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# TOPOLOGICAL FIELD THEORY: ZERO-MODES AND RENORMALIZATION

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Abstract: We address the issue of the non-triviality of the observables in various Topological Field Theories by means of the explicit introduction of the zero-modes into the BRST algebra. Supersymmetric quantum mechanics and Topological Yang-Mills theory are dealt with in detail. It is shown that due to the presence of fermionic zero-modes the BRST algebra may be dynamically broken leading to non trivial observables albeit the local cohomology being trivial. However the metric and coupling constant independence of the observables are still valid. A renormalization procedure is given that correctly incorporates the zero-modes. Particular attention is given to the conventional gauge fixing in Topological Yang-Mills theories, with emphasis on the geometrical character of the fields and their rôle in the non-triviality of the observables.

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## 1) INTRODUCTION

In a series of articles Witten [1,2] expressed various topological invariants as correlation functions in Topological Field Theories. These theories possess a supersymmetry and it was left open by Witten if the supersymmetry is a BRST symmetry. This was answered in the affirmative shortly afterwards in [3,4,5,6]. The BRST algebra put forward in [3,4] for the 4-dimensional Yang-Mills theory, is the algebra proposed by Witten augmented with conventional gauge transformations (so that the field content is that of Witten plus the usual Faddeev-Popov ghosts, anti-ghosts and Lagrange multiplier fields).

In another development Horne [7] has obtained an alternative formulation of the four dimensional Topological Field Theory, by keeping the supersymmetry of [1] and BRST gauge fixing the gauge invariance. In this way there are two BRST-supersymmetries leaving the action invariant. Consequently the field content is much larger than in the schemes of [3, 4, 5, 6]. It was subsequently established in [8] that the classical BRST cohomology of [3, 4, 5, 6] is trivial implying that there are no nontrivial invariants to be obtained in this way. It was proposed instead to look at the basic cohomology associated with the BRST algebra and argued that Horne's action satisfies the requirements needed to ensure that there are non trivial topological invariants.

On the other hand in the case of isolated instantons (zero dimensional moduli space) the partition function has been computed using the actions proposed in [3, 4] and Horne's action. The results agree [9] and coincide with Witten's calculation, so at this level there is no contradiction between the two approaches. Of course in the calculation of topological invariants when the moduli space has dimension greater than zero, the argument presented in [8] becomes applicable and all the invariants of interest are $Q$ exact.

Though recently there has been a large amount of work in this field there remain three major issues to resolve,

(*i*) The above mentioned triviality of the observables (topological invariants),

(*ii*) The explicit incorporation of the zero-modes into the BRST algebra so as to properly define the invariance of the model and

(*iii*) A definition of a renormalization procedure that takes (*ii*) into account *

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* These limitations of the conventional formulations have been pointed out in one form or another by R.Stora [10]

In this work the above issues are addressed. As for $(i)$ the main results are that the topological BRST symmetry may be dynamically broken. This mechanism is identical to that discussed by Witten for conventional supersymmetry and extended by Fujikawa to the case of BRST symmetry [11]. A consequence of the symmetry breaking is that cohomologically trivial quantities might have non-zero expectation values, since the BRST symmetry is not a good symmetry of the theory (this statement was expressed differently in [8] where the basic cohomology argument was put forward implying that the BRST symmetry was not adequate for a good description of the topological invariants). An important aspect of this is that even though the vacuum is not BRST invariant it is still possible to establish the coupling constant and metric independence of the topological correlation functions. The non-triviality of the observables in gauge theories can also emerge because of the non-global nature of the trivialization: this is shown by direct calculation, but also in Horne's approach where the gauge dependence is made explicit at the formal level. (ii) is accounted for by a general algorithm, that leads to a resolution of (iii).

The plan of the paper is as follows. In the next section supersymmetric quantum mechanics is dealt with in some detail as it is a simple model that ellucidates many of the relevant features. Section 3 is concerned with Topological Yang-Mills Theory in four dimensions. Special aspects of the conventional gauge invariance gauge fixing are discussed in detail.

## 2) Supersymmetric Quantum Mechanics

### 2.1) Fermion Zero-Mode and Breaking of the Topological Supersymmetry

The identification of N=2 supersymmetric quantum mechanics as a Topological Field Theory was made in [12]. Subsequently this correspondence has been analyzed by many authors [13]. The issue here is the question of the breakdown of the topological supersymmetry. Witten [11] argued that dynamical breaking of supersymmetry occurs in supersymmetric quantum mechanics when the potential energy has an even number of zeroes. The breakdown of the symmetry is a non-perturbative effect, arising from the existence of instantons. Salomonsom and van Holten [14](see also [15]) soon after, exhibited this mechanism in some detail. As we have the freedom to use a convenient gauge we will not allude to these works in the sequel.

The BRST algebra of interest is

$$\{Q, x\} = \psi \quad \{Q, \overline{\psi}\} = B$$

$$\{Q, \psi\} = 0 \quad \{Q, B\} = 0$$

$$Q^2 = 0$$

where the fields $(x, B, \psi, \overline{\psi})$ are assigned the ghost numbers (0,0,1,-1) respectively. Her we follow Witten's notations so that Q, means a BRST transformation.

The action is taken to be

$$S = \int_{-\infty}^{+\infty} dt \quad \{Q, \overline{\psi}\,(dx/dt + V'(x))\}$$

$$= \int_{-\infty}^{+\infty} dt \quad B\,(dx/dt + V'(x)) - \overline{\psi}\,(d/dt + V''(x))\,\psi$$

A term $\{Q, \alpha\overline{\psi}\,B\} = \alpha\,B^2$ could have been added, leading to a more conventional form for the action [11,14] without changing the results we are to obtain. However the form of the action chosen here allows a simple analysis. We consider the path integral in the $x(t)$ field between $x_i$ and $x_f$, two minima of the potential $V'(x)$. The advantage of the gauge choice $\alpha = 0$ is that the path integral only takes values on the classical paths, for integrating over the B field leads to the constraint

$$dx/dt + V'(x) = 0$$

which is the instanton equation. Notice that the anti-instanton $dx/dt - V'(x) = 0$ does not appear here. The solutions denoted $x_c(t)$ of the instanton equation must be such that $x(-\infty) = x_i$ and $x(+\infty) = x_f$. There is an associated zero-mode (for a given $x_c(t)$),

$$x_0 = \lambda\, V\,'(x_c)$$

and by supersymmetry a fermionic zero-mode,

$$\psi_0 = \sigma V'(x_c)$$

where $\lambda$ is a bosonic parameter and $\sigma$ its Grassman superpartner. Notice that both $\lambda$ and $\sigma$ are independant of the time $t$. The zero-modes correspond to the arbitrariness in choosing the centre of the instanton. The instanton moduli space is one dimensional and the fermionic zero-mode represents a tangent to this at a given point $x_c$. This means that $x_c(t)$ is a member of the one parameter family of solutions,

$$x_c(t-\lambda)$$

where $\lambda$ is the centre of the instanton; the zero-mode described above is simply the first term in a Taylor expansion about $\lambda = 0$. The $\psi$ zero-mode is obtained from the general solution by the supersymmetry transformation,

$$\{Q, \lambda\} = \sigma$$

and is given by

$$\psi_c = \sigma\, dx_c(t-\lambda)/d\lambda.$$

Due to the presence of the fermionic zero-mode the partition function vanishes as the ghost number of the vacuum is no longer zero. This is the case as there are no normalizable $\overline{\psi}$ zero-modes, and correspondingly no B zero-modes (as one sees by inspection, or via the supersymmetry transformations). To get a nonzero expectation value we need a function that has Grassman degree one (i.e. that saturates the fermion zero-mode). Let us evaluate $\langle \psi V\,'(x)\rangle$,

$$\langle \psi V\,'(x)\rangle = \int DxD\psi\, DBD\overline{\psi} \quad e^{-S}\, \psi V\,'(x)$$

$$= \int DxD\psi \quad \delta(\dot{x} + V\,')\; \delta(\dot{\psi} + V\,''\psi)\, \psi V\,'(x)$$

We have been able to perform the B and $\overline{\psi}$ integrations precisely as those fields do not suffer from zero-modes. The delta functions ensure that only the classical trajectories will contribute, that is the integration is restricted over the moduli space and its tangent space (at a point). Expanding about the arbitrary classical solution $x_c(t-\lambda)$ and the fermionic zero-mode $\psi_c$ we arrive at

$$\langle \psi V\ '(x) \rangle = \int dx_c d\psi_c \quad \frac{det\,(d/dt + V\ ''(x_c))}{|\,det\,(d/dt + V\ ''(x_c))\,|}\ \psi_c V'(x_c)$$

Under the assumption that no sign ambiguitie arises, the ratio of determinants is one. Furthermore on changing integration variables to $(\lambda, \sigma)$ the Jacobian is unity (as required for the supersymmetry), so that finally one has,

$$\begin{aligned} \langle \psi V\ '(x) \rangle &= \int d\lambda\, d\sigma \quad \sigma\, dx_c(t-\lambda)/d\lambda\ V\ '(x_c(t-\lambda)) \\ &= \int d\lambda \quad dV(x_c(t-\lambda))/d\lambda \\ &= V(-\infty) - V(+\infty) \end{aligned}$$

One gets (minus) the 'winding number' of the instanton, and as such it is about the most topological quantity that is available in supersymmetric quantum mechanics. Yet we have calculated a BRST exact correlation function, $\psi V'(x) = \{Q, V(x)\}$, but

$$\langle \psi V\ '(x) \rangle = \langle\, \{Q, V(x)\}\, \rangle \neq 0.$$

Note that

(i) We have not really incorporated the zero-modes $\lambda$ and $\sigma$ into the set of transformations for the fields. This has been 'implicitly' done in line with all other works on Topological Field Theory. To remedy this we give a complete treatment of the zero-modes in section (2.3).

(ii) In conventional field theoretic calculations a renormalization prescription must be given so that determinants can be properly defined. So we face the requirement of obtaining mathematical formulae as integrals over moduli space while from the physics point of view this is the cause of the symmetry breaking and ought to be avoided. The resolution of this situation is the subject matter of section (2. 4).

(iii) Even though it is BRST exact the correlator computed above is the winding number of the instanton. Hence a topological invariant may be evaluated from a BRST exact term, and this follows from the fact that the fermionic zero-modes break dynamically the symmetry (as the moduli-space is non-compact). This situation persists in all (Witten type) Topological Field Theories which are defined over moduli spaces of non zero dimension. In fact all Q invariant forms $mod\,d$ of degree n ( we count dt, $\psi$, $\dot{\psi}$, ...as being of degree $(1,0)$, $(0,1)$, $(0,1)$, $\ldots$) are $d+Q$ exact. This is the subject to which we now turn.

## 2.2) Trivality of the BRST Cohomology, Observables and Metric Independence.

The cohomology at work in Topological Field Theories is generated by $Q \, mod \, d$. In the example that we are considering the base space is not compact (the real line). This means that we may pick up end point contributions from integrals over $d$ exact forms. This is the analogue of the situation in gauge theories on non compact manifolds where there are Higgs fields, as for example in the three dimensional model proposed in [6,16]. Now any $f(x,\psi)$ that is $Q$ closed is automatically $Q$ exact or a constant (this is a direct consequence of the fact that $H^1(R) = 0$ and $H^0(R) = R$, which extends to Grassman valued forms). The same statement is true for the cohomology generated by $d+Q$: namely if $\mathcal{F}$ is such that $(d+Q)\,\mathcal{F} = 0$ when $\mathcal{F}$ is an n form of the type $(1, n-1) \oplus (0,n)$ then $\mathcal{F} = (d+Q)\,\mathcal{H}$ for some $\mathcal{H}$. Let us decompose $\mathcal{F}$ as,

$$\mathcal{F} = G\,dt + K$$

where G is a $(0, n-1)$ form while K is $(0,n)$. As de-Rham cohomology on the line is trivial we may express $G\,dt$ as $dJ$ for some J (explicitly this is $J = \int^t dsG(s)$). We have then that,

$$(d+Q)\,\mathcal{F} = 0 = (d+Q)\,(dJ + K)$$
$$= -d\,(QJ - K) + QK$$

so that,

$$QJ = K + d\phi \quad , \quad QK = 0$$

However $QJ$ is a $(0,n)$ form as is K , hence $\phi = 0$ and $QJ = K$ which is consistent with $QK = 0$. Thus $\mathcal{F} = (d+Q)\,J$.

This establishes that any topological invariant will be trivial in the local cohomology [8]. However it might be non trivial when evaluated in the path integral, as we saw above. Indeed $\psi\,V'(x) = \{Q, V(x)\}$ is one of the terms in $\mathcal{F} = (d+Q)\,V(x)$.

What about metric independence? The standard argument in Topological Field Theory to prove that the topological invariants are indeed metric as well as coupling constant independent relies on the vacuum being BRST invariant: typically for a metric and coupling constant independent operator $\mathcal{O}$ (recall that $Q$ also enjoys these properties), that is also BRST closed,

$$\frac{\delta\langle\mathcal{O}\rangle}{\delta g} = \langle\mathcal{O}\,\frac{\delta\{Q,W\}}{\delta g}\rangle$$

$$= \langle\{Q,\frac{\delta(\mathcal{O}\,W)}{\delta g}\}\rangle$$

$$= 0$$

where $g$ denotes either the metric or the coupling constants, and the action is expressed as $S = \{Q,W\}$ . The last line in the equation is a consequence of the presumed invariance of the vacuum under the action of $Q$. Now the $g$ independence of the partition function is guaranteed regardless of whether the symmetry is broken or not. If the symmetry is not broken the above deduction is correct, on the other hand if the symmetry is broken the partition function vanishes (owing to the the fermion zero-modes) for all values of $g$. Then the stress energy tensor vanishes accordingly, and this asserts the topological nature of the theory. But what of the other objects that one would like to evaluate? Consider the expectation value of the operator $\mathcal{O}$

$$\delta\langle\mathcal{O}\rangle = \langle\mathcal{O}\,\delta\,S\rangle$$

To get a non zero expectation value, the ghost number of $\mathcal{O}$ must equal the number of fermionic zero-modes. In this instance one may think of $\mathcal{O}$ as providing the correct measure on the moduli space. With this measure the BRST operator is not broken, for the vacuum no longer carries a ghost number and $\prime$ is itself BRST invariant. This means in turn that with $\delta S = \{Q,\delta W\}$ then

$$\delta\langle\mathcal{O}\rangle = \langle\mathcal{O}\{Q,\delta W\}\rangle$$

$$= \langle\{Q,\delta W\}\rangle'$$

$$= 0$$

with the prime denoting the new (bosonic) vacuum. Thus Witten's original proof is still valid within our interpretation of symmetry breaking.

## 2.3) Inclusion of Zero-Modes In the BRST Algebra.

To date our analysis has been rather heuristic. In this section we show how it can be put on a better footing through the explicit incorporation of the zero-modes. To express the topological invariants one wishes to keep the zero-modes, and consequently to have a vanishing partition function and a breakdown of the symmetry: this is the mathematical side. On the other hand, for physicists, the BRST supersymmetry is a guiding light on the renormalization of the theory, and allows for a correct definition of the determinants. The following construction is designed to keep both camps happy.

Indeed let us split off the zero-modes explicitly, and impose on the rest of the fields to be orthogonal to the zero-modes. We first expand about the general instanton solution and set

$$x(t) = x_c(t-\lambda) + x_q(t)$$

$$\psi(t) = -\sigma \frac{dx_c(t-\lambda)}{d\lambda} + \psi_q(t)$$

The field content and BRST action are given by,

$$\{Q, x_q(t)\} = \psi_q(t) \quad \{Q, \psi_q(t)\} = 0$$

$$\{Q, \overline{\psi}(t)\} = B(t) \quad \{Q, B(t)\} = 0$$

$$\{Q, \lambda\} = \sigma \quad \{Q, \sigma\} = 0$$

$$\{Q, \overline{\sigma}\} = \tau \quad \{Q, \tau\} = 0$$

with statistics for the fields $(x_q, \psi_q, \overline{\psi}, B, \lambda, \sigma, \overline{\sigma}, \tau)$ given by $(0, 1, -1, 0, 0, 1, -1, 0)$.

The new $t$ independent fields are introduced so as to be able to gauge fix the zero-modes in $x_q(t)$ and $\psi_q(t)$ . Notice that we have not exhibited those degrees of freedom enjoyed by the zero-modes, *i.e.* $\delta x_q(t) = -\rho dx_c(t-\lambda)/d\lambda$. Unlike the conventional collective coordinate change of variables [17] this is not necessary here. We have more freedom to manouver and have taken advantage of this to minimize the field content*.

The action of interest is,

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* a complete display of the invariance properties with the next least field content reads $\{Q, x_q\} = -\rho dx_c/d\lambda + \psi_q, \{Q, \psi_q\} = \overline{\rho} dx_c/d\lambda - \rho\sigma d^2 x_c/d\lambda^2, \{Q, \rho\} = \overline{\rho}, \{Q, \overline{\phi}\} = \gamma$ where $\overline{\rho}$ and $\gamma$ do not transform and with action $S = \int \{Q, \overline{\psi}(dx/dt + V'(x)) + \overline{\sigma} x_q dx_c/d\lambda + \overline{\phi}\psi_q dx_c/d\lambda\}$.

$$S = \int_{-\infty}^{+\infty} dt \quad \{Q, \overline{\psi}(\frac{dx}{dt} + V\,'(x)) + \overline{\sigma}\, x_q \,\frac{dx_c}{dt}\}$$

$$= \int_{-\infty}^{+\infty} dt \quad B\,(\frac{dx}{dt} + V\,'(x)\,) + \overline{\psi}\,(\frac{d}{dt} + V''(x)\,)\psi$$

$$+\tau\, x_q \,\frac{dx_c}{dt} - \overline{\sigma}\,\psi_q \,\frac{dx_c}{dt} + \overline{\sigma}\, x_q \,\frac{d^2 x_c}{dt^2}\,\sigma$$

The integration over the $B$ field implies that $x$ satisfies the instanton equation. The boundary conditions on $x_q(t)$, namely $x_q(-\infty) = x_q(+\infty) = 0$ then show that it is pure zero-mode, while the integral over $\tau$ asserts that $x_q(t)$ has no zero-mode component. Thus $x_q(t) = 0$. It follows that the $\overline{\psi}$ integration enforces $\psi$ to be also a pure zero-mode, while the integral over $\overline{\sigma}$ finally establishes that $\psi_q$ vanishes (the last step could have been forgone as the vanishing of $x_q(t)$ implies the same for $\psi_q(t)$ by supersymmetry). As in section 2.1 the partition function degenerates into an integration over $\lambda$ and $\sigma$.

It follows that the field content is adequate to give a well defined path integral. Besides the results of the previous sections 2.1 and 2.2 are reobtained in a more rigorous way. Indeed the calculation of $\langle \psi\, V'(x) \rangle$ is replaced here by $\langle (\psi_q + \sigma\,\frac{dx_c}{d\lambda})\, V'(x_c + x_q) \rangle$. Once more the expectation value may be expressed as an expectation of a BRST commutator, and again we conclude that the symmetry is broken. Another advantage of the present formalism useful in the following is the clear distinction between zero and non zero-modes.

## 2.4) Renormalization

For a general Topological Field Theory a well defined renormalization program may be implemented even though there is a breakdown of the BRST symmetry. The main point is that determinants that need to be given a good definition are always associated with non zero-mode integrations. Let the non zero-modes be the quantum fields under consideration, and treat the zero-modes as external fields (this makes sense because of the demarkation made in the previous sub-section). One takes,

$$Z(\lambda^i,\sigma^i,J_a,K_a)=\int \mathcal{D}\Phi_a \quad e^{-S(\Phi_a,\lambda^i,\sigma^i)+\int \Phi_a J^a+\{Q,\Phi_a\}K^a(-1)^f}$$

that is the form of a general Topological Field Theory in any dimension (the $\lambda$'s parametrize the zero-modes and $f$ stands for the fermion number of the source $K_a$ ). The BRST algebra is guaranteed to close in the presence of the zero-modes. Let $\Phi_a$ be the field content when the zero-modes are not explicitly incorporated, then

$$\{Q,\Phi\}=K(\Phi) \quad and \quad \{Q,K(\Phi)\}=0$$

if

$$Q^2=0$$

Now let $\Phi=\Phi_q+\Phi_c(\lambda^i)$ where $\Phi_c(\lambda^i)$ are the classical solutions we are trying to model (so that not all the fields present will have such a classical component). The transformation for $\Phi$ remains as above while that for $\Phi_q$ is determined from it once the rule for $\lambda^i$ is given. Let us take

$$\{Q,\lambda^i\}=\sigma^i \quad \{Q,\sigma^i\}=0$$

The transformation for $\Phi_q$ is determined as follows,

$$\{Q,\Phi_q+\Phi_c\}=K(\Phi_q+\Phi_c)$$

so that

$$\{Q,\Phi_q\}=K(\Phi_q+\Phi_c)-\sigma^i\frac{d\Phi_c}{d\lambda^i}$$

The action of $Q$ again gives zero on the right hand side, so that the BRST operator is seen to be nilpotent on the enlarged set of fields. Once more anti-ghosts and multiplier fields need to be added with the transformation rules,

$$\{Q,\overline{\sigma}^i\}=\tau^i \quad \{Q,\tau^i\}=0.$$

An appropriate gauge fixing term that could be added to the original action would be

$$\{Q,\overline{\sigma}^i\Phi^a_q\Phi^a_{c,i}\}.$$

Having thus explained the content of $Z$ in general, let us pass to the one-particle irreducible generating functional,

$$-\log Z(\lambda^i,\sigma^i,J_a,K_a)=\Gamma(\lambda^i,\sigma^i,\phi^a,K_a)-\int J_a\phi^a$$

$$J^a = \frac{\delta\Gamma}{\delta\phi_a}$$

The BRST invariance of this theory may be expressed as,

$$[\sigma^i \frac{d\Gamma}{d\lambda^i} + \int \frac{\delta\Gamma}{\delta\phi^a}\frac{\delta\Gamma}{\delta K_a}] = 0$$

we also write this as,

$$\{Q, \Gamma\} = 0$$

Perturbation theory is then understood to be performed in the presence of the external fields. The aim of the renormalization programme consists in choosing allowed counterterms of the quantum fields so that the symmetry survives at the quantum level. Now if we find at the (regularized) quantum level that

$$\{Q, \Gamma\} = X$$

then $\{Q, X\} = 0$ as $Q^2 = 0$, and as a consequence of the triviality of the BRST cohomology we obtain $X = \{Q, Y\}$, for some $Y$. So one renormalizes by shifting $\Gamma$ to $\Gamma - Y$ thus re-establishing the symmetry.

## 3) Topological Yang Mills Theory

### 3.1) Classical Cohomology

In the previous section we showed how to incorporate the zero-modes explicitly in the BRST algebra for any Topological Field Theory. As there are some interesting extra features in gauge models, we present now the fields and BRST algebra of the Topological Yang-Mills Theory as introduced in [3,4],

$$\{Q, A\} = \psi + d_A c \quad \{Q, \psi\} = d_A\phi - [\psi, c]$$

$$\{Q, \phi\} = [\phi, c] \quad \{Q, c\} = -\phi - c^2$$

$$\{Q, \chi_+\} = b_+ - [\chi_+, c] \quad \{Q, b_+\} = \eta + [\chi_+, \phi]$$

$$\{Q, \bar{c}\} = b \quad \{Q, b\} = 0$$

$$\{Q, \bar{\phi}\} = \eta + [\bar{\phi}, c] \quad \{Q, \eta\} = [\bar{\phi}, \phi] - [\eta, c]$$

$$Q^2 = 0$$

The fields $(A, \psi, \phi, \chi_+, \overline{\phi}, \eta, c, \bar{c}, b_+, b)$ are forms of degree (1, 1, 0, $2_+$, 0, 0, 0, 0, 0, $2_+$, 0) and ghost number (0, 1, 2, -1, -2, -1, 1, -1, 0, 0) respectively (a $2_+$ designates a self dual 2-form). The argument given in [8] is that with the following field redefinitions,

$$\psi' = \psi + d_A c \quad b_+ - [\chi_+, c]$$

$$\phi' = \phi + c^2 \quad \eta' = \eta + [\overline{\phi}, c]$$

the $Q$ algebra boils down to

$$\{Q, A\} = \psi' \quad \{Q, \psi'\} = 0$$

$$\{Q, \chi_+\} = b'_+ \quad \{Q, b'_+\} = 0$$

$$\{Q, c\} = \phi' \quad \{Q, \phi'\} = 0$$

$$\{Q, \overline{\phi}\} = \eta' \quad \{Q, \eta'\} = 0$$

which has vanishing cohomology and vanishing cohomology *mod d*. Indeed, consider the Donaldson invariants given in [1] as the terms in the expansion according to the ghost and form rank of

$$tr\mathcal{F}^2 \quad where \quad \mathcal{F} = F_A + \psi + \phi$$

The observables satisfy the equation

$$(d + Q)\, tr\, \mathcal{F}^2 = 0$$

However $tr\, \mathcal{F}^2$ may be expressed as

$$tr\, \mathcal{F}^2 = (d + Q) tr\, ((A + c)\mathcal{F} - \frac{1}{3}(A + c)^3)$$

which shows that the proposed invariants are $(d + Q)$ exact. It has been suggested by Kanno (who also gives the relationship of these objects to a Weil algebra structure [18]) and also by Maillet and Niemi [19] that the right hand side of the expression being a generalized Chern-Simons term, has only a local meaning (much as $trF_A^2$ is locally represented by $trd(AF_A - \frac{1}{3}A^3)$). We will in the sequel come back to this point of view and to the question of the breakdown of the supersymmetry.

## 3.2) Quantum Action

The action is to be chosen as a BRST commutator. So the reader may feel that if we stray from 'good' gauge choices we might be led into difficulties, and she would in fact be right. To see the possible pitfalls, we recall that we are interested in the submanifold of $\mathcal{A}/\mathcal{G}$ (the space of (irreducible) connections modulo the gauge group) $\mathcal{M}$ of gauge inequivalent self dual connections. However the path integral is to be performed over the whole space $\mathcal{A}$. One needs to reduce this integration down to $\mathcal{A}/\mathcal{G}$, so that a good gauge slice must be chosen.m Firstly when expanding about a classical solution $A_0$, the background gauge gives a good gauge slice locally [20], that is any connection close enough to the background connection is gauge equivalent to one on the slice and no two connections on the slice (both close enough to $A_0$) are gauge equivalent. The Gribov problem [21] precludes this from being true globally [22]. The gauge function has to be chosen so that the gauge condition obtained is

$$d_{A_0} * (A - A_0) = 0$$

(recall $\mathcal{A}$ is an affine space with $A - A_0 \in \Omega'$ (Lie G), Lie G being the Lie algebra of the Lie group G and $\Omega'$ the space of forms over M with value in Lie G) where $A_0$ is a (fixed) background instanton on $\mathcal{M}$.

Now $d * (A - A_0) = 0$ will not do, yet the algebraic manipulations of the BRST gauge fixing in that case or in the case $d_{A_0} * (A - A_0) = 0$ are identical. Clearly gauge choices must be made in line with the geometrical construction that one wishes to decribe. In particular one is really never just quantizing a zero action (or $TrF_A^2$) but rather at the same time stipulating a whole host of geometrical constructs (see also [23]).

For Yang-Mills theory the geometrical set up, as explained in [3] and [6], is as in the paper of Atiyah and Singer [22]: One considers the principal bundle($(P \times \mathcal{A})/\mathcal{G}, G, M \times \mathcal{A}/\mathcal{G}$). $P$ is a principal bundle over the compact oriented Riemannian four manifold $M$ with structure group $G$, $\mathcal{A}$ is the space of connections on $P$ (which we take as irreducible) and $\mathcal{G}$ is the group of gauge transformations of $P$ connected to identity. The action $\mathcal{G}$ on $P \times \mathcal{A}$ is given by $(p, A) \rightarrow (\phi(p), \phi.A)$ for $\phi \in \mathcal{G}$. Under the assumption that $\mathcal{G}$ leaves a point of the fibre of $P$ fixed then $\mathcal{N} = (P \times \mathcal{A}/\mathcal{G})$ is a principal bundle with group $G$ and base $\mathcal{N}/G = M \times \mathcal{A}/\mathcal{G}$. There is a natural connection on $\mathcal{N}$ with curvature $\mathcal{F}_0$ which is a horizontal two form (a natural bigrading of a $(p, q)$-form refers to a p-form on M and a q-form on $\mathcal{A}/\mathcal{G}$) with values in Lie G. In local coordinates on [5 $M \times \mathcal{A}/\mathcal{G}$, $(p, A)$ the $(2,0)$ component of $\mathcal{F}_0$ is $F_{\mu\nu}(m)$, m being the projection of p ;the (1,1) component is $\delta A_\mu$ where $\delta A \epsilon \Omega^1 \bigotimes LieG$ and $d_A * \delta A = 0$ while the (0,2) component is $(d_A * d_A)^{-1}[\delta A^\mu, \delta A'_\mu]$. Notice that the $\delta A_\mu$ are taken to be tangent to a local cross section of the orbit space $\mathcal{A}/\mathcal{G} \rightarrow \mathcal{A}$ so that they lie in the gauge fixed direction. The final step which takes us to Donaldson's theory is to restrict our attention to the base space $M \times \mathcal{M}$ where $\mathcal{M}$ is the instanton moduli space, $\mathcal{M} \subset \mathcal{A}/\mathcal{G}$.

The aim of Topological Field Theory is to encode this information into a supersymmetric model, which is facilitated by the natural bigrading of the differential forms. First we need to restrict our attention to $\mathcal{A}/\mathcal{G}$: the appropriate choice of gauge is

$$d_{A_0} * (A - A_0) = 0$$

The second ingredient is the tangent space to a point $A \epsilon \mathcal{A}/\mathcal{G}, T(\mathcal{A}/\mathcal{G}, A)$ (for a point $A$ 'near' $A_0$) that can be modeled by Grassman valued one-forms (Lie G valued) $\psi$ so that the bigrading is now that of differential form number and ghost number. But as already noted $\delta A \, \epsilon \, T(\mathcal{A}/\mathcal{G}, A)$ satisfies $d_A * \delta A = 0$ which translates to

$$d_A * \psi = 0$$

Lastly we restrict ourselves to the self dual solutions

$$(1 - *) \, F_A = 0$$

These three canonical gauge choices may be implemented with the following action (in the sequel the trace is to be understood)

$$S = \int \{Q, \chi_+ F_A + \bar{c} \, d_{A_0} * (A - A_0) + \bar{\phi} \, d_A * \psi\}$$

Note that the gauge conditions have not been smeared by terms of the form $\{Q, \alpha\chi b_+ + \beta\bar{c}\, b + \gamma\bar{\phi}\eta\}$. This may be done but as for supersymmetric quantum mechanics will not alter the results. One gets

$$S = \int (b_+ \, F_A - \chi_+ \, d_A\psi + b \, d_{A_0} * (A - A_0) - \bar{c} \, d_{A_0} * \psi$$

$$-\bar{c} \, d_{A_0} * d_A \, c + \bar{\phi} \, d_A * \psi + \bar{\phi} \, [\psi, \psi] + \bar{\phi} \, d_A * d_A \, \phi)$$

When there is no $\psi$ zero-mode $(dimH^0(M,P) = dimH^1(M,P) = dimH^2_+(M,P) = 0)$, this action has been successfully used to compute the partition function of isolated instantons [1]. Here we are interested in the situation where

$dim\,\mathcal{M} = dim H^1(M,P) \neq 0$ while $dim H^0(M,P) = dim H^2_+(M,P) = 0$. (When $dim H^0(M,P) \neq 0$ one has reducible connections which can be dealt with, however $dim H^2_+(M,P) = 0$ may not be relaxed, as it is essential for Taube's proof of the existence of (anti-) self dual solutions[24]).

The integration over the b and $b_+$ fields (there are no zero-modes associated with these fields owing to the above cohomological restrictions) lead to the delta function conditions

$$\int_{\mathcal{A}} \mathcal{D}A \quad \delta((1-*)\,F_A)\,\delta(d_{A_0} * (A - A_0)).$$

They restrict the integration to be over $\mathcal{M}$. If we set $A = a + \delta a$ where $a \,\epsilon\, \mathcal{M}$ then the integral becomes,

$$\int_{\mathcal{M}} \quad |det^{-1}\,T_0|$$

where $T_0 : \Omega^1 \rightarrow \Omega^0 \bigoplus \Omega^2_+$ via $T_0(\delta a) = (d_{A_0} * \delta a, (1 - *)\, d_a\, \delta a)$. The determinant is defined via Schwarz's prescription [25] as $detT_0 = det^{-\frac{1}{2}} T_0\, T_0^{\dagger}$ with $T_0^{\dagger}$ the adjoint operator (this will not be needed in the following).

Likewise the integration over $\chi$ and $\eta$ leads to

$$\int_{T(\mathcal{A}/\mathcal{G},A)} \quad \delta((1-*)d_A\psi)\,\delta(d_A * \psi)$$

which is an integral over the tangent space at a point. With the previous constraint this reduces to

$$\int_{T(\mathcal{M},a)} \mathcal{D}\psi_0 \quad detT$$

where T is given by $T(\psi) = (d_a * \psi, (1 - *)\, d_a\psi)$ and differs from $T_0$ only in the gauge fixing term.

An expectation value of some functional of the fields $\Phi$ (but not depending on b,$b_+$,$\chi$ or $\eta$) boils down to

$$\langle \Phi(A, \psi, \phi, \overline{\phi}, c, \overline{c}) \rangle = \int da d\psi_0 \mathcal{D}\phi \mathcal{D}\overline{\phi} \mathcal{D}c \mathcal{D}\overline{c} \quad \Phi(a, \psi_0, \phi, \overline{\phi}, c, \overline{c})\, e^{S_0}\, \frac{detT}{|detT_0|}$$

with

$$S_0 = \int (-\overline{c}\, d_{A_0} * \psi_0 - \overline{c}\, d_{A_0} * d_a\, c + \overline{\phi}\, d_a * d_a\, \phi + \overline{\phi}\, [\psi_0, \psi_0])$$

### 3.3) Geometrical Significance of $\langle\phi\rangle$ and $\langle c\rangle$

From the gauge conditions $F_A$ and $\psi$ have the interpretations of being the (2,0) and (1,1) components of the curvature on $M \times \mathcal{A}/\mathcal{G}|_{\mathcal{M}}$ respectively. One of the remarkable properties of the Topological Field Theory showed by Witten is that $\langle\phi\rangle$ plays the role of (0,2) component of the curvature. This is a direct consequence of $\overline{\phi}$ coupling to $[\psi_0, \psi_0]$ so that, as the ratio of determinants is unity (this will be proved shortly),

$$\langle\phi\rangle = (d_a * d_a)^{-1}\,[\psi_0^{\mu}, \psi_{0\mu}]$$

Here the necessity of the gauge choice $d_A * \psi = 0$ becomes apparent. Had $d * \psi = 0$ been imposed instead then the identification of the (0,2) component of the curvature would not have been possible.

Now the ghost field also has a non zero vacuum expectation value. This follows from the the antighost $\bar{c}$ coupling to $d_{A_0} * \psi_0$. $\langle c\rangle$ has an important geometrical meaning for the theory at hand. In Witten's original formulation the gauge degrees of freedom were not accounted for. The transformation rule for the field A is $\{Q, A\} = \psi$. Yet on the moduli space the gauge is $d_{A_0} * (A - A_0) = 0$ while the BRST transformation of this (in Witten's model) should be $d_{A_0} * \psi = 0$ which clearly is not the correct gauge chosen for $\psi$. This means that on the moduli space the BRST transformation of A can only equal $\psi$ up to a gauge transformation (that takes $\psi$ to the same gauge surface as A). This is not a flaw in Wittens model as his statements are true modulo gauge transformations.

Here the gauge transformation that is required is $\langle c\rangle$: indeed if $\psi_0$ satisfies $d_A * \psi_0 = 0$ then we can find a $\psi' = \psi_0 + d_A\Lambda$ such that $d_{A_0} * \psi' = 0$. Solving for $\Lambda$ gives

$$\begin{aligned}\Lambda &= (d_{A_0} * d_A)^{-1} d_{A_0} * \psi_0 \\ &= \langle c\rangle\end{aligned}$$

The last line follows from inspection of the action $S_0$. Hence the left hand and right hand sides of the supersymmetry equation $\{Q, A\} = \psi + d_A\, c$ agree on $\mathcal{M}$. For a clear description of the geometrical situation concerning $\mathcal{A}$ and $\mathcal{A}/\mathcal{G}$ see [23].

$\langle c\rangle$ does not play the role of the natural connection $w_A$ on the bundle $\mathcal{N}$ as

$$w_A(\psi) = (d_A * d_A)^{-1} d_A * \psi$$

Rather if we make the shift of variables [8] given in section 3.1, the new action becomes,

$$S = \int b_+F_+ + \chi_+ d_A\psi + \eta d_A * (\psi - d_A c)$$

$$+\bar{c}d_{A_0} * \psi + \overline{\phi} d_A * d_A\phi + \overline{\phi}[\psi, \psi - d_A c] - \overline{\phi} d_A * [\psi, c]$$

and now $\langle c\rangle = w_A(\psi) = (d_A * d_A)^{-1} d_A * \psi$. But the drawback is that $\psi$ is gauge fixed at $A_0$. As we will see in section 4 this is almost Horne's action (modulo some missing terms which will be shown not to influence the path integral but which make an important interpretational contribution).

Returning to the previous form of the action, the mismatch of gauge fixing conditions appears in the determinants that are still to be evaluated. If we evaluate the expectation value of a functional that does not depend on $\overline{\phi}$, or $c$, the integration becomes,

$$\langle\Phi(A,\psi,\phi)\rangle = \int da\, d\psi_0 \quad \Phi\left(a,\psi_0,\langle\phi\rangle\right) \; \frac{det\, d_{A_0} * d_a}{det\, d_a * d_a} \; \frac{detT}{|detT_0|}$$

The ratio of determinants is one;

$$\frac{det\, d_{A_0} * d_a}{|detT_0|} = \frac{det\, d_a * d_a}{detT}$$

Indeed while for isolated instantons $detT = \pm|detT|$ for each seperated instanton, for instantons on $\mathcal{M}$ there is no such sign ambiguity. The overall sign may be declared once and for all to be positive. Now the ratio on the left hand side has a path integral representation as the gauge fixing of,

$$\int \mathcal{D}\chi_+ \mathcal{D}\Sigma \quad e^{\int \chi_+ d_a \Sigma}$$

in the gauge $d_{A_0}\Sigma = 0$, while the right hand side is the same path integral in the gauge $d_a * \Sigma = 0$. By conventional gauge independence arguments the path integral is the same (up to a possible factor) in either gauge. Hence

$$\langle\Phi(A,\psi,\phi)\rangle = \int da\, d\psi_0 \quad \Phi\left(a,\psi_0,\langle\phi\rangle\right)$$

and this is the situation that Witten arrives at in [1]. So we have rederived Witten's results at the level of $\mathcal{M}$.

### 3.4) Inclusion of Zero-Modes In Topological Yang-Mills Theory

A correct treatment of the zero-modes is just as straightforward as in supersymmetric quantum mechanics. Let

$$A = A_q + A_c(\lambda_i)$$

$$\psi = \psi_q + \sigma^i A_{c,i}(\lambda_i)$$

where $\lambda^i$ and $\sigma^i$ have no space-time dependence, and let

$$(1-*)F_{A_c(\lambda)} = 0 \quad d_{A_0} * [A_c(\lambda) - A_0] = 0 \quad with \quad A_c(0) = A_0$$

$$A_{c,i}(\lambda) = \frac{dA_c(\lambda)}{d\lambda^i} \quad d_{A_0} * A_{c,i} = 0$$

$$\int_M A_{c,i} A_{c,j} = \delta_{ij}$$

The algebra is completed with the rules,

$$\{Q, \lambda^i\} = \sigma^i \quad \{Q, \sigma^i\} = 0$$

Clearly the BRST operator is nilpotent on the enlarged set of fields. Two space-time independent fields $(\overline{\sigma}^i, \tau^i)$ still need to be added so as to gauge fix the zero-modes of $A_q$ and $\psi_q$. They transform as

$$\{Q, \overline{\sigma}^i\} = \tau^i \quad \{Q, \tau^i\} = 0$$

and

$$Q^2 = 0.$$

The action must be dictated by geometry plus a term to gauge fix the zero-modes in $A_q$ and $\psi_q$, so that we are not overcounting zero-modes. We thus consider the following action,

$$S = \int \{Q, \chi F_A + \bar{c}\, d_{A_0} * (A - A_0) + \overline{\phi}\, d_A * \psi + \overline{\sigma}^i A_q A_{c,i}\}$$

$$= \int (b_+ F_A - \chi_+ d_A \psi + b\, d_{A_0} * (A - A_0) - \bar{c}\, d_{A_0} * \psi$$

$$-\bar{c}\, d_{A_0} * d_A\, c + \eta\, d_A * \psi + \overline{\phi}\, [\psi, \psi] + \overline{\phi}\, d_A * d_A\, \phi$$

$$+\tau^i\, A_q\, A_{c,i} - \overline{\sigma}^i\, (\psi_q + d_A c)\, A_{c,i} - \overline{\sigma}^i\, A_q\, A_{c,ij}\, \sigma^j)$$

Consider first the constraints on the $A_q$ field. The $b_+$ integration leads to $A_q$ satisfying the linearized equation $(1 - *)\, d_{A_c(\lambda)} A_q = 0$, while the $b$ integration implies $d_{A_0} * A_q = 0$ whence we may expand $A_q$ in the complete set $A_{c,i}$ as $A_q = \theta^i\, A_{c,i}$. Now the term $\int \tau^i\, A_q\, A_{c,i}$ in the action becomes $\int \tau^i\, \theta^j\, A_{c,i}\, A_{c,j} = \tau^i\, \theta_i$. The $\tau^i$ integration sets $\theta^i$ to zero. Consequently $A_q$ vanishes, so that in the rest of the action $A$ may be replaced by $A_c(\lambda)$. Before proceeding with the analysis for the $\psi_q$ field we need to introduce a new complete set of functions $U_i(\lambda)$ satisfying,

$$(1 - *)\, d_{A_c(\lambda)}\, U_i(\lambda) = 0 \qquad \int U_i(\lambda)\, U_j(\lambda) = \delta_{ij}$$

$$and \quad d_{A_c(\lambda)} * U_i = 0$$

The $U_i$ and $A_{c,i}$ are related by a (non-local) gauge transformation,

$$A_{c,i} = \Pi_0 U_i \quad with \quad \Pi_0 = 1 - d_{A_c} (d_{A_0} * d_{A_c(\lambda)})^{-1} d_{A_0} *$$

The integral over $\chi_+$ renders the $\psi_q$ field a pure zero-mode. The $\eta$ integral implies that

$$\psi_q = \xi^i U_i(\lambda) - \sigma^i A_{c,i}(\lambda)$$

Notice that not only $\bar{c}$ couple to a zero-mode but so too does c (specifically in the action $\overline{\sigma} d_{A_c} c A_{c,i}$). Integrating over these ghosts one generates a term

$$\int d_{A_c} * A_{c,i} (d_{A_0} * d_{A_c})^{-1} d_{A_0} * \psi_q \overline{\sigma}^i$$

Add $\int \overline{\sigma}^i \psi_q A_{c,i}$ already present in the action to obtain,

$$\int \overline{\sigma}^i A_{c,i} \Pi_0 \psi_q$$

$$= \int \overline{\sigma}^i\, A_{c,i}\, (\varsigma^j - \sigma^j)\, A_{c,j}$$

$$= \overline{\sigma}^i\, (\varsigma_i - \sigma_i)$$

Finally the integral over $\overline{\sigma}^i$ enforces that $\varsigma^i = \sigma^i$. The difference with the supersymmetric quantum mechanics model is that the $\psi_q$ zero-mode is not set to zero, instead $\psi_q = \sigma^i\,(U_i(\lambda) - A_{c,i}(\lambda))$. $\psi_q$ carries the discrepency between the two gauge conditions. After all these manouvers the path integral takes the form,

$$Z = \int d\lambda\, d\sigma$$

all the determinants having cancelled. Notice that the formal manipulations of the previous sections can now be justified and shown to lead to correct results: indeed one may formally evaluate Donaldson's invariants [26] in this way.

## 4) CONCLUSION AND DISCUSSION

Let us conclude with some remarks concerning the issue of triviality of the observables:

(i) We saw in supersymmetric quantum mechanics that if the dimension of the moduli space is non-zero then the expectation value of a BRST exact operator may not vanish when the moduli space is not compact. Part of Donaldson's work [26] displays that for (anti)- selfdual instantons over a 4-manifold M, the moduli space $\mathcal{M}$ may be compactified by taking the union $\mathcal{M} \cup M$ (M forming the boundary of $\mathcal{M}$). So here too boundary terms for BRST exact objects may be important.

(ii) The ghost $\langle c\rangle$ takes one from the gauge slice determined by the reference point $A_0$, to the intrinsic slice at an arbitrary point A. Because of the Gribov problem the section defined by $A_0$ is not global. Suppose A lies in the overlap of the neighborhoods of $A_0$ and $A_0'$ with local sections defined by them. Then with reference to the two sections the expectation value of c will be

$$\langle c\rangle = \frac{1}{d_{A_0} * d_A} d_{A_0} * \psi$$

$$\langle c\rangle' = \frac{1}{d_{A_0'} * d_A} d_{A_0'} * \psi$$

so that $\langle c\rangle$ depends clearly on the reference point and thus is not globally defined. Notice however that the gauge fixing for $\psi$ does not require an external reference point $A_0$ but is done point by point on $\mathcal{A}$ (indeed $\psi = QA_q - d_A c$

has a gauge invariant meaning). However those Q-exact topological invariants of interest are exact on functions that depend on c. But $\langle c\rangle$ has only local meaning, hence this is another way of seeing that the observables are indeed not trivial, as advocated in [18,19].

(iii) Let us finally show the correspondance in the $\delta$-function gauge of Horne's action with the usual approaches [3,4]. We work directly in the component formalism. Let

$$\delta\chi_{\alpha\beta} = b^{+}_{\alpha\beta}$$

$$\delta A_\alpha = \psi_\alpha$$

$$\delta\xi = \phi$$

$$\delta\overline{\phi} = \eta$$

whereas $b^{+}_{\alpha\beta}, \psi_\alpha, \phi, \eta$ do not transform.

The action reads

$$S = \{\delta, \chi_{\alpha\beta}F_{\alpha\beta} + \overline{\phi} d_A * (\psi - d_A\xi)\}$$

but due to conventional gauge invariance still needs to be gauge fixed. To this end introduce the fields $\overline{c}, b, \overline{\rho}, \overline{\epsilon}, \epsilon$ and $\rho$ with

$$\delta\overline{c} = b$$

$$\delta\overline{\rho} = \overline{\epsilon}$$

$$\delta\epsilon = \rho$$

whereas $b, \overline{\epsilon}, \rho$ do not transform. The gauge transformations $Q^g$ read

$$Q^g A_\alpha = D_\alpha\epsilon \quad Q^g\psi_\alpha = D_\alpha\rho + [\psi_\alpha, \epsilon]$$

$$Q^g\epsilon = -\epsilon^2 \quad Q^g\rho = [\rho, \epsilon]$$

$$Q^g\xi = \rho + [\xi, \epsilon] \quad Q^g\phi = [\phi, \epsilon] + [\xi, \rho]$$

$$Q^g\overline{\rho} = \overline{c} \quad Q^g\overline{c} = 0$$

$$Q^g\overline{\epsilon} = b \quad Q^g b = 0$$

whereas all other fields transform homogeneously in $\epsilon$. $Q^g$ clearly commutes with $\delta$. One adds to the action the gauge fixing term

$$\{\delta, \{Q^g, \overline{\rho} d_{A_0} * (A - A_0)\}\}$$

Now just "unshift" the $\psi, b^+, \phi, \eta$ fields by the inverse field redefinitions advocated in [8], namely $\psi \to \psi + D\xi$ , $b^+ \to b^+ + [\chi, \xi]$, $\phi \to \phi - 1/2[\xi, \xi]$ and $\eta \to \eta + [\phi, \xi]$. Then the gauge fixed action becomes

$$S = b^+ F_A - 2\chi d_A \psi + \eta d_A * \psi - \overline{\phi} d_A * d_A \phi + \overline{\phi}[\psi, \psi] + b d_{A_0} * (A - A_0) +$$

$$\overline{c} d_{A_0} * (\psi + d_A \xi) + \overline{\epsilon} d_{A_0} * d_A \epsilon - \overline{\rho} d_{A_0} * d_A \rho + \overline{\rho} d_{A_0} [\psi + d_A \xi, \epsilon].$$

Notice that a correct gauge fixing term is now obtained for the $\psi$ field, and $\xi$ enters as $c$ does in the formulations of sections (3.2) and (3.3). The ghosts $\rho, \overline{\rho}, \epsilon$ and $\overline{\epsilon}$ play no rôle in the computation of expectation values that do not include them explicitely: indeed assigning an arbitrary charge to $\rho$, the opposite to $\overline{\rho}$, one sees that the interaction term $\overline{\rho} d_{A_0} * [\psi + d_A \xi, \epsilon]$ carries this charge, and thus contributes to expectation values of fields with the appropriate opposite charge. Furthermore the determinants arising from $\rho\overline{\rho}$ and $\epsilon\overline{\epsilon}$ integrations cancel each other. It follows that the action derived here and the ones of the previous sections in essence agree. To write down a topological invariant which is $\delta$ and $Q^g$ closed (but possibly neither $\delta$ nor $Q^g$ exact) we might try $\phi^2$ in Horne's variables. This will not do as it is not $Q^g$ invariant. Instead $(\phi - 1/2[\xi, \xi])^2$ is both $\delta$ and $Q^g$ closed: we then see the essence of the shift of variables back into the form presented in [3,4]. Let us return to the question of triviality: $(\phi)^2$ in the shifted variables is expressable as

$$\phi^2 = \delta(\phi\xi - \xi^3/3)$$

This is part and parcel of the triviality expressed in (3.1). While before one had to resort to explicit computation to exhibit the local nature of the right hand side of this equation (an auxiliary argument), it is now clear that $(\phi\xi - 1/3\xi^3)$ is not $Q^g$ invariant. So even though $\rho, \overline{\rho}, \epsilon$ and $\overline{\epsilon}$ do not enter in the evaluation of expectation values, they lead to their non triviality: so one should not underestimate the importance of these extra ghosts. This exhibits the usefulness of the basic cohomology [8]. While here we have not included the zero-modes, this can be easily done along the lines discussed in previous sections without changing the present analysis. In this work we have only discussed two models, but the generality of the analysis is clear.

Indeed this technics may be applied to all topological field theories including whose introduced in [27].

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