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Winding of Planar Brownian Curves

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Abstract

We compute the joint probability for a brownian curve to wind n times around a prescribed point and to enclose a given algebraic area. An estimate from below of the arithmetic area is obtained.

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Since the pioneering work of Edwards [1], the study of path integrals in the presence of topological constraints has aroused considerable interest. On the one hand these techniques are of direct relevance for polymer physics, on the other hand they also allow to throw a bridge with some rigorous investigations of mathematicians. Consider for instance the two dimensional brownian motion on the punctured plane $P - \{0\}$. The problem of finding the asymptotic probability distribution of the total angle $\theta(t)$ wound at time t around 0 was first addressed by Spitzer [2] who showed that $X = 2\theta(t)/\ln t$ is distributed according to a Cauchy law for $t \rightarrow +\infty$. This result was then extended by Yor et al. [3] to the case of n prescribed points. This question has also been reexamined by Rudnick et al. [4] who showed that by removing a disc from the plane, instead of a point, the asymptotic distribution changes drastically from a Cauchy law to an exponential law (which thus leads to finite moments). The winding number distribution was also discussed by Wiegel in the context of polymer entanglements [5].

An apparently unrelated problem concerns the probability distribution of the area enclosed by a planar brownian curve. First raised by Levy [6] and solved magisterially by the use of Fourier-Wiener series, this problem was more recently reexamined by Brereton et al. [7], Khandekar et al. [8] and Duplantier [9].

The purpose of this paper is to extend this approach to the case of the joint probability distribution for a closed planar brownian walk to wind n times around a prescribed point and enclose a given algebraic area (the initial = final point been left unspecified). Interestingly enough, this quantity is related to the two body partition function of a gas of particles obeying fractional statistics (anyons).

The plan of the paper is as follows: for pedagogical reasons we first rederive Wiegel's results concerning the probability $P(A)$ for a closed planar brownian curve to enclose after a time τ a given algebraic area A. This quantity can be expressed in terms of the partition function of a charged particle embedded in a constant magnetic field. This partition function diverges as the total area of the plane but an adequate normalization leads back to the finite $P(A)$. We then consider the probability $P_\omega(n)$ for the same brownian curve to wind n times around a given point. In this case the correspondance involves a vortex laying at O and carrying a magnetic flux ϕ. Here an harmonic regulator ω is needed to compute the partition function of a charged particle moving in the vortex field. In the limit where the regulator vanishes, one finds that the probability for zero winding is 1, when the regulator is infinitesimal, we will show that $P_\omega(n)$ behaves as $1/n^2$ when n is non-vanishing. We finally evaluate the joint probability $P_\omega(n, A)$ for the brownian curve to enclose a given area A and wind n times around a given point. This requires to compute the partition function of a charged particle embedded in a magnetic field superposed with a vortex field. By an appropriate reinterpretation of our results we then give an estimate from below of the total arithmetic area enclosed by the curve.

Let us first consider $P(A)$ and review the basic material at hand. The Wiener integral representation of the transition probability for a random walk to start from $\vec{r}'$ and end at $\vec{r}''$ after a "time" τ is

$$P(\vec{r}'', \vec{r}') = \frac{1}{2\pi\tau} \exp - \frac{(\vec{r}'' - \vec{r}')^2}{2\tau} = N \int_{\vec{r}(0)=\vec{r}'}^{\vec{r}(\tau)=\vec{r}''} \exp\left(- \frac{1}{2} \int_0^\tau \dot{\vec{r}}^2(s) ds\right) [D\vec{r}] \tag{1}$$

where N is a normalization factor. In these units the average end to end square distance between $\vec{r}'$ and $\vec{r}''$ is $\langle [\vec{r}(\tau) - \vec{r}(0)]^2 \rangle = 2\tau$. Now the problem is to express the probability for such a curve when it goes back to the starting point $\vec{r}' = \vec{r}''$ to enclose a given algebraic area A. Following Edwards, one way is to impose in the Wiener integral the constraint

$$A = \frac{1}{2} \int_0^\tau (\vec{r} \times \dot{\vec{r}}).\vec{k} ds \tag{2}$$

where $\vec{k}$ is the unit vector orthogonal to the plane. One thus gets

$$P(\vec{r}',\vec{r}',A) = N\int_{-\infty}^{+\infty} d\lambda \frac{\exp(-i\lambda A)}{2\pi}\int_{\vec{r}(0)=\vec{r}'}^{\vec{r}(\tau)=\vec{r}'}[D\vec{r}]\exp-\int_0^\tau (\frac{\dot{\vec{r}}^2(s)}{2} - i\frac{\lambda}{2}(\vec{r}\times\dot{\vec{r}}).\vec{k})ds \quad (3)$$

where we have used the identity $2\pi\delta(x) = \int_{-\infty}^{+\infty}\exp(i\lambda x)d\lambda$. Invoking the usual correspondance with quantum mechanics we observe that the "action" appearing in the path integral describes a particle of unit charge and mass moving in a constant magnetic field of strength $+\lambda$ orthogonal to the plane.

If we now assume that the initial point is not prescribed, i.e. that the closed brownian curve can wander everywhere in the plane, the resulting probability for it to enclose the area A reads $P(A) = \int_{plane} P(\vec{r}',\vec{r}',A)d^2\vec{r}'$. In accordance with the Feynman-Kac formula it involves the partition function $Z(\lambda)$ of a particle in a constant magnetic field [10] with for "temperature" $1/kT = \tau$, namely

$$P(A) = N\int_{-\infty}^{+\infty} d\lambda \frac{\exp(-i\lambda A)}{2\pi}Z(\lambda) = N\int_{-\infty}^{+\infty} d\lambda \frac{\exp(-i\lambda A)}{2\pi}\frac{S\lambda}{\sinh(\tau\lambda/2)}\frac{1}{4\pi} \quad (4)$$

As it is well known the partition function diverges as the surface S of the plane (due to the translation invariance of the system, the degeneracy on each Landau level is infinite). However this problem can be handled by a proper normalization. One gets

$$P(A) \equiv \frac{P(A)}{\int_{-\infty}^{+\infty} P(A)dA} = \frac{\pi}{2\tau}\frac{1}{\cosh^2(\frac{\pi A}{\tau})} \quad (5)$$

which is nothing but the usual result [8,9] but here simply obtained as the Fourier transform of the partition function of an electric charge in a constant magnetic field.

We now consider the problem of the winding of a closed brownian curve around a fixed point. The corresponding probability will be shown to be the Fourier transform of the partition function of a charged particle in a vortex field. We start from the same path integral transition probability $P(\vec{r}'',\vec{r}')$ but impose instead a constraint expressing that the total angle wound at time τ around O is $2\pi n$. This gives

$$n = \frac{1}{2\pi}\int_0^\tau \dot{\theta} ds \tag{6}$$

where θ is the polar angle between $\vec{r}'$ and $\vec{r}''$ around the point O. After inserting the Kronecker constraint $\delta_{n,\frac{1}{2\pi}\int_0^\tau \dot{\theta} ds} = \int_0^1 d\xi \exp i\xi 2\pi(n - \frac{1}{2\pi}\int_0^\tau \dot{\theta} ds)$ we get

$$P(\vec{r}',\vec{r}',n) = N\int_0^1 d\xi \exp(i2\pi\xi n)\int_{\vec{r}(0)=\vec{r}'}^{\vec{r}(\tau)=\vec{r}'}[D\vec{r}] \exp - \int_0^\tau (\frac{\dot{\vec{r}}^2(s)}{2} + i\xi\dot{\theta})ds \tag{7}$$

The "action" appearing in the path integral now describes a particle of unit charge and mass moving in a vortex field localized at the origin and carrying a flux $\phi = -2\pi\xi$. Again we are interested in a brownian curve that can wander everywhere in the plane. The resulting probability for it to wind n times around the origin reads $P(n) = \int_{plane} P(\vec{r}',\vec{r}',n)d^2\vec{r}'$ which now involves the partition function $Z(\xi)$ of a particle in a vortex with for "temperature" $1/kT = \tau$. Thus one has

$$P(n) = N\int_{-\infty}^{+\infty} d\xi \frac{\exp(i2\pi\xi n)}{2\pi} Z(\xi) \tag{8}$$

The partition function $Z(\xi)$ is easily shown to diverge, since the particle may wander everywhere in the whole plane. A suitable regularisation is to assume that the particle is attracted to the point O by an harmonic force. The corresponding Hamiltonian then reads

$$2H = (\vec{p} + \frac{\phi}{2\pi r}\vec{u}')^2 + \omega^2 r^2 \tag{9}$$

where $\vec{u}'$ is the orthoradial unit vector perpendicular to the radial vector $\vec{r}$. It is interesting to point out that this Hamiltonian can be viewed as the two body relative Hamiltonian of a system of anyons carrying both an electric charge and a magnetic flux and interacting with an harmonic force. As discussed in ref.[11] where the second virial coefficient for an anyon gas was computed by means of this harmonic well regulator, the spectrum of H is

(n and m integers, $n \geq 0$) $E_{n,m} = \omega(|m-\xi| + 1 + 2n)$ where ξ stands for the fractional part of $\phi/2\pi$ (in accordance with a general result of N.Byers and C.N. Yang [12], the spectrum is indeed periodic in ξ of period 1). The partition function reads:

$$Z_\omega(\xi) = \frac{\cosh(\frac{1}{kT}\omega)(\xi - \frac{1}{2})}{2\sinh(\frac{1}{kT}\omega)\sinh(\frac{1}{2kT}\omega)} \tag{10}$$

It follows that

$$P_\omega(n) = N\int_0^1 d\xi \frac{\exp(i2\pi\xi n)}{2\pi}\frac{\cosh r\omega(\xi - \frac{1}{2})}{2\sinh(r\omega)\sinh(\frac{r\omega}{2})} \tag{11}$$

By a proper normalization one finally obtains

$$P_\omega(n) = \frac{\int_0^1 d\xi \exp(i2\pi\xi n)\cosh r\omega(\xi - \frac{1}{2})}{\sum_{n=-\infty}^{n=+\infty}\int_0^1 d\xi\exp(i2\pi\xi n)\cosh r\omega(\xi - \frac{1}{2})} \tag{12}$$

that reads

$$P_\omega(n) = \frac{2r\omega\sinh(\frac{r\omega}{2})}{\cosh(\frac{r\omega}{2})(r^2\omega^2 + 4\pi^2 n^2)} \tag{13}$$

a result already obtained by Wiegel in a different way. When ω is infinitesimal $P_\omega(n)$ behaves as

$$\frac{r^2\omega^2}{4\pi^2 n^2} \tag{14}$$

and $P_\omega(0)$ as

$$1 - \frac{r^2\omega^2}{12} \tag{15}$$

In the limit $\omega \to 0$ one thus has $P(0) = 1$. This result is not a surprise, it reflects the fact that for $\omega = 0$, the particle is no more attached to the origin, thus a typical closed curve will have a vanishing winding number. The set of curves with a non vanishing winding number being of zero measure, it thus follows that $P(0) = 1$. It would be interesting to distinguish in the $n = 0$ sector, curves which do not enclose the origin from curves which do enclose the origin but an equal number of times clockwise and anticlockwise. Such a distinction can not however be reached within the scope of this analysis.

Results in (14,15) can still be given a simple interpretation if we adopt a different point of view. Indeed, the integration $\int d^2\vec{r}' \ldots$ used in calculating $P(n)$ means that we count all the closed curves that begin at every point $\vec{r}'$ and wind n times around the origin. Equivalently, we can say that we consider all the closed curves beginning at a given fixed point, say the origin, and count all the points of the plane that are wound around n times by those curves. Thus, for each closed curve, we can define different winding sectors, each one being labelled by the winding number n of any point inside it. As an example, see Fig.1 (Notice that the 0-sector is very different of the others because it consists, first, in small "islands" lying inside the envelope of the curve and, also in an infinite part, the rest of the plane).

Calling S_n the arithmetic area of the n-sector, it follows that :

$$\lim_{\omega \to 0} P_\omega(n) \propto < S_n > ,$$

where $< \ldots >$ means averaging over all possible closed curves of a given average length. Further justification of the above relationship will be given in (22-25). Then, (14) gives for the arithmetic area $< S_n >$:

$$< S_n > = \frac{c}{n^2} \quad (n \neq 0) \tag{16}$$

It will be shown below that $c = \tau/2\pi$ (27).

Of course, we observe that, for all, $n \neq 0$:

$$\frac{< S_n >}{< S_0 >} = \lim_{\omega \to 0} \frac{P_\omega(n)}{P_\omega(0)} = 0 , \tag{17}$$

which follows trivially from the finiteness of the average length of the curve : the area of the 0-sector is obviously infinite.

To summarize the above discussion, we may adopt two different points of view.

i) the standard one, eq.(12) which leads to

$$\lim_{\omega\to 0} P_\omega(n) = \delta_{n,0}$$

ii) the one of eq.(16) which leads to finite positive quantities for all n except for $n = 0$ where we have a divergence.

Notice that i) does not give any information about the brownian curve we want to study.

Now, we calculate the probability, $P_\omega(n, A)$ for the curve to enclose a given algebraic area A and wind n times around a given point. Imposing simultaneously the constraints (2) and (6), we have to solve the problem of a charged particle moving in a vortex field (flux : $-2\pi\xi$) placed at the origin and a uniform magnetic field, $+\lambda$, perpendicular to the plane. Moreover, we use the same harmonic well regulator as before.

The energy levels are given by :

$$E_{M,n} = \epsilon'(|M-\xi| + 2n + 1 - (M-\xi)\frac{\lambda}{\lambda'}) \tag{18}$$

where M and n are integers ($n \geq 0$) ,

$$\lambda'^2 = \lambda^2 + (2\omega)^2$$
$$\epsilon' = \frac{\lambda'}{2}$$

In obvious notations, the partition function writes :

$$Z_\omega(\lambda,\xi) = \frac{1}{4\sinh\frac{\lambda' t}{2}}\left[\frac{e^{\frac{t}{2}(\lambda+\lambda')(\xi-\frac{1}{2})}}{\sinh\frac{t}{4}(\lambda+\lambda')} + \frac{e^{\frac{t}{2}(\lambda-\lambda')(\xi-\frac{1}{2})}}{\sinh\frac{t}{4}(\lambda'-\lambda)}\right] \tag{19}$$

Following the same approach as for $P_\omega(n)$, we get, N' being a normalization factor :

$$\begin{aligned} P_\omega(n,A) &= N'\int_0^1 d\xi \int_{-\infty}^{+\infty} d\lambda \exp(i2\pi\xi n - i\lambda A) Z_\omega(\lambda,\xi) \\ &\equiv N' X_\omega(n,A) \end{aligned} \tag{20}$$

The normalized probability will thus read

$$\mathcal{P}_\omega(n, A) = \frac{X_\omega(n, A)}{\mathcal{N}}$$

where $\mathcal{N} = \sum_n \int dA X_\omega(n, A) = \frac{\pi}{\sinh \omega\tau} \coth \frac{\omega\tau}{2}$.

We can calculate exactly the quantity $X_\omega(n, A)$. We get the following result when $(A > 0, n \leq 0)$ or $(A < 0, n \geq 0)$:

$$X_\omega(n, A) = \frac{4\pi^3}{\tau} \left(\sum_{m=1}^{+\infty} \frac{(-1)^{m+1} m^2 \exp(-\omega_m \frac{2|A|}{\tau})}{\omega_m((\omega_{2n})^2 + 4\pi|n|\omega_m)} \right)$$

where we have defined :

$$\omega_p \equiv \sqrt{(\omega\tau)^2 + p^2\pi^2}$$

The expressions are rather lengthy for the other cases. $((A > 0, n > 0)$ or $(A < 0, n < 0))$ and are omitted here. Let us consider the limit $\omega \to 0$. Defining

$$X(n, A) \equiv lim_{\omega \to 0} X_\omega(n, A) \ ,$$

we get :

$$\begin{aligned} X(n, A) \ &= \frac{1}{\tau} \left\{ \frac{1}{|n|} \left(\sum_{\substack{m=1 \\ m \neq |n|}}^{+\infty} (-1)^m e^{-\frac{2\pi|A|m}{\tau}} \left(\frac{m}{m - |n|}\right) \right) \right. \\ &\left. + e^{-\frac{2\pi n A}{\tau}} (-1)^n \left(\frac{1}{|n|} - \frac{2\pi|A|}{\tau}\right) \right\} \end{aligned}$$

if $(A > 0, n > 0)$ or $(A < 0, n < 0)$

and

$$X(n, A) = \frac{1}{\tau} \left\{ \frac{1}{|n|} \sum_{m=1}^{+\infty} (-1)^{m+1} e^{-\frac{2\pi|A|m}{\tau}} \left(\frac{m}{m + |n|}\right) \right\} \tag{21}$$

if $(A > 0, n < 0)$ or $(A < 0, n > 0)$. The limit $A \to 0$ will be studied in (28,29).

Of course, the relationship :

$$X(n, A) = X(-n, -A) \qquad \text{holds.}$$

We can show that, for $n \neq 0$, the quantity $X(n, A)$ is finite and positive and it becomes infinite for $n = 0$. This is in agreement with the detailed discussion given above : in particular one has $\lim_{\omega \to 0} \mathcal{P}_\omega(0, A) = \mathcal{P}(A)$.

The variations of $X(n, A)$ as a function of n are displayed in Fig.2 for 3 different values of A (=5 ; 0.5; 0). τ has been taken equal to 2π and the dashed curves have been drawn to guide the eye. (More important than the magnitudes of thoses curves are their shapes). First, we observe an asymmetry when $A > 0$:

$$X(n, A) > X(-n, A) \quad \text{for all} \quad n > 0 \ .$$

Moreover, for sufficiently high positive values of A (for instance, $A = 5$), the $n = 1$ value largely dominates the others :

$$X(1, A) >> X(n, A) \quad \text{for all } n \neq 0; 1$$

(See the end of this paper (30-31) for a detailed discussion of this particular point). Finally, as $|A|$ decreases, the asymmetry progressively disappears. Of course, for $A = 0$, the symmetry $n \leftrightarrow -n$ is recovered :

$$X(n, A) = X(-n, A) \quad \text{when } A = 0 \ .$$

Now, we deal with the interpretation of $X(n, A)$.

Using Eq.(20), in the limit $\omega \to 0$, we can establish the following relationship :

$$\sum_{n=-\infty}^{+\infty} nX(n, A) = \frac{1}{\tau} A\mathcal{P}(A) \tag{22}$$

where $\mathcal{P}(A)$ is defined in (5).

This equation has at the level of the partition function the following counterpart

$$\lim_{\omega \to 0} \left[\frac{\partial}{\partial \xi} Z_\omega(\lambda, \xi)|_{\xi=0} = \frac{-2\pi}{\tau \mathcal{N}} \frac{\partial}{\partial \lambda} Z_\omega(\lambda, \xi = 0) \right] \tag{23}$$

where $\frac{\partial Z}{\partial \xi}|_{\xi=0}$ is defined below (23').

To obtain Eq.(22) we have used the following identities :

$$n e^{2i\pi\xi n} = \frac{1}{2i\pi}\frac{\partial}{\partial\xi}(e^{2i\pi\xi n}) \ ,$$

$$\sum_n e^{2i\pi\lambda n} = \sum_m \delta(\lambda - m)$$

and the expression of the derivative of the partition function :

$$\frac{1}{2}\cdot \lim_{\omega\to 0}(\frac{\partial}{\partial\xi}Z_\omega(\lambda,\xi)|_{\xi=0^+} + \frac{\partial}{\partial\xi}Z_\omega(\lambda,\xi)|_{\xi=0^-}) = \frac{-1}{2\sinh(\frac{\tau}{2}\lambda)}\left(1 - (\frac{\tau}{2}\lambda)\coth(\frac{\tau}{2}\lambda)\right) \quad (23')$$

(The above equation (22) has also been directly obtained from the expressions (21) after ... a rather tedious calculation !)

Thus, following eq.(22), we can define the new quantity :

$$< S(n,A) >= \tau\frac{1}{P(A)}X(n,A) \quad (24)$$

which gives :

$$\sum_{n=-\infty}^{+\infty} n < S(n,A) >= A \quad (25)$$

So, the interpretation of $< S(n,A) >$ is very clear : this quantity is the mean value of the arithmetic area of the n-sector ($n \neq 0$), the total algebraic area enclosed by the curve being fixed and equal to A.

Going further, we can calculate, A still being fixed, the average of the total arithmetic area enclosed by the curve (except for the 0-sector !) :

$$\begin{aligned} \textstyle\sum_{n\neq 0} < S(n,A) > &= \frac{1}{P(A)}\left\{\log(1 + e^{-\frac{2\pi|A|}{\tau}}) + (\frac{1}{1+e^{\frac{2\pi|A|}{\tau}}}) \cdot \frac{2\pi|A|}{\tau}\right\} \\ &\equiv \sum_{n\neq 0} < S(n,-A) > \end{aligned} \quad (26)$$

At this stage, it is worth noticing that the use of simultaneous conditions on n (winding number) and A (algebraic area) gives access to the arithmetic area : a rather unexpected result.

Finally, averaging $< S(n,A) >$, given in (24), over A we get :

$$< S_n > \equiv \int dA \mathcal{P}(A) < S(n,A) > = \frac{\tau}{2\pi} \cdot \frac{1}{n^2} \,. \tag{27}$$

leading to $c = \tau/2\pi$ (16). Thus, the mean area of the n-sector ($n \neq 0$) is completely determined.

To close our analysis, we consider some asymptotic behaviour of the quantities $< S(n,A) >$.

First, taking the limit $A \to 0$ with some care, we obtain :

$$< S(n,0) > = < S(-n,0) > = \frac{1}{\mathcal{P}(0)} \left\{ \frac{1}{2|n|} + (-1)^{n+1} \log 2 + \sum_{q=1}^{|n|} \frac{(-1)^{q+1-n}}{q} \right\} \,, \tag{28}$$

which gives, when $n = 1, 2, 3, \ldots$ the funny sequence of numbers $(\log 2 - \frac{1}{2})$, $(\frac{3}{4} - \log 2)$, $(\log 2 - \frac{2}{3}), \ldots$ (we have omitted the factor $\frac{1}{\mathcal{P}(0)}$). Moreover, it can be shown that, when $n \to \pm\infty$:

$$< S(n,0) > \sim \frac{\text{const}}{\text{n}^2} \,. \tag{29}$$

A behaviour close to the one of $< S_n >$ (we recall that $< S_n >$ is obtained after an averaging over A).

Now, considering the limit $A \to +\infty$, we get :

$$\frac{< S(n,A) >}{< S(1,A) >} \underset{A \to +\infty}{\longrightarrow} 0 \quad \text{for all } n \neq 0, 1 \tag{30}$$

Analogously, when $A \to -\infty$, we have :

$$\frac{< S(n,A) >}{< S(-1,A) >} \underset{A \to -\infty}{\longrightarrow} 0 \quad \text{for all } n \neq 0, -1 \tag{31}$$

This means that a curve of a given average length reduces to a single ring when it is constrained to enclose an infinite area. All this is consistent with (26).

Indeed, we have

$$\frac{\sum_{n\neq 0} < S(n,A) >}{|A|} \underset{A\to\pm\infty}{\longrightarrow} 1 , \qquad (32)$$

from which we conclude that the arithmetic area becomes equal to the absolute value of the algebraic area when $A \to \pm\infty$. The limit (32) is a strong indication that the 0-sector lies completely outside the envelope of the curve when $|A| \to \infty$.

Figure Captions

Fig.1 : A closed curve with its various winding sectors, the so-called n-sectors. Each n-sector is labelled by the winding number, n, of any point inside it. The 0-sector has an infinite area.

Fig.2 : The (unnormalized) "probability" $X(n, A)$ as a function of n for 3 different values of A. We can see that the asymmetry $n \leftrightarrow -n$ progressively disappears when $|A|$ decreases.

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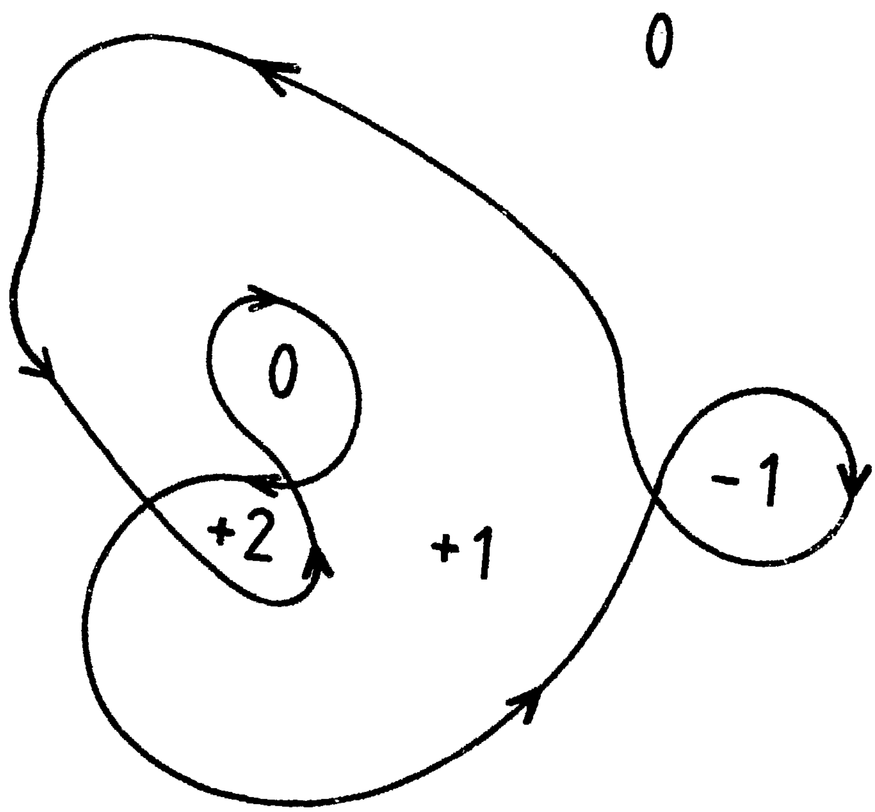

Fig. 1

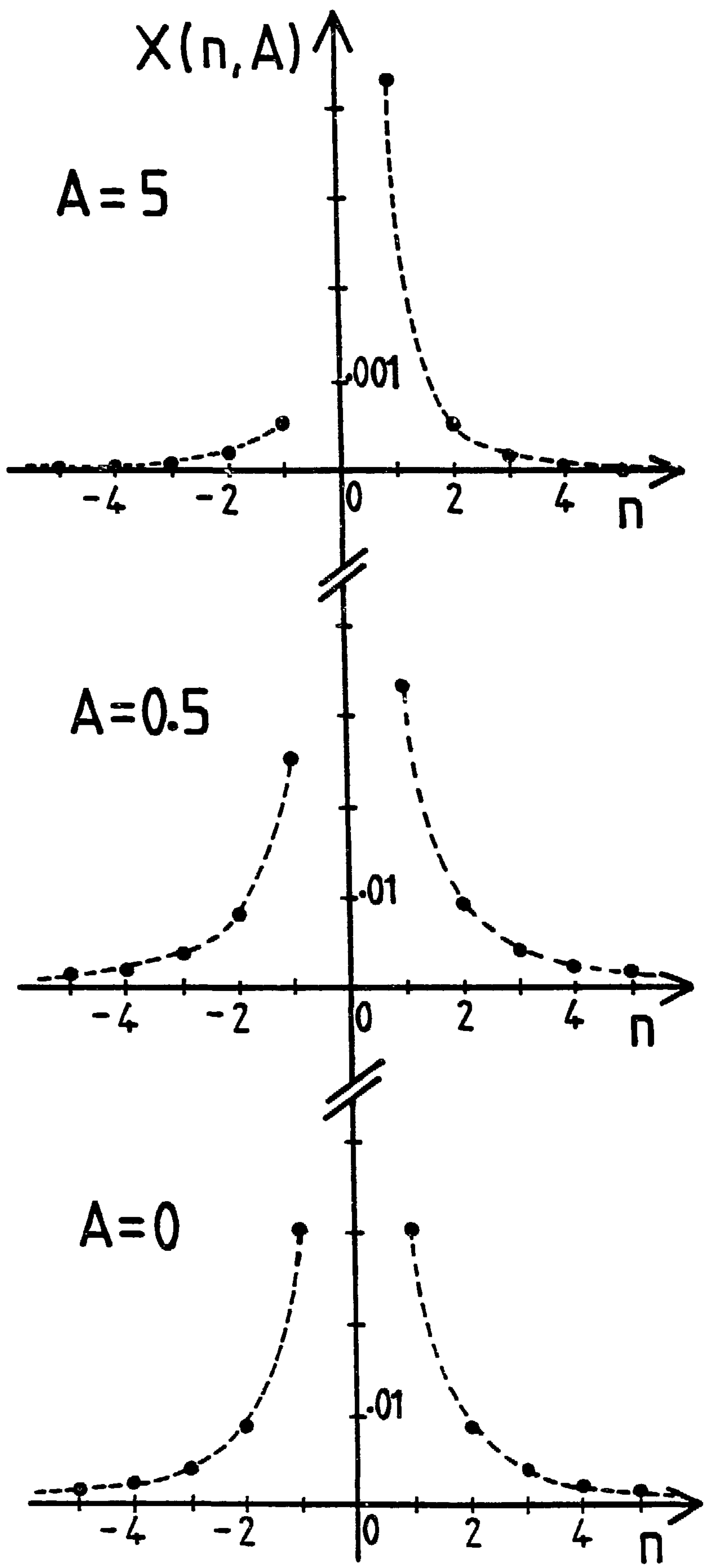

Fig. 2