisimple and it is related, via (3.3), to the Lorentz algebra. Defining

$$\gamma_\pm = \frac{1 \pm i\gamma_1}{2} \qquad (7.8)$$

one gets

$$\gamma_{\pm}^2 = \gamma_{\pm}, \quad \gamma_{+}\cdot\gamma_{-} = 0, \quad \gamma_{\pm}\cdot\gamma_{3} = \gamma_{3}/2, \quad \gamma_{3}^2 = 0 \qquad (7.9)$$

Therefore γ_3 and γ_+ (or γ_-) generate a non semi simple subalgebra of (7.4) without identity. Such subalgebra is related, via (3.3), to the subalgebra of the Poicaré algebra constituted by the translations P_μ and the SU(2) generated by

$$N_i = \frac{J_i + iK_i}{2} \qquad i=1,2,3 \qquad (7.10)$$

This is an example where the equations (3.3) work for a Jordan algebra without identity.

8. DISCUSSION AND CONCLUSION

So far we have related Jordan algebras to the construction of Lie algebras and shown that they possess a more fundamental structure in the sense that they can be thought as building blocks for constructing certain Lie algebras. These Lie algebras in turn, inherit the algebraic properties of the Jordan algebra. We now wish to comment on the generalization of our construction of sections 3 and 4 to infinite dimensional Lie algebras.

To start with, we should like to point out that a loop algebra can be straightfowardly fitted into the construction of section 3 and 4 since the underlying algebra given by

$$a_m . b_n = (a.b)_{m+n} \qquad ; m,\ n \in \mathbb{Z} \tag{8.1}$$

trivially satisfy the Jordan identity whenever the zero modes do. From the analogy with the loop Lie algebra, we realize (8.1) as

$$a_m = \lambda^m a \qquad , \lambda \in \mathbb{C} \tag{8.2}$$

and dub it the Loop Jordan. From (8.2), eq. (3.3) become

$$\left[T^i_m(a),\ T^j_n(b)\right] = i\varepsilon_{ijk} T^k_{m+n}(a.b) + \delta^{ij} D^{m+n}_{ab} \tag{8.3a}$$

$$\left[D^n_{ab},\ T^i_m(c)\right] = -T^i_{m+n}\Big((a,c,b)\Big) \tag{8.3b}$$

$$\left[D^m_{a,b}\ ,\ D^n_{c,d}\right] = D^{m+n}_{\mathcal{D}_{ab}(c),d} + D^{m+n}_{c,\mathcal{D}_{ab}(d)} \tag{8.3c}$$

where $T^i_m(a) = \lambda^m T^i(a)$ and $D^m_{a,b} = \lambda^m D_{a,b}$.

For Kac-Moody algebras the very same construction of section 4 can be applied when the generators of the Lie algebra T^a are replaced by currents $T^a(z) = \sum_n T^a_n z^{-n}$, $a = 1, \ldots, \dim g$. In such cases, the abelian subspaces L_1 and L_{-1} define the infinite dimensional Jordan algebra (8.1). In this context, the central element K belongs to the subalgebra D and we have

$$\left[T^i_m(a),\ T^i_n(b)\right] = D^{m+n}_{a,b} + Km\langle a,b\rangle\delta_{m+n,0}$$

where $\langle a,b\rangle$ was defined in (3.13). However K seems to have no relation with the Jordan algebra. This point has to be better understood. A promissing picture exist for the case of Clifford algebras in relation to the work of Jimbo and Miwa[22], as we mention below.

On the other hand, fermionic fields play the role of building blocks for constructing representations of Kac-Moody algebras. In fact, they may be decomposed into an infinite dimensional Jordan algebra. Consider for instance a pair of real independent Fermi fields

$$H^{(1)}(z) = \sum b_n^{(1)} z^{-n} \qquad \text{and} \qquad H^{(2)}(z) = \sum b_n^{(2)} z^{-n}$$

where $\{b_m^{(i)}, b_m^{(j)}\} = \delta_{ij}\delta_{m+n,0}$ and $n,m \in \mathbb{Z}$ or $m,n \in \mathbb{Z}+1/2$ according to Ramond or Neveu-Schwarz type respectively. In either way, each of those fields define an infinite dimensional Clifford algebra with generators

$$\gamma_m^+ = (b_m + b_{-m})/\sqrt{2} \quad \text{and} \quad \gamma_m^- = (b_m - b_{-m})/i\sqrt{2} \qquad m \in \mathbb{Z}^+ \quad \text{or} \quad m \in \mathbb{Z}^+ + 1/2$$

satisfying $\{\gamma_m^\pm, \gamma_n^\pm\} = \delta_{m,n}$ and $\{\gamma_m^+, \gamma_n^-\} = 0$. Through our construction of section 3, such Jordan algebra leads to the Lie algebra $SO(\infty)$. It contains an $\hat{U}_1$ Kac-Moody algebra as a subalgebra since both contain a pair of Fermi fields (see Jimbo and Miwa (22)). Similarly, other representations of Kac-Moody algebras can be obtained embedded in $SO(\infty)$ when more fermions are introduced following ref. (22).

We believe that our results may provide new insights towards constructing new representations of Kac-Moody algebras beyond those obtained from the quark model and the vertex operator construction[8]. In particular, we are now involved with the problem of constructing sistematically the conformal subalgebras os exceptional algebras and its relation to free fermions.

The fourth line of the Freudenthal magic square actually corresponds to a generalization of our construction in which instead of three we have seven subspaces $T_i(\alpha)$, $i=1,\ldots,7$, corresponding to the imaginary units of octonions. The construction is based upon decomposing the exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8 into 7 subspaces according to two U_1 generators. Finally we should mention that the construction of section 3 also applies to Lie and Jordan super algebras. The very same formula, (3.3) holds true in such case with Lie bracket and Jordan product being replaced by their supersymmetric counterparts. Those results shall be reported elsewhere.

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APPENDIX

The commutation relations for a simple Lie algebra L, in the Chevalley basis, are given by[21]

$$[H_a, H_b] = 0 \qquad a,b = 1,2,\dots,\text{rank } L \tag{I.1}$$

$$[H_a, E_\alpha] = (2\alpha_a.\alpha/\alpha_a^2)E_\alpha \tag{I.2}$$

$$[E_\alpha, E_\beta] = \begin{cases} (q+1)\varepsilon(\alpha,\beta)E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \quad (I.3a) \\ H_\alpha \equiv n_a H_a & \text{if } \alpha+\beta = 0 \quad (I.3b) \\ 0 & \text{otherwise} \quad (I.3c) \end{cases}$$

where α_a are the simple root of L, n_a are the integers in the expansion $\alpha/\alpha^2 = n_a\alpha_a/\alpha_a^2$, and q is the highest positive integer such that $\beta - q\,\alpha$ (or $\alpha - q\,\beta$) is a root. The cocycles $\varepsilon(\alpha,\beta)$ take the values ± 1 and are obviously antisymmetric

$$\varepsilon(\alpha,\beta) = -\varepsilon(\beta,\alpha) \tag{I.4}$$

The structure constants of L are completely determined from its root system except for the cocycles which are found by using Jacobi identities. If α,β and γ are roots adding up to zero, the Jacobi identity for their corresponding step operators yield

$$\epsilon(\alpha,\beta) = \epsilon(\beta,\gamma) = \epsilon(\gamma,\alpha) \ , \ \alpha + \beta + \gamma = 0. \qquad (1.5)$$

Further relations are found by considering Jacobi identities of three step operators corresponding to roots adding up to a fourth root. Now the Jacobi identity yields relations involving products of two cocycles. However in many situations there are only two non vanishing terms in the Jacobi identity. In order to classify all cases where none of the three terms vanish we consider roots α,β and γ so that $\alpha+\beta$, $\alpha+\gamma$, $\beta+\gamma$ and $\alpha+\beta+\gamma$ are roots as well. We shall denote the long roots by μ, ν, ρ,... and the short ones by e, f, g, ... From the properties of the roots one gets that $2\frac{\mu.\nu}{\mu^2}$, $2\frac{\mu.e}{\mu^2}$, $2\frac{e.f}{e^2}$ = 0, ±1 (see chapter 3 of ref. (21)). Let us consider the possible cases:

i) If $\mu+\nu$ is a root $(\mu+\nu)^2/\mu^2 = 2 + 2\mu.\nu/\mu^2$. Since $\mu+\nu$ can not be longer than μ one gets $2\frac{\mu.\nu}{\mu^2} = -1$. So, $\mu+\nu$ is a long root and if $\mu+\nu+\rho$ is also a root one gets by the same argument that $2(\mu + \nu).\rho/\mu^2 = -1$. Therefore $\mu+\rho$ and $\nu+\rho$ can not be roots simultaneously.

ii) If $\mu+e$ is a root $(\mu + e)^2/\mu^2 = 1 + \frac{e^2}{\mu^2} + 2\frac{\mu.e}{\mu^2}$. Then $2\frac{\mu.e}{\mu^2} = -1$ and $\mu+e$ is a short root. Therefore, if $\mu+e+\nu$ is a root, $\frac{2(\mu + e).\nu}{\nu^2} = -1$ and consequently $\mu + \nu$ and $\nu + e$ can not be both roots.

iii) Analogously if e+f and $\mu+e+f$ are roots and has $\frac{2(e + f).\mu}{\mu^2}$ =

-1, independent of e+f being either short or long. So, it is impossible for μ+e and μ+f to be both roots.

iv) If e+f is a root there exists three possibilities:

a) $2\frac{e.f}{e^2} = -1$ and e + f is a short root;

b) $2\frac{e.f}{e^2} = 1$ and $\frac{(e+f)^2}{e^2} = 3$ (can only happen if $L=G_2$);

c) e.f=0 and $\frac{(e+f)^2}{e^2} = 2$ (can only happen if L = B_n, C_n or F_4).

Suppose e, f and g are short roots of G_2 and e + f + g is a root. If the sum of any two of them is a root them $\frac{2(e+f).g}{g^2} = 0, \pm 2$. But this is impossible since it should be -3 if e + f was a long root and ±1 if it were short. Consider now the case where e, f and g are short roots of B_n, C_n, or F_4 and e + f + g is a root. If the sum of any two of then is a short root then $\frac{2(e+f).g}{g^2} = -2$. But this is in contradition whith the assumption e+f is short. Therefore one of them, lot us say e + f, must be a long root and consequently e.f = 0. In order to e + f + g be a root one needs $\frac{2e.g}{g^2} = \frac{2f.g}{g^2} = -1$. Since the short roots of B_n are either orthogonal or antiparallel such possibility can only occur if L = C_n or F_4.

Given two roots α and β, let p be the highest positive integer such that $\beta + p\alpha$ is a root. This integer is related to q introduced in (I.3a) by $\frac{2\alpha.\beta}{\alpha^2} = q-p$. The roots $\beta + n\alpha$ $(-q \le n \le p)$

constitute the α-root string through β[21]. For the algebras B_n, C_n and F_4 the maximum number of roots in such strings is 3. Therefore if $\alpha + \beta$ is a root of such algebras, q for the pair α,β is equal to one only if $\alpha.\beta = 0$. In the other cases $q = 0$.

Consider the Jacobi identity for three step operators corresponding to short roots e, f and g such that the sum of any two of then and e + f + g are roots. According to the discusion above this can only happen with the algebras C_n or F_4 with $e.f = 0$ and $\frac{2e.g}{g^2} = \frac{2e.f}{g^2} = -1$. The three terms of such identity are non vanishing and since q = 1 only for the pair e, f, we have that the coefficient of the term containing $\epsilon(e,f)$ is 2 and the other two coefficients are equal to 1. Consequently the Jacobi identity for such operators implies

$$\epsilon(e,f)\,\epsilon(e + f,g) = \epsilon(g,e)\,\epsilon(f,g + e) = \epsilon(f,g)\,\epsilon(e,f+g). \qquad (1.6)$$

In all other cases, the Jacobi identity for three step operators corresponding to non proportional roots α, β and γ such that $\alpha + \beta + \gamma$ is a root, has only two terms. The reason, as we have shown above, is that the sum of a given pair of roots, let us say $\alpha + \gamma$, is not a root. In such cases the Jacobi identity implies

$$\epsilon(\alpha,\beta)\,\epsilon(\alpha + \beta,\gamma) = \epsilon(\beta,\gamma)\,\epsilon(\alpha,\beta + \gamma). \qquad (1.7)$$

Consider the α-root string through β[21] containing the roots $\beta+n\alpha$ with $-q \leq n \leq p$ (see definition above). From the Jacobi identity for the step operators E_α , $E_{-\alpha}$ and $E_{\beta+n\alpha}$ with $1 \le n \le p$, one obtains p relations involving the cocycles and the integers p and q. By adding them up one obtains

$$\varepsilon(\alpha+\beta,-\alpha)\varepsilon(\alpha,\beta) = -1$$

Then using (I.5)

$$\varepsilon(\alpha,\beta) = -\varepsilon(-\alpha,-\beta) \tag{I.8}$$

This relation implies the mapping $H_\alpha \longrightarrow -H_\alpha$ and $E_\alpha \longrightarrow -E_{-\alpha}$ is an automorphism of order two of the Lie algebra L.

The relations (I.4), (I.5), (I.7) and (I.8) are sufficient to find all consistent choices of signs for the cocycles appearing in (I.3a). In the case of the algebras C_n and F_4 one has, in addition, to consider the relations (I.6).

Notice the relations (I.4-8) are invariant under the transformation $\varepsilon(\alpha,\beta) \rightarrow \rho(\alpha)\rho(\beta)\rho(\alpha+\beta)\varepsilon(\alpha,\beta)$ where $\rho = \pm 1$ and $\rho(\alpha)\rho(-\alpha) = 1$. This is related to the fact one can always replace E_α by $\rho(\alpha)E_\alpha$.

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SU(n+1)	$\alpha_1 + \alpha_2 + \ldots + \alpha_n$
SO(2n+1)	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + 2\alpha_n$
SP(n)	$2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$
SO(2n)	$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2	$3\alpha_1 + 2\alpha_2$

TABLE 1 - Highest root for simple Lie algebras

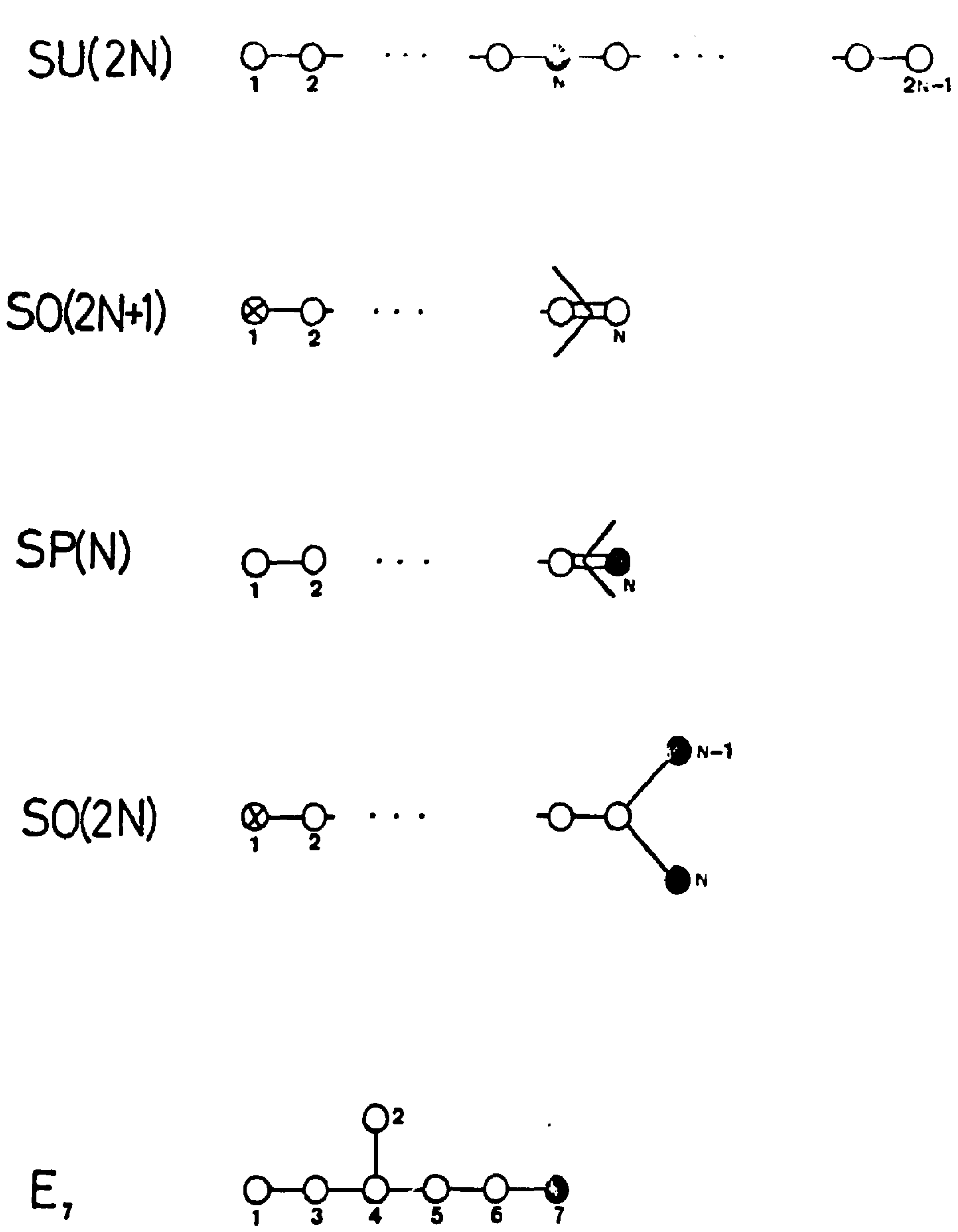

fig. 1

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{Q}$	$\mathbb{O}$
$\mathbb{R}$	SO(3)	SU(3)	Sp(3)	F_4
$\mathbb{C}$	SU(3)	SU(3)xSU(3)	SU(6)	E_6
$\mathbb{Q}$	SP(3)	SU(6)	SO(12)	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{Q}$
$\mathbb{R}$	SO(n)	SU(n)	Sp(n)
$\mathbb{C}$	SU(n)	SU(n)xSU(n)	SU(2n)
$\mathbb{Q}$	SP(n)	SU(2n)	SO(4n)

Table 2: Freudenthal Magic Square