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DYNAMICAL PHASE TRANSITIONS IN SPIN MODELS AND AUTOMATA

DERRIDA B.
CEA Centre d'Etudes Nucleaires de Saclay, 91 - Gif-sur-Yvette (FR).
Service de Physique Theorique

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DYNAMICAL PHASE TRANSITIONS IN SPIN MODELS AND AUTOMATA

Bernard DERRIDA

Service de Physique Théorique† de Saclay

91191 Gif-sur-Yvette Cédex France

Systems in statistical mechanics are usually defined as collections of N objects which interact through an Hamiltonian $\mathcal{H}$. Then one can study the dynamics of these systems by introducing dynamical rules which are compatible with the Hamiltonian $\mathcal{H}$.

One can consider more general systems (automata, neural network models, spin models with non symmetric interactions) which are defined only by dynamical rules without reference to any Hamiltonian. The aim of these lectures is to describe some properties of such systems and a few methods which can be useful to study them. The outline of these lectures is the following:

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† Laboratoire de L'institut de Recherche Fondamentale du Commissariat à l'Energie Atomique

1. INTRODUCTION

Let us start with a few examples of dynamical rules for the time evolution of spin models or automata.

1.A. Examples :

1.A.1. System governed by an Hamiltonian : the Ising Model.

The system consists of N Ising spins $S_i = \pm 1$ which are located on a regular lattice. The energy $\mathcal{H}(C)$ of a spin configuration

$$C = \{S_1, S_2, ..., S_N\} \tag{1}$$

is given by

$$\mathcal{H}(C) = -\sum_{i,j} J_{ij}\; S_i S_j - \sum_i h_i\; S_i \tag{2}$$

where J_{ij} is the interaction between spin i and spin j and h_i is the local field on site i.

To describe the properties of this system in thermal equilibrium at temperature T, one writes that each configuration is occupied with a probability $P_{eq}(C)$

$$P_{eq}(C) = Z^{-1} \exp(-\mathcal{H}(C)/T) \tag{3}$$

where Z is the partition function

$$Z = \sum_C \exp(-\mathcal{H}(C)/T) \tag{4}$$

There are several ways of introducing dynamical rules for this system. Let me first describe the so called sequential heat bath dynamics.

- *The sequential heat bath dynamics*

One starts with some initial spin configuration C_0. Then to make the system evolve from a configuration C_t at time t to a configuration $C_{t+\Delta t}$ at time $t + \Delta t$ (usually one chooses $\Delta t = 1/N$ so that each spin is updated on average once per unit time) one goes through the following steps :

1. Choose a site i at random $1 \leq i \leq N$
2. Compute the probability $p_i(t)$

$$p_i(t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\sum_j \frac{J_{ij}S_j(t) + h_i}{T}\right) \tag{5}$$

3. Choose a random number $z_i(t)$ uniformly distributed between 0 and 1
4. Update the spin S_i according to the following rule

$$S_i(t + \Delta t) = \text{sign}\,(p_i(t) - z_i(t)) \tag{6}$$

and leave the other spins unchanged $S_j(t + \Delta t) = S_j(t)$ for $j \neq i$. One can repeat this procedure again and again. We see that each configuration $C_{t+\Delta t}$ depends on the previous configuration C_t through (5) and on some noise (the choice of the site i to update and the random number $z_i(t)$)

$$C_{t+\Delta t} = F(C_t, \text{Noise}_t) \tag{7}$$

This is an example of *stochastic dynamics*.

The probability $P_t(C)$ of finding the system in a configuration $C = \{S_i\}$ at time t satisfies the following Master equation :

$$P_{t+\Delta t}(C) = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{2} + \frac{S_i}{2}\tanh\left(\sum_j \frac{J_{ij}S_j + h_i}{T}\right)\right][P_t(C) + P_t(C_i)] \tag{8}$$

where C_i is the configuration obtained from C by flipping the spin S_i. One can easily check that the equilibrium (3) is a fixed point of the Master equation (8). In doing that one can notice that it is essential for $P_{\text{eq}}(C)$ to be the fixed point of (8) that the interactions J_{ij} are symmetric

$$J_{ij} = J_{ji} \tag{9}$$

The matrix (8) which gives $P_{t+\Delta t}(C)$ as a function of the $P_t(C')$ satisfies the conditions of the Perron Frobenius theorem[1] (all elements are positive or zero and there is a path from any C to any C'). This implies that there is a single eigenvalue 1 for which the eigenvector is $P_{\text{eq}}(C)$ and that all the other eigenvalues have a modulus less than 1. Therefore, one knows that

$$P_t(C) \longrightarrow P_{\text{eq}}(C) \qquad \text{as} \quad t \longrightarrow \infty. \tag{10}$$

There exist other dynamical rules[2] than the sequential heat bath dynamics (for example the Metropolis algorithm) which lead to the same equilibrium (3). In order to speed up the computer simulation, it is tempting to update several spins simultaneously. Then one can show that the heat bath dynamics always lead to the equilibrium (3) provided that the spins which are updated simultaneously do not interact directly (i.e. i and j can be updated in parallel if $J_{ij} = 0$).

- *Parallel heat bath dynamics*

In the full parallel dynamics all the spins are updated at each time step. So one starts with some initial configuration C_0 and the configuration C_{t+1} is obtained from C_t by

$$S_i(t+1) = \text{sign}\left[\frac{1}{2} + \frac{1}{2}\tanh\left(\sum_j \frac{J_{ij}S_j(t) + h_i}{T}\right) - z_i(t)\right] \tag{11}$$

where the $z_i(t)$ are N random numbers uniformly distributed between 0 and 1.

Here again, one can write a Master equation for $P_t(C)$:

$$P_{t+1}(C) = \sum_{C'}\prod_{i=1}^{N}\left[\frac{1}{2} + \frac{S_i}{2}\tanh\left(\sum_j \frac{J_{ij}S'_j + h_i}{T}\right)\right] P_t(C') \tag{12}$$

and using the Perron Frobenius theorem[1], one knows that in the long time limit

$$P_t(C) \longrightarrow \tilde{P}_{\text{eq}}(C) \tag{13}$$

because all the eigenvalues of (12) except 1 have a modulus less than 1.

Because all the spins are updated simultaneously, the equilibrium $\tilde{P}_{\text{eq}}$ is no longer given by (3) but by[3]

$$\tilde{P}_{\text{eq}}(C) = \text{Constant} \prod_{i=1}^{N} e^{h_i S_i/T}\cosh\left(\sum_j \frac{J_{ij}S_j}{T} + \frac{h_i}{T}\right) \tag{14}$$

Here again, to check that (14) is the fixed point of (12), one needs to use the symmetry (9) of the interactions.

1.A.2. Systems without Hamiltonian

- ***Neural network models; non symmetric spin glasses***

The simplest neural network models are defined as systems of N neurons $S_i(t) = \pm 1$ which interact through synapses J_{ij}. One can choose for neural network models the same stochastic dynamics as for spin models :

$$S_i(t + \Delta t) = \text{sign}\left[\frac{1}{2} + \frac{1}{2}\tanh\left(\sum_j \frac{J_{ij}S_j(t) + h_i}{T}\right) - z_i(t)\right] \tag{15}$$

with $\Delta t = 1/N$ for sequential dynamics and $\Delta t = 1$ for parallel dynamics and where $z_i(t)$ is a random number uniformly distributed between 0 and 1. The only difference with spin models is that the interactions are in general non symmetric

$$J_{ij} \neq J_{ji} \tag{16}$$

Although the time evolution of $P_t(C)$ is still given by (8) or (12), and that we know from the Perron Frobenius theorem that $P_t(C) \longrightarrow P_{eq}(C)$, the equilibrium is no longer given by (4) or (14) and in general one does not know the expression giving $P_{eq}(C)$ in terms of the J_{ij} and the h_i.

So for systems with non symmetric interactions, the study of the equilibrium does not look simpler than the study of the non equilibirum.

- *Networks of automata*

Automata are other examples of systems wich evolve without Hamiltonian. In the simplest cases, they are defined as systems of N spins $S_i = \pm 1$ with K input sites $j_1(i), ..., j_K(i)$ and a Boolean function f_i associated to each site i. Then the system evolves according to the following rule

$$S_i(t + 1) = f_i(S_{j_1(i)}(t), ..., S_{j_K(i)}(t)) \tag{17}$$

Since the functions f_i and the input sites $j_1(i)...j_K(i)$ of each site i are fixed, the system is deterministic. Therefore (17) defines a map in phase space

$$C_{t+1} = F(C_t) \tag{18}$$

An example of automata is clearly the case of spin models or neural networks with parallel dynamics at zero temperature. Then (17) takes the following form

$$S_i(t+1) = \text{sign}\left(\sum_j J_{ij} S_j(t) + h_i\right) \tag{19}$$

1.B. Attractors and valleys

In the above examples, we have seen two kinds of dynamics : deterministic dynamics and stochastic dynamics:

1.B.1. Deterministic dynamics (Ising model or neural networks with parallel dynamics at zero temperature, automata).

Then, the configuration C_{t+1} depends only on the previous configuration

$$C_{t+1} = F(C_t) \tag{20}$$

For deterministic dynamics (eq.(20)), the time evolution always ends up by becoming periodic. This is because phase space is finite (2^N different configurations). After a time $t > 2^N$, the system must have visited twice the same configuration, say at times t_1 and t_2

$$C_{t_1} = C_{t_2}$$

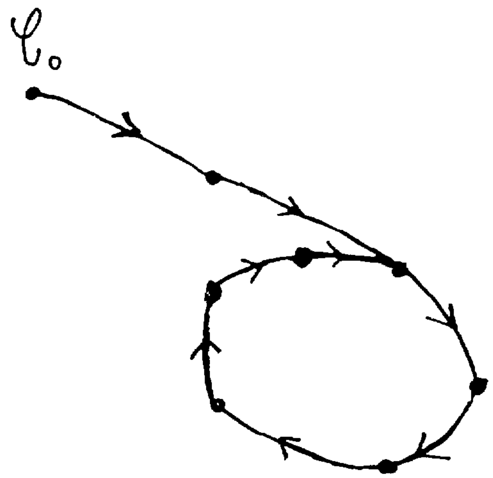


FIGURE 1

The trajectory of a configuration in phase space

Since the dynamics are deterministic, one has for any later time τ

$$C_{t_1+\tau} = F^\tau(C_{t_1}) = F^\tau(C_{t_2}) = C_{t_2+\tau} \tag{21}$$

There are in general (and even for finite N) several periodic attractors : phase space is broken into the *basins of attraction* (that we will call *valleys*) of these different attractors. The only a priori restriction on the period is that it is less than 2^N.

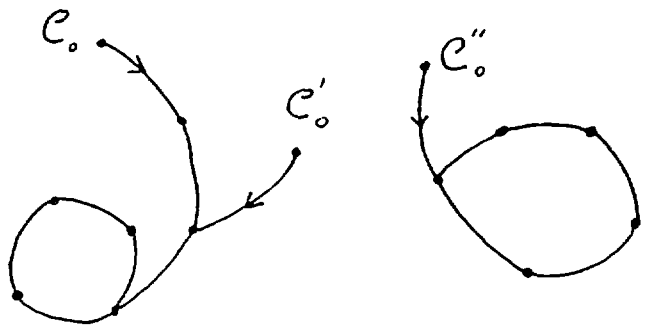


FIGURE 2
Two randomly chosen configurations can either belong to the same valley ($\mathcal{C}_0$ and $\mathcal{C}'_0$) or to distinct valleys ($\mathcal{C}_0$ and $\mathcal{C}''_0$)

It is probably in the case of deterministic dynamics that the difference between systems which evolve according to an Hamiltonian (zero temperature parallel dynamics with symmetric interactions $J_{ij} = J_{ji}$) and systems which evolve without Hamiltonian (non symmetric J_{ij}, automata) is the most apparent. For systems which evolve according to an Hamiltonian, one can introduce an energy function (often called a Lyapunov function) which decreases with time. The consequence is that the cycle length of Hamiltonian systems is always very small. We are now going to show that it is at most 2 for spin models at zero temperature with parallel dynamics and symmetric interactions ($J_{ij} = J_{ji}$).

The spins evolve according to

$$S_i(t+1) = \text{sign}\left(\sum_j J_{ij} S_j(t)\right) \tag{22}$$

If one defines $E(t)$ by

$$E(t) = -\sum_{ij} J_{ij} S_i(t) S_j(t+1) \tag{23}$$

One can easily see (using the symmetry $J_{ij} = J_{ji}$) that there are two ways of writing $E(t)$

$$E(t) = -\sum_i g_i(t)S_i(t+1) = -\sum_i g_i(t+1)S_i(t) \tag{24}$$

where

$$g_i(t) = \sum_j J_{ij}S_j(t) \tag{25}$$

Because the dynamics (22) can be written as

$$S_i(t+2) = \text{sign}(g_i(t+1)) \tag{26}$$

One can see from (24) that

$$E(t+1) = -\sum_i |g_i(t+1)| \tag{27}$$

by comparing (24) and (27) one gets that

$$E(t+1) - E(t) = -\sum_i |g_i(t+1)|\,[1 - S_i(t)S_i(t+2)] \tag{28}$$

So $E(t)$ is a decreasing function of time. Because phase space is finite, it exists a time t_0 such that $E(t_0+2) = E(t_0)$ which implies from (28) that

$$S_i(t_0+2) = S_i(t_0) \qquad \forall \;\; i \tag{29}$$

This means that the system has reached a cycle 2.

One should notice that the same problem with non symmetric J_{ij} ($J_{ij} \neq J_{ji}$) can have any cycle length. The difference between symmetric and non symmetric interactions is that with non symmetric J_{ij}, $E(t)$ cannot be written in the two forms as in (24).

1.B.2. Stochastic dynamics (Ising model or neural networks at non zero temperature)

For stochastic dynamics the configuration $\mathcal{C}_{t+1}$ depends on the previous configuration $\mathcal{C}_t$ and on some stochastic variable Noise_t

$$\mathcal{C}_{t+1} = F(\mathcal{C}_t, \text{Noise}_t) \tag{30}$$

Usually stochastic dynamics allow a finite system to go from any configuration to any other configuration. This means that the phase space of a finite system consists of a single valley. It is only in the thermodynamic limit ($N \longrightarrow \infty$) that one can observe well defined valleys.

For finite systems at non zero temperature one can only define quasi attractors and the number of these attractors depends on the time scale. To illustrate this fact let us take a simple example : the infinite range ferromagnetic Ising model which is governed by the following Hamiltonian

$$\mathcal{H} = -\frac{1}{N} \sum_{i<j} S_i S_j \tag{31}$$

As usual with infinite range models one has to normalize the interactions with N in order to keep a temperature scale independent of N.

To describe the time evolution of this system, one could write the Master equation (8). However, because the system is infinite range one can write directly a Master equation for the total magnetization M of the system :

$$\begin{aligned} P_{t+1/N}(M) &= \frac{N+M+2}{2}\left[\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{M+1}{NT}\right)\right] P_t(M+2) \\ &+ \frac{N+M}{2}\left[\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{M-1}{NT}\right)\right] P_t(M) \\ &+ \frac{N-M}{2}\left[\left(\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{M+1}{NT}\right)\right)\right] P_t(M) \\ &+ \frac{N-M+2}{2}\left[\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{M-1}{NT}\right)\right)\right] P_t(M-2) \end{aligned} \tag{32}$$

One can diagonalize this matrix for finite N (here $N = 100$). If one calls $(\lambda_i)^{1/N}$ its eigenvalues (one introduces this $1/N$ because $\Delta t = 1/N$), one gets for the largest eigenvalues the temperature dependence shown on figure 3 :

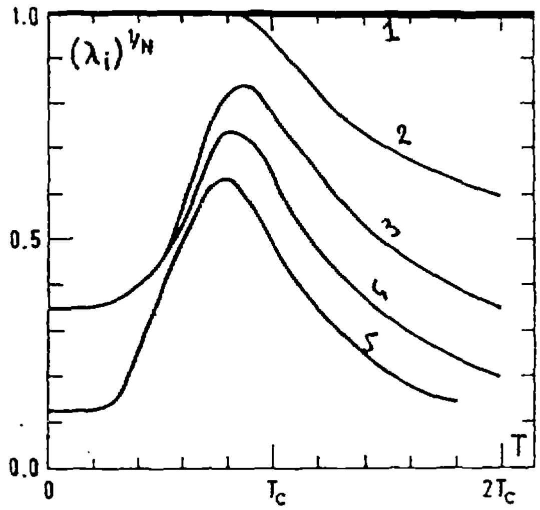


FIGURE 3

The eigenvalues of the Master Equation (32) versus temperature

We see that there is always an eigenvalue 1 corresponding to the normalization $\sum_M P_t(M) = 1$. For $T > 1$, the other eigenvalues are far from 1 : there is a single valley and the system is paramagnetic.
For $T < 1$, the second eigenvalue λ_2 is very close to 1 (it would be degenerate to 1 only for $N = \infty$). This means that for time scales t such that $(\lambda_2/\lambda_1)^t$ close to 1, the system behaves as if there were two attractors. It is only for times such that $(\lambda_2/\lambda_1)^t \ll 1$ that one observes a single valley.

We see with this example that for stochastic dynamics at non zero temperature the number of attractors depends on the time scale. In the limit $t \longrightarrow \infty$, for a finite system, one always has a single valley. However for short enough times (if $(\lambda_2/\lambda_1)^t$ is close enough to 1), the system behaves as if there were several attractors.

1.C. Mean fields models

It is possible to build a family of models with non symmetric interactions for which the dynamics can be calculated analytically. These models are defined as systems of N Ising spins $S_i(t) = \pm 1$ which evolve in time. For each site i, K input sites $j_1(i), ..., j_K(i)$ are chosen at random among the N sites. Then the spins evolve according to

$$S_i(t+1) = \text{sign}\left[\frac{1}{2} + \frac{1}{2}\tanh\left(\sum_j \frac{J_{ij}S_j(t) + h_i}{T}\right) - z_i(t)\right] \tag{33}$$

where in (33) the sum over j runs over the K input sites of site i and $z_i(t)$ is as above a random number uniformly distributed between 0 and 1.

The only reason[14,40] which makes these models exactly soluble is that the K inputs of each site are chosen at random among the N sites. This implies that in the limit $N \longrightarrow \infty$, the K^t sites which belong to the tree of ancestors of a given site i are all different (this is true as long as $t \ll \log N$).

Since to compute the value of a spin $S_i(t)$ at time t from the initial condition $\{S_j(0)\}$, only the K^t ancestors of site i are involved and because with probability 1, there is no loop in this tree of ancestors, at each time step, the inputs $j_1(i)...j_K(i)$ of each site i are not correlated (one assumes that the spins $S_i(0)$ are not correlated at time $t = 0$). Therefore if one define $m_i(t)$ by

$$m_i(t) = \langle S_i(t) \rangle \tag{34}$$

where in (34), the average $\langle \quad \rangle$ means an average over the initial condition $\{S_i(0)\}$ and over the history (i.e. the $z_i(t)$), one can deduce from (33) :

$$m_i(t+1) = \sum_{\sigma_1=\pm1} \cdots \sum_{\sigma_K=\pm1} \prod_{r=1}^{K} \frac{1+m_{j_r}(t)\sigma_r}{2} \tanh\left(\sum_j \frac{J_{ij_r}\sigma_r + h_i}{T}\right) \tag{35}$$

Equation (35) expresses the fact that the ancestors $S_{j_r}(t)$ of site i are not correlated at time t. This is a consequence of the fact that the tree of ancestors has no loop and the spins are not correlated at $t = 0$. The equation (35) was first obtained in the case of the Kauffman model[7] for which the time evolution (33) is replaced by a random Boolean function. It is however much more general since its validity depends only on the absence of correlations in the initial condition and of the fact that the inputs of each site are chosen at random.

In order to illustrate these equations, let us discuss two examples :

1.C.1. The ferromagnetic case

- It is the special case where the K interactions J_{ij} of each site i are equal

$$J_{ij} = J \tag{36}$$

Then by choosing the initial condition with $m_i(t)$ independent of i, one sees from (35) that at any later time t, the solution remains uniform :

$$m_i(t) = m(t) \tag{37}$$

and that the magnetization $m(t)$ evolves according to

$$m(t+1) = \sum_{n=0}^{K} \binom{K}{n} \left(\frac{1+m(t)}{2}\right)^n \left(\frac{1-m(t)}{2}\right)^{K-n} \tanh\left(\frac{(2n-K)J+h}{T}\right) \tag{38}$$

It is important to notice that the fact that the inputs $j_1(i)...j_K(i)$ are chosen at random is essential to derive (38). This model is clearly a model with non symmetric interactions since when a site j is an ancestor of a site i, it is very unprobable for large N that i is an ancestor of j.

The corresponding model with symmetric interactions which would be a usual tree with $J_{ij} = J_{ji} = J$ is a much more difficult problem for which one does not know the analytic expression of the magnetization as a function of time.

From (38), one sees that in zero field ($h = 0$), the system can either be in a paramagnetic phase (if $m = 0$ is the attractive fixed point) or in a ferromagnetic phase (if there is a pair of attractive fixed points $\pm m^*$ with $m^* > 0$). The transition temperature T_c below which the system is ferromagnetic can be computed from (38) by linearizing around $m = 0$ and one finds that T_c is solution of

$$1 = \sum_{n=0}^{K} \binom{K}{n} \frac{(2n-K)}{2^K} \tanh\left(\frac{(2n-K)J}{T_c}\right) \tag{39}$$

1.C.2. Chaotic phase in a non symmetric spin glass[6,8]

Consider again a system a N spins $S_i(t) = \pm 1$ with K random inputs $j_1(i)...j_K(i)$ associated to each site i. Let us describe the dynamical properties of this system in the case of parallel dynamics at zero temperature (deterministic dynamics)

$$S_i(t+1) = \operatorname{sign}\left(\sum_j J_{ij} S_j(t) + h_i\right) \tag{40}$$

where the interaction J_{ij} and the local fields h_i are randomly distributed according to Gaussian distributions

$$\rho(J_{ij}) = \frac{1}{\sqrt{2\pi}} \exp - \left(\frac{J_{ij}^2}{2} \right) \tag{41}$$

$$\rho(h_i) = \frac{1}{\sqrt{2\pi\Delta}} \exp - \left(\frac{h_i^2}{2\Delta} \right) \tag{42}$$

We are going to show that there exists a critical value Δ_c of Δ such that for $\Delta < \Delta_c$ the system has a chaotic behavior. Let us compare two configurations $\{S_i(t)\}$ and $\left\{\tilde{S}_i(t)\right\}$ which evolve according to the same rules (40) :

$$\begin{aligned} S_i(t+1) &= \text{sign}\left(\sum_j J_{ij} S_j(t) + h_i \right) \\ \tilde{S}_i(t+1) &= \text{sign}\left(\sum_j J_{ij} \tilde{S}_j(t) + h_i \right) \end{aligned} \tag{43}$$

To do so, we can define the distance $d(t)$ between these 2 configurations

$$d(t) = \frac{1}{2N} \sum_{i=1}^{N} \left| S_i(t) - \tilde{S}_i(t) \right| \tag{44}$$

This distance is the fraction of spins which are different in the two configurations. One can show that the time evolution of this distance is given by

$$d(t+1) = \sum_{n=0}^{K} \begin{pmatrix} K \\ n \end{pmatrix} [1 - d(t)]^{K-n} [d(t)]^n \, \frac{2}{\pi} \tan^{-1} \left(\sqrt{\frac{n}{K - n + \Delta}} \right) \tag{45}$$

Formula (45) can easily be understood. At time t, a given site i has a probability $\begin{pmatrix} K \\ n \end{pmatrix} (1-d)^{K-n} d^n$ of having n inputs different in the two configurations. Then for such a site, one can write (44) as

$$\begin{aligned} S_i(t+1) &= \text{sign}(X + Y) \\ \tilde{S}_i(t+1) &= \text{sign}(X - Y) \end{aligned} \tag{46}$$

where X and Y are Gaussian variables of width $\Delta + K - n$ and n. Then the probability that $S_i(t+1) \neq \tilde{S}_i(t+1)$ is given by

$$\text{Prob}\left(S_i(t+1) \neq \tilde{S}_i(t+1)\right) = \text{Prob}(|X| < |Y|) = \frac{2}{\pi}\tan^{-1}\left(\sqrt{\frac{n}{K-n+\Delta}}\right) \tag{47}$$

To obtain (47), we have used the fact that X and Y are sums of independent random variables. This is justified because the inputs $S_{j_1}(t)...S_{j_K}(t)$ are uncorrelated.

From (45), we see that $d = 0$ is always a fixed point. Clearly, if two configurations are identical, they remain identical for ever. When Δ varies, this fixed point becomes unstable for $\Delta < \Delta_c$ where Δ_c is solution of

$$1 = \sum_{n=0}^{K} \binom{K}{n} n \, \frac{2}{\pi} \tan^{-1}\left(\sqrt{\frac{1}{K-1+\Delta_c}}\right) \tag{48}$$

and a new attractive fixed point d^* appears. Since $d = 0$ is unstable, one can conclude that two configurations which are initially very close have trajectories which diverge implying that the system is chaotic.

It is interesting to notice that for these simple mean field models, the distance between 2 configurations converges to a fixed value d^* independent of their initial distance.

2. NETWORKS OF RANDOM AUTOMATA

In this chapter we are going to consider a single class of models : networks of random automata. The system consists of N spins $S_i(t) = \pm 1$. Each spin S_i receives inputs from K other sites $j_1(i)...j_K(i)$ and evolves according to a random Boolean function f_i of K variables

$$S_i(t+1) = f_i\left(S_{j_1(i)}(t), ..., S_{j_K(i)}(t)\right) \tag{49}$$

There are 2^{2^K} different Boolean functions of K variables : for example for $K = 1$, there are 4 Boolean functions g_1, g_2, g_3 and g_4 :

σ	$g_1(\sigma)$	$g_2(\sigma)$	$g_3(\sigma)$	$g_4(\sigma)$
−	+	+	−	−
+	+	−	+	−

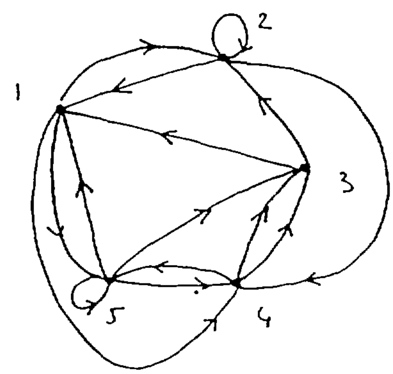
Here, we choose for each site i a random Boolean f function with a probability $p^n(1-p)^{2^K-n}$ where n is the number of times that the function takes the value -1. (In the above example g_1 would be chosen with probability $(1-p)^2$, g_2 and g_3 with probability $p(1-p)$ and g_4 with probability p^2). For $p = 1/2$, all the Boolean functions have equal weights.

In this chapter we will first consider the mean field case (Sections A, B and C) for which the inputs are $j_1(i)...j_K(i)$ of each site i are chosen at random among the N sites. Then we will discuss the finite dimensional case where the inputs are the neighbors of a site on a regular lattice (Section D).

2.A. The Kauffman model

The Kauffman model was introduced as a model for cell differentiation. It describes a system of N genes $S_i = \pm 1$ (+1 if the gene is on and −1 if it is off). The activity of a gene i is influenced by K other genes $(j_1(i), j_2(i)...j_K(i))$.Each gene is influenced by other genes which are usually far away and it is a reasonable approximation to assume that $j_1(i)...j_K(i)$ are chosen at random among the N genes. Then another simplifying assumption is to take a discrete time and random Boolean functions f_i to make evolve the gene activities.

So in the Kauffman model the $S_i(t)$ evolve according to (49) with random Boolean functions f_i and random inputs $j_1(i)...j_K(i)$.



i	$j_1(i)$	$j_2(i)$	$j_3(i)$
1	2	3	5
2	3	2	1
3	4	4	5
4	2	5	1
5	1	5	4

FIGURE 4
Example of the connections in the Kauffman model for $N = 5$ sites and $K = 3$

The first results[9−11] obtained by Kauffman concerned the periods of the attractors. Kauffman studied numerically how the period of an attractor depends on the system size N. He found two regimes when $p = 1/2$: for $K \leq 2$

$$\text{Period} \sim N^x \tag{50}$$

whereas for $K > 2$, the period increases exponentially with N

$$\text{Period} \sim e^{N\alpha} \tag{51}$$

Up to now, there does not exist any exact analytical calculation of the period except for $K = 1$ [12,13]. In particular the value of α (in eq.(51)) as a function of K has not yet been calculated analytically (see the conclusion for an attempt).

Because this model belongs to the class of mean field models described in 1.C. (the inputs are chosen at random), there are some quantities for which the time evolution can be calculated exactly. As in example 1.C.2., one can calculate the time evolution of the distance $d(t)$ between two configurations $\{S_i(t)\}$ and $\left\{\tilde{S}_i(t)\right\}$:

$$d(t) = \frac{1}{2N} \sum_{i=1}^{N} \left| S_i(t) - \tilde{S}_i(t) \right| . \tag{52}$$

By using exactly the same reasoning as in 1.C.2., one obtains[14,15] that

$$d(t+1) = 2p(1-p)\left(1 - (1 - d(t))^K\right) \tag{53}$$

Here again, the probability that a spin i has all its inputs the same in the two configurations is $(1 - d(t))^K$. For the spin i to be different at time $t + 1$, first we need that not all the inputs are the same (probability $= 1-(1-d)^K$ and then that f gives two different outputs (probability$=2p(1-p)$). From (54) we see that there is a critical line

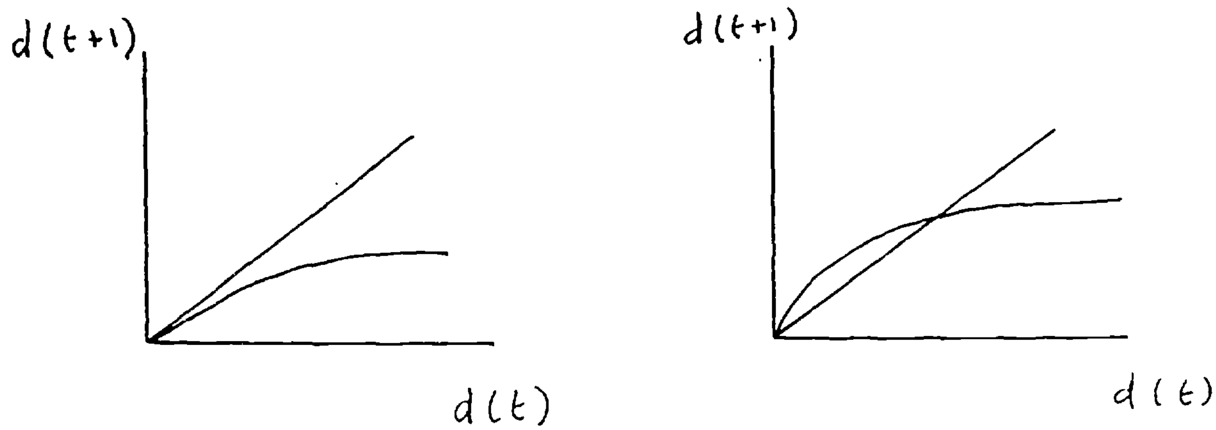
$$K_c(p) = \frac{1}{2p(1-p)} \tag{54}$$

For $K < K_c$, the only (attractive) fixed point is $d = 0$. So for any initial distance $d(0)$, the final distance vanishes. The system forgets completely its initial condition since different initial configurations always end up by becoming identical. By choosing $\left\{\tilde{S}_i(0)\right\} = \{S_i(1)\}$, we see that for almost all i one has in the limit $t \longrightarrow \infty$

$$S_i(t) = \tilde{S}_i(t) = S_i(t+1) \tag{55}$$

and therefore (almost) all spins are fixed in time. Since almost all spins are fixed in time, one can say that the system is in *its frozen phase.*

For $K > K_c$, the fixed point $d = 0$ becomes unstable and there appears a new attractive fixed point $d^* \neq 0$. Two initial conditions, even if they are very close, remain different. In the long time limit their distance $d(t)$ does not depend on their initial distance and converges to d^*, the attractive fixed point of (53). Since close trajectories in phase space have the tendency to diverge, one can say that $K > K_c(p)$ corresponds to the *chaotic phase*.



Frozen phase $K < K_c(p)$

Chaotic phase $K > K_c(p)$

FIGURE 5

2.B. Distribution of activities in the Kauffman model[7,16]

There are other quantities which can be computed exactly for the Kauffman model.

If one defines $m_i(t)$ as

$$m_i(t) = \langle S_i(t) \rangle \tag{56}$$

where $\langle \ \ \rangle$ means an average over an ensemble of initial conditions and if the spins are chosen to be uncorrelated in the initial condition, i.e. the probability of starting with some initial condition $\{S_i\}$ is

$$\prod_{i=1}^{N} \left(\frac{1 + m_i(0) S_i}{2} \right) \tag{57}$$

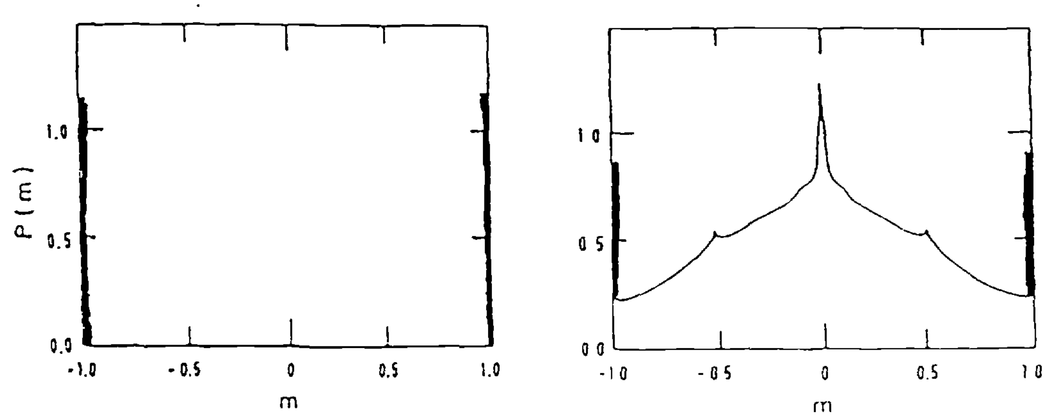
we have seen in 1.C. that almost all the $m_i(t)$ will satisfy (in the limit $N \longrightarrow \infty$)

$$m_i(t+1) = \sum_{\sigma_1 = \pm 1} \cdots \sum_{\sigma_K = \pm 1} \prod_{r=1}^{K} \left(\frac{1 + m_{j_r}(t)\, \sigma_r}{2} \right) f_i(\sigma_1, ..., \sigma_r) \tag{58}$$

From (58) one can write a recursion[7] for the distribution of activities $P_t(m)$ which is defined by

$$P_t(m) = \frac{1}{N}\sum_{i=1}^{N}\delta(m - m_i(t)) \tag{59}$$

In the limit $t \longrightarrow \infty$, $P_t(m)$ converges to a stationnary distribution $P_\infty(m)$ which is the solution of an integral equation[7]. For the Kauffman model one finds two sorts of shapes :



Frozen phase

Chaotic phase ($K = 3$, $p = 1/2$)

FIGURE 6

In the frozen phase $K < K_c(p)$, $P_\infty(m)$ consists of the sum of two delta functions at $+1$ and at -1 : almost all the spins have a value independent of their initial conditions.

For $K > K_c(p)$, in addition to the two delta functions at $m = \pm 1$, one has a continuous part in $P_\infty(m)$.

There is another definition of $P_\infty(m)$ which gives exactly the same distribution. If one defines m_i as the time average of $S_i(t)$ for a fixed initial condition

$$m_i = \lim_{t_0 \longrightarrow \infty} \frac{1}{t_0}\sum_{t=1}^{t_0} S_i(t) \tag{60}$$

and if $P(m)$ is the distribution of these m_i

$$P(m) = \frac{1}{N}\sum_{i=1}^{N}\delta(m - m_i) \tag{61}$$

one can show[7] that $P(m)$ satisfies exactly the same integral equation as $P_\infty(m)$ and therefore has the shapes given in Figure 6. The delta functions at $m = \pm 1$ correspond to the spins which do not move whereas the continuous part in $P(m)$ represents the moving spins.

2.C. The effect of noise

For all the automata models defined by (49), one can be interested to study the effect of an external random noise. To do so, one replaces the time evolution (49) by a stochastic rule :

$$\begin{aligned} S_i(t+1) &= f_i\left(S_{j_1(i)}(t), \ldots, S_{j_K(i)}(t)\right) \quad \text{with probability } \frac{1}{2} + \frac{1}{2}\tanh\frac{1}{T} \\ S_i(t+1) &= -f_i\left(S_{j_1(i)}(t), \ldots, S_{j_K(i)}(t)\right) \quad \text{with probability } \frac{1}{2} - \frac{1}{2}\tanh\frac{1}{T} \end{aligned} \tag{62}$$

The parameter T in (62) plays the role of a temperature. Clearly in the zero temperature limit, one recovers the deterministic case (49).

To implement (62), one chooses at each time step N random numbers $z_i(t)$ uniformly distributed between 0 and 1 and one updates $S_i(t+1)$ by

$$S_i(t+1) = \text{sign}\left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{T}\right) f_i\left(S_{j_1}(t), \ldots, S_{j_K}(t)\right) - z_i(t)\right) \tag{63}$$

We see that as in other stochastic cases, a configuration $C_{t+1} = \{S_i(t+1)\}$ depends on the previous configuration $C_t = S_i(t)$ and on the thermal noise $\text{Noise}_t = z_i(t)$.

$$C_{t+1} = F(C_t, \text{Noise}_t) \tag{64}$$

So the configuration C_t of the system at time t depends on its initial condition C_0 and on the thermal noise between times 0 and $t-1$. To study the respective effects of the initial condition and of the thermal noise, one can measure the distance $d(t)$ between two configurations (52), but there are several distances which can be measured:

2.C.1. Uncorrelated noise

First one can measure the distance between two configurations which are subjected to uncorrelated noises i.e.

$$\begin{aligned} S_i(t+1) &= \text{sign}\left[\frac{1}{2} + \frac{1}{2}\tanh\frac{1}{T} f_i\left(\{S_j(t)\}\right) - z_i(t)\right] \\ \tilde{S}_i(t+1) &= \text{sign}\left[\frac{1}{2} + \frac{1}{2}\tanh\frac{1}{T} f_i\left(\left\{\tilde{S}_j(t)\right\}\right) - \tilde{z}_i(t)\right] \end{aligned} \tag{65}$$

For the Kauffman model and again because the inputs of each site i are uncorrelated, one can obtain an exact expression for the time evolution of $d(t)$. One finds that for $p = 1/2$

$$d(t+1) = \frac{1}{2}\left(1 - (1 - d(t))^K \tanh^2 \frac{1}{T}\right) \tag{66}$$

In the limit $T \longrightarrow 0$, one recovers (53) (for $p = 1/2$). There is not much to say on $d(t)$ in this case because there is always an attractive fixed point of (66) which changes smoothly with T without any trace of a phase transition.

2.C.2. Same noise

Another way of defining the distance between two configurations consists of using the same noise for the two configurations

$$z_i(t) = \tilde{z}_i(t) \tag{67}$$

One can again calculate the time evolution of the distance $d(t)$ in this case and one finds for $p = 1/2$

$$d(t+1) = \left(1 - (1 - d(t))^K\right) \frac{1}{2} \tanh \frac{1}{T} \tag{68}$$

$d = 0$ is always a fixed point : two identical configurations remain identical if they are subjected to the same thermal noise. As for uncorrelated noise one recovers (53) in the limit $T \longrightarrow 0$.

One can see in (68) that there is a dynamical phase transition at a temperature T_c given by

$$\frac{K}{2} \tanh \frac{1}{T_c} = 1 \tag{69}$$

For $T > T_c$, $d = 0$ is an attractive fixed point and two different initial configurations always become identical. The system forgets quickly its initial condition.

For $T < T_c$, $d = 0$ becomes an unstable fixed point and a new attractive fixed point $d^* \neq 0$ appears. The trajectories of two initially close configurations (if their initial distance is non zero) diverge and their distance converges to d^*.

2.C.3. Damage spreading

Even if the initial distance is zero (for example if the two configurations differ initially by a finite number of spins), there is a finite probability that in the long time limit the two configurations end up at the distance d^*. If one defines the survival probability $P(T)$ that two configurations which differ initially by a single spin are still different in the long time limit, one can show that $P(T) \neq 0$ for $T < T_c$ and that $P(T)$ is the non zero solution of

$$P(T) = 1 - \exp\left(-\frac{K}{2} P(T) \tanh \frac{1}{T}\right) \tag{78}$$

This expression can be understood by writing a recursion for $Q_m(t)$, the probability that 2 configurations which differ initially by m spins have become identical before or at the t$^{\text{th}}$ time step. For $p = 1/2$, the recursion has the following form

$$Q_m(t+1) = (c_m)^N + \sum_{n \geq 1}^{N} \frac{N!}{n!(N-n)!} (1 - c_m)^n c_m^{N-n} Q_n(t) \tag{79}$$

where c_m is the probability that a randomly chosen site will be identical at time $t+1$ knowing that the two configurations differ at time t by m spins

$$c_m = \left(1 - \frac{m}{N}\right)^K + \left(1 - \left(1 - \frac{m}{N}\right)^K\right) \left(\frac{1}{2} + \frac{1}{2}(1 - \tanh \frac{1}{T})\right) \tag{80}$$

For large N and m finite, (79) becomes

$$\begin{aligned} Q_m(t+1) = {} & \exp\left(-\frac{mK}{2} \tanh \frac{1}{T}\right) \\ & + \sum_{n \geq 1} \left(\frac{mK}{2} \tanh \frac{1}{T}\right)^n \frac{1}{n!} \exp\left(-\frac{mK}{2} \tanh \frac{1}{T}\right) Q_n(t) \end{aligned} \tag{81}$$

If at time t and for finite m, we choose the following form of the $Q_m(t)$

$$Q_m(t) = [a(t)]^m \tag{82}$$

where $a(t)$ is a parameter, one gets from (81), that $Q_m(t+1)$ keeps this m dependence with $a(t+1)$ given by

$$a(t+1) = \exp\left[-\frac{K}{2}(1 - a(t)) \tanh \frac{1}{T}\right] \tag{83}$$

In the limit $t \longrightarrow \infty$, $a(t)$ converges to a fixed point a^* (with $a^* = 0$ if $T > T_c$ and $a^* \neq 0$ if $T < T_c$). It is then clear that

$$P(T) = 1 - a^* \tag{84}$$

since $1 - a^*$ is the probability that two configurations which differ by a single spin will never meet and this leads to (78).

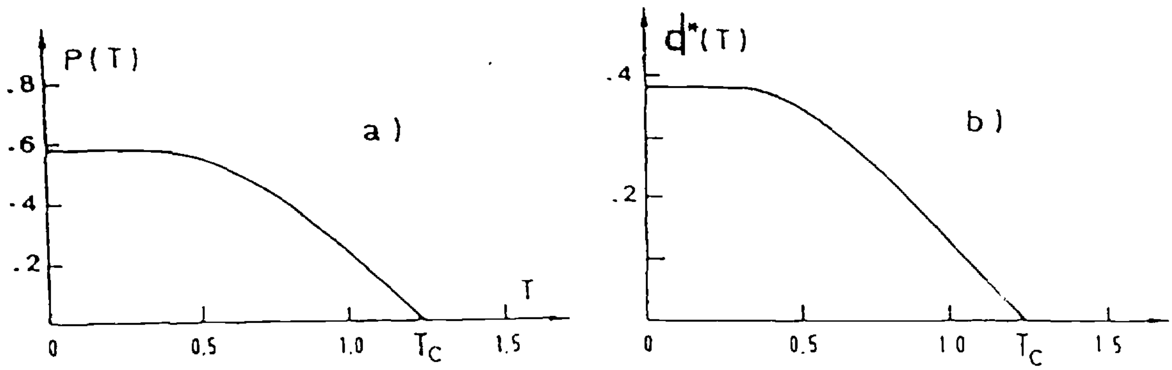


FIGURE 7a and 7b
The survival probability $P(T)$ and the distance $d^*(T)$ versus temperature for $K = 3$ and $p = 1/2$. They both vanish at a dynamical phase transition $T_c \simeq 1.243$ solution of (69).

2.D. Random automata in finite dimension

For the Kauffman model (deterministic version as in sections 2.A and 2.B) the evolution of the distance $d(t)$ and the distribution of local magnetisations can be calculated exactly. This is because the Kauffman model is a mean field model (Section 1.C.).

One can wonder what would be the effect of correlations when the system has some spatial structure. The simplest way of generalizing the Kauffman model to a finite dimensional model is to put the spins $S_i(t)$ on a regular lattice (of coordination number K) and to consider that the inputs $j_1(i)...j_K(i)$ are the neighbors of site i on the lattice. So the model in finite dimension is a system of N spins $S_i(t)$ on a regular lattice. For each i:

(1) the inputs $j_1(i)...j_K(i)$ are the nearest neighbors of i on the lattice

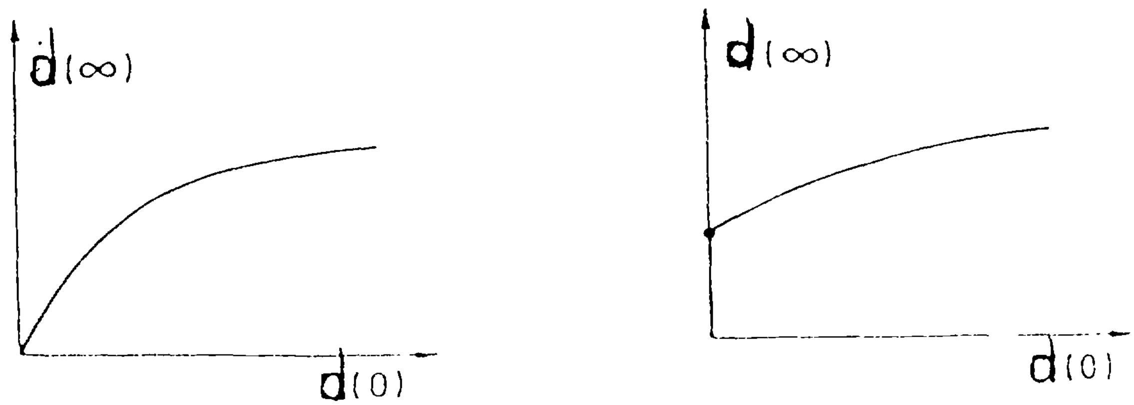
(2) the function f_i is a random Boolean function of K variables.

Since the number K of inputs is fixed, we can only vary the parameter p (introduced at the beginning of section 2) to see both the frozen and the chaotic phases. Because of the symmetry $p \longrightarrow 1 - p$, one needs only to study the range $0 \leq p \leq .5$.

For these finite dimensional systems, no analytical work has been done so far and all the known results have been obtained by numerical simulations. Let me describe here some results obtained for the 2d square lattice[17–23].

- *The distance*

The first quantity one can measure is the distance $d(t)$ between two different configurations C_t and $\tilde{C}_t$ as defined in equation (52). An important difference with the Kauffman model discussed in section 2.A. is that in finite dimension for $0 < p \leq .5$, the distance $d(\infty)$ in the long time limit depends on the initial distance $d(0)$.



$p < p_c$ frozen phase

$p > p_c$ chaotic phase

FIGURE 8
The distance $d(t)$ for $t \longrightarrow \infty$ versus the initial distance

The transition between the chaotic and the frozen phase can nevertheless be seen by looking at how $d(\infty)$ depends on $d(0)$. When $d(0) \longrightarrow 0$, one observes[17] a threshold at

$$p_c \simeq .26 \tag{85}$$

(More recent numerical simulations[19] give a higher estimate $p_c \simeq .29$).

For $p < p_c$ (the frozen phase), $d(\infty)$ vanishes as $d(0) \longrightarrow 0$ whereas for $p > p_c$ (the chaotic phase) $d(\infty)$ has a non zero limit as $d(0) \longrightarrow 0$. The behavior of $d(\infty)$ as a function of $d(0)$ is very reminiscent of the magnetisation as a function of the magnetic field h for a ferromagnet. Above T_c, m vanishes as $h \longrightarrow 0$ whereas below T_c, m has a finite limit in the limit $h \longrightarrow 0$.

- *The damage spreading*

In the chaotic phase, a small difference has the tendency to spread over the whole system. This means that if one considers two configurations which differ at $t = 0$ by a single spin, they have a finite probability of differing by a macroscopic number of spins at time $t = \infty$. Critical exponents related to this problem of damage spreading have been calculated numerically[19,23].

Figure 9 obtained by AU Neumann illustrates the damage spreading phenomenon in the case of the $2d$ Kauffman model. One starts for a sample of 100×100 spins with 2 configurations which are identical except for a small square of 10×10 at the center. After many time steps, one sees that the damage has spread over the whole system in the chaotic phase $p > p_c$ whereas it remains localized in the frozen phase $p < p_c$

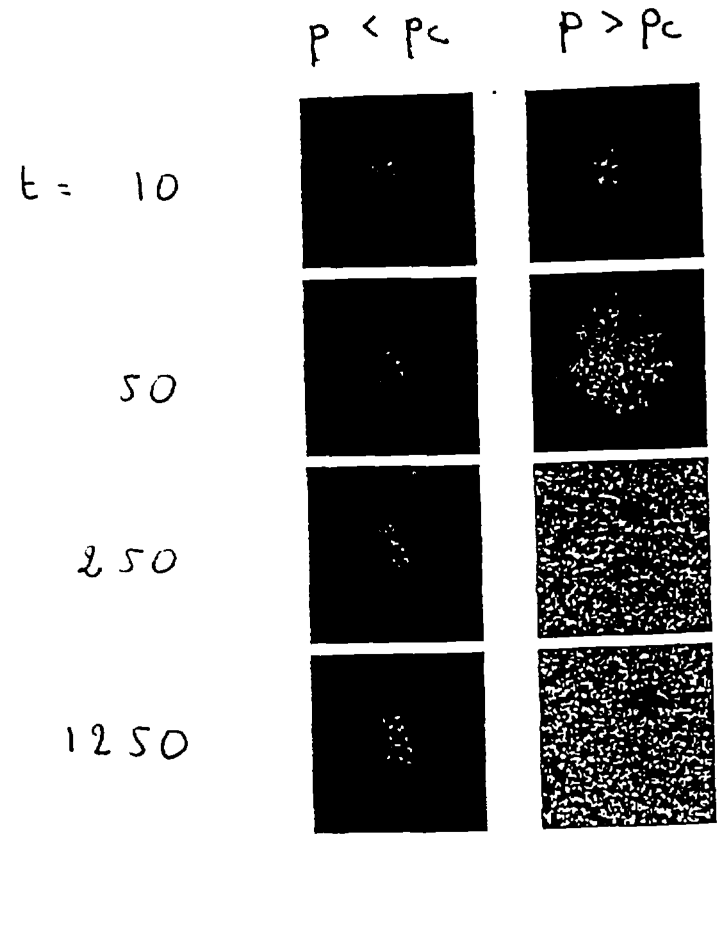


FIGURE 9

Damage spreading for the 2d Kauffman model (figure by A.U. Neumann)

- *The distribution of activities*

One can also measure the distribution of local magnetisations $P(m)$ as defined by equations (60) and (61)[22]. One starts with a random initial configuration and one measures for each site i the magnetisation m_i averaged over a long (but finite $\sim 1000 steps$) time for a large system.

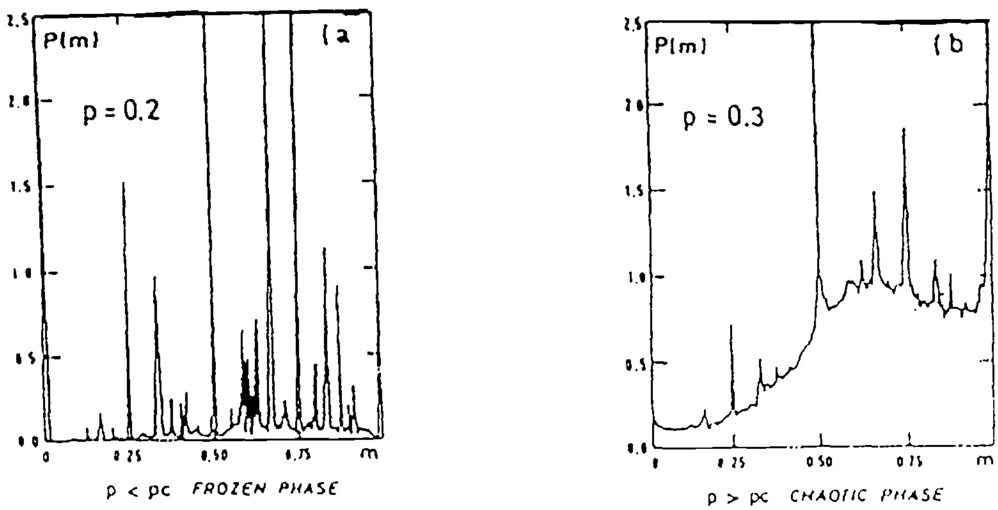


FIGURE 10

$P(m)$ versus m for the 2 dimensional case

For $p < p_c$ (frozen phase), $P(m)$ is a sum of delta functions at simple rational values of m. This corresponds to either spins which remain fixed for ever (after a few time steps) or to spins which belong to small clusters which have a periodic behavior with a short period.
In the chaotic phase ($p > p_c$) we see that in addition to the delta functions at simple rational values of m, there is a continuous part in $P(m)$. This continuous part is due to the presence of an infinite cluster of spins with no period (i.e. an infinite period).

We have seen that the distance $d(t)$ and the distribution of magnetisations $P(m)$ have more structure in the finite dimensional case than in the Kauffman model discussed in section 2.A, ($d(\infty)$ depends on $d(0) and$ $P(m)$ has an infinite number of delta functions at all rational values of m). There is nevertheless a clear numerical evidence for a transition from the frozen to the chaotic phase (the fact that $d(\infty)$ does not vanih as $d(0) \longrightarrow 0$ and the presence of a continuous part in $P(m)$ are two clear signatures of the chaotic phase).

The effect of noise on automata in finite dimension has been studied numerically[24] and the results are very similar to those shown in Figure 8 (for the mean field case).

3. SPIN MODELS

3.A. Distance method in presence of noise

For deterministic dynamics, we have seen in section 2 that one can observe dynamical phase transitions between a chaotic and a frozen phase by measuring the distance between two configurations which evolve according to the same rules.

For stochastic dynamics, we can recover these dynamical phase transitions provided that we use the same thermal noise to update the two configurations (2.C).

One can then try to see how these dynamical phase transitions are related to usual equilibrium phase transitions of systems which possess an Hamiltonian[25].

Consider a system of N Ising spins $S_i = \pm 1$ for which the Hamiltonian $\mathcal{H}$ is

$$\mathcal{H} = -\sum_{i,j} J_{ij} S_i S_j \tag{86}$$

Two configurations $\{S_i(t)\}$ and $\left\{\tilde{S}_i(t)\right\}$ which evolve according to the heat bath dynamics are subjected to the same thermal noise (same random number $z_i(t)$)

$$\begin{aligned} S_i(t+\Delta t) &= \text{sign}\left[\frac{1}{2} + \frac{1}{2}\tanh\left(\sum_j \frac{J_{ij}S_j(t)}{T}\right) - z_i(t)\right] \\ \tilde{S}_i(t+\Delta t) &= \text{sign}\left[\frac{1}{2} + \frac{1}{2}\tanh\left(\sum_j \frac{J_{ij}\tilde{S}_j(t)}{T}\right) - z_i(t)\right] \end{aligned} \tag{87}$$

and we measure the distance $d(t)$:

$$d(t) = \frac{1}{2N}\sum_{i=1}^{N}\left|S_i(t) - \tilde{S}_i(t)\right| \tag{88}$$

In the limit $T \longrightarrow \infty$, each time that a spin i is updated, it becomes identical in the two configurations. Therefore

$$d(t) = d(0)\mathrm{e}^{-t} \tag{89}$$

So at high temperature the two configurations subjected to the same noise attract each other and one expects $d(t)$ to vanish in the long time limit.

At low temperature, if the system has several low temperature phases, there is always the possibility that each configuration falls into a different valley and therefore one expects that $d(t)$ remains non zero, at least for far enough initial conditions to allow the two configurations to fall into different valleys.

One can then wonder whether the only mechanism for two configurations (subjected to the same noise) to remain different (in the limit $t \longrightarrow \infty$) is the appearence of infinite barriers between valleys. The distance method has already been used for several systems and two sorts of effects have been found which give a non vanishing distance in the long time limit:

1. The two configurations belong to two valleys in phase space separated by high energy barriers, like in the case of ferromagnets[25-27].

2. The chaotic character of the dynamics makes two initially close trajectories diverge (automata, spin glasses, neural networks)[25,6].

We are now going to compare the dynamical phase transition observed by looking at the distance (between two configurations subjected to the same noise) and the equilibrium phase transition for two systems: First the ferromagnetic Ising model for which we will show that the dynamical phase transition coincides with the phase transition at equilibrium. Then we will consider the case of spin glasses and see how the equilibrium phase transition could be related to the dynamical properties of the distance.

3.2 The ferromagnet

When one compares two configurations subjected to the same thermal noise at a temperature T, one can measure two quantities: the survival probability $P(T)$, which is the probability that the two configurations are still different (after a long time t) and the average distance $D(T) = \langle d \rangle$ between these two configurations (where the average is done over the cases for which the two configurations are still different).

- *The mean field ferromagnet*

For the mean field model defined by (31)

$$\mathcal{H} = -\frac{1}{N} \sum_{i<j} S_i S_j \tag{90}$$

one can calculate[28] analytically the survival probability $P(T)$ and the distance $D(T)$ for sequential or parallel dynamics.

The survival probability depends on how the two initial conditions are chosen. For sequential dynamics, one finds that

$$P(T) = \frac{2}{\pi} \tan^{-1} \left[\left(2 \, \frac{1-T}{T} \right)^{1/2} \right] \tag{91}$$

when the two initial configurations $\mathcal{C}_0$ and $\tilde{\mathcal{C}}_0$ are opposite ($\mathcal{C}_0$ is random) and

$$P(T) = \frac{2}{\pi} \tan^{-1} \left[(1-T)^{1/2} \right] \tag{92}$$

when the two initial conditions $\mathcal{C}_0$ and $\tilde{\mathcal{C}}_0$ are independent random configurations.

When the two configurations remain different in the long time limit, one can calculate their distance $D(T)$ and one finds[28] that this distance is equal to the magnetization $m(T)$

$$D(T) = m(T) \tag{93}$$

which is the fixed point ($m > 0$) of

$$m = \tanh\left(\frac{m}{T}\right) \tag{94}$$

For this mean field model, we see that above $T_c (= 1)$, the two configurations meet quickly, so $P(T)$ vanishes. Below T_c, there is a finite probability $P(T)$ that one configuration falls into the + phase and the other in the − phase: When the temperature changes, the relative effects of the initial conditions and of the thermal noise changes. A higher temperature increases the probability that the two configurations fall into the same valley.

The fact that the distance $D(T)$ is exactly equal to the magnetization is more surprising. One expects that if one configuration falls into the + phase and the other in the − phase, their distance is comparable to $m(T)$. But the exact equality (93) is not a priori obvious. We will see below that the equality (93) holds in finite dimension for any system which contains only ferromagnetic bonds.

- *The barrier heights*

Before doing that, let me mention that the comparison of two or more configurations subjected to the same noise can be used in ferromagnetic systems to measure barrier heights.

For any finite system, two configurations subjected to the same noise will ultimately meet. One can define the time τ_2 for two configurations to meet. In the low temperature phase, this time τ_2 increases with the system size because, with a finite probability, each configuration is in a

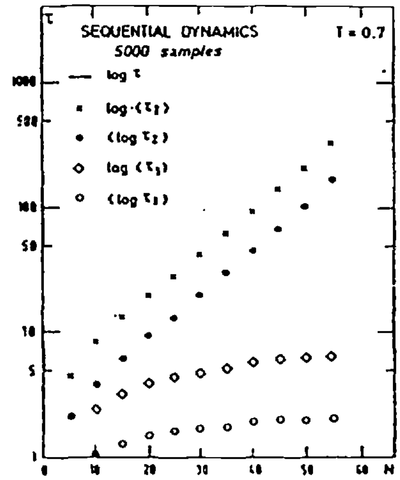


FIGURE 11
Sequential dynamics : in the ferromagnetic phase ($T = 0.7 < T_c = 1$), the time τ_2 for two configurations to meet as a function of N. The slope of τ_2 is a measure of the barrier height. The time τ_3 that three configurations remain different is much shorter because there are only two valleys.

different phase and the time for them to meet is the time for one of the two configurations to jump from one phase to the other.

Fig.11 gives τ_2 as a function of the system size for the mean field model (90). We see that τ_2 increases exponentially with N and the slope of log τ_2 versus N gives a measure of the free energy barrier between the two phases[28].

One can compare more than 2 configurations. The time τ_3 shown in figure 1 represents the time for two among 3 configurations to meet. This time is much shorter because, the mean field ferromagnet possesses only 2 valleys. Therefore, three configurations cannot remain different for very long. Of course, τ_3 could increase with N for systems (like the 3 Potts model) which have at least three low temperature phases.

In the case of automata, with noise (Section 2.B), the times τ_2, τ_3, τ_4... can be measured and one finds that they all increase exponentially with N in the chaotic phase[29].

- ***The finite dimensional case***

The fact that the distance $D(T)$ is equal to the magnetization can be understood for any

Ising model with ferromagnetic interactions. The derivation of this result was explained to me by J.L. Lebowitz[30].

Let us choose for the two initial configurations $\{S_i(0)\}$ and $\left\{\tilde{S}_i(0)\right\}$, two configurations which satisfy

$$S_i(0) \leq \tilde{S}_i(0) \qquad \forall i \tag{95}$$

Because the bonds are ferromagnetic, one can check from the dynamical rule (87), that at any later time, this property remains valid

$$S_i(t) \leq \tilde{S}_i(t) \tag{96}$$

Therefore the distance $d(t)$ defined by (88) becomes

$$d(t) = \frac{1}{2N} \sum_{i=1}^{N} \left(\tilde{S}_i(t) - S_i(t)\right) = \frac{1}{2}\left[\tilde{m}(t) - m(t)\right] \tag{97}$$

We see that the distance is related to the magnetization. In particular if the two configurations fall into the same phase, $d(t) \longrightarrow 0$ because $m(t)$ and $\tilde{m}(t)$ converge to the same value, whereas if one configuration falls into the + phase and the other in the − phase, $d(t)$ is equal to the spontaneous magnetization.

This analogy between quantities measured by comparing two configurations and magnetic properties can be extended to correlation functions[27].

3.C. Distances in the spin glass problem

One can study the time evolution of the distance between pairs of configurations subjected to the same thermal noise in all kinds of systems. In this section, I will describe what is observed in the case of spin glasses.

- *The Sherrington Kirkpatrick model*[31,32]

The Sherrington Kirkpatrick model (SK model) plays the role in spin glass theory of the mean field model (90) discussed in the previous section for the ferromagnet.

The Hamiltonian of the SK model is

$$\mathcal{H} = -\frac{1}{\sqrt{N}} \sum_{i<j} J_{ij} S_i S_j \tag{98}$$

where in (98) the J_{ij} are random variables ($J_{ij} = \pm 1$ or J_{ij} Gaussian with $\langle J_{ij}^2 \rangle = 1$).

For any distribution of interactions J_{ij} which satisfies $\langle J_{ij}^2 \rangle = 1$, the transition temperature T_c of the SK model defined by (98) is

$$T_c = 1 \tag{99}$$

To compare the time evolution of these two configurations subjected to the same thermal noise, one can consider the following three situations.

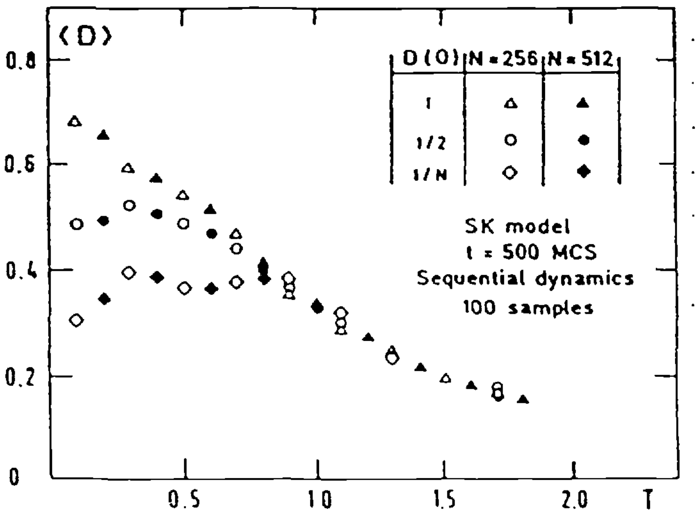


FIGURE 12
Sherrington Kirkpatrick model : The distance $D(T)$ versus temperature T for three initial distances; opposite initial conditions ($d(0) = 1$), uncorrelated initial conditions ($d(0) = 1/2$), identical initial conditions except for one spin ($d(0) = 1/N$).

a) The two initial configurations C_0 and $\tilde{C}_0$ are opposite, C_0 being random ($S_i(0) = -\tilde{S}_i(0)$ for all i). Therefore $d(0) = 1$. (The data corresponding to this case will be represented by triangles in the figures).

b) The two initial configurations C_0 and $\tilde{C}_0$ are random and independent: $d(0) = 1/2$ (squares in the figures).

c) The two initial configurations C_0 and $\tilde{C}_0$ differ by a single spin: $d(0) = 1/N$ (diamonds in the figures).

For each of these three initial conditions, one can measure the survival probability $P(T)$ and the distance $D(T)$. One finds that the survival probability $P(T)$ is 1 at all temperature (for large enough systems) for each of the three initial situations. The results[33] for $D(T)$ are shown in figure 12.

We see that the distance $D(T)$ does not vanish at any temperature. Numerical calculations performed at higher temperature indicate that $D(T)$ would vanish at $T = \infty$ and that $D(T) \sim 1/T^2$.

So the temperature where $D(T)$ vanishes is not related to the phase transition in the SK model. However, as it can be seen in Fig.12, one observes two regions for the distance. A high temperature regime where $D(T)$ does not depend on the initial distance and a low temperature regime where $D(T)$ presents remanence effects and does depend on the initial distance. The temperature which separates these two regimes seems to be close to $T_c = 1$. However the data of Fig.12 are not of good enough quality to be certain that the equilibrium transition temperature $T_c = 1$ is the place where the remanence effects in the distance $D(T)$ start to appear.

- *The 3 dimensional spin glass*

The same calculation can be repeated for finite dimensional systems[25]. In 3d with $\pm J$ bonds. one finds that below a temperature $T_1 \simeq 4J$, the survival probability $P(T) \neq 0$. For initial conditions a and b, one finds that $P(T) = 1$ (as for the automata in presence of noise of Section 2.C) whereas $P(T)$ is a function of temperature for the case c (two initial configurations which differ by a single spin) which is very similar to what was obtained for automata (Fig.7a). So the survival probability $P(T)$ in the case of 3d spin glasses has in all cases a similar temperature dependence as in the automata model.

The distance $D(T)$ obtained for the 3d $\pm J$ spin glass is given in figure 13. We see clearly 3 regimes :

- A high temperature regime $T > T_1$ where the distance $D(T)$ vanishes.

- An intermediate regime $T_2 < T < T_1$ where the distance $D(T)$ is non zero but is independent of the initial distance.

- A low temperature regime $T < T_2$ where the distance $D(T)$ depends on the initial distance.

From figure 13, one can estimate that $T_1 \simeq 4.$ and $T_1 \simeq 1.5$. These estimates may have some dependence on the system size and on the time t of the simulation and a good finite size scaling analysis[26] would be necessary to improve these estimates.

The main question about these results is how to compare them with what is already known in the 3d spin glass[34,35]. Unlike in the ferromagnetic case, these does not exist here any proof that

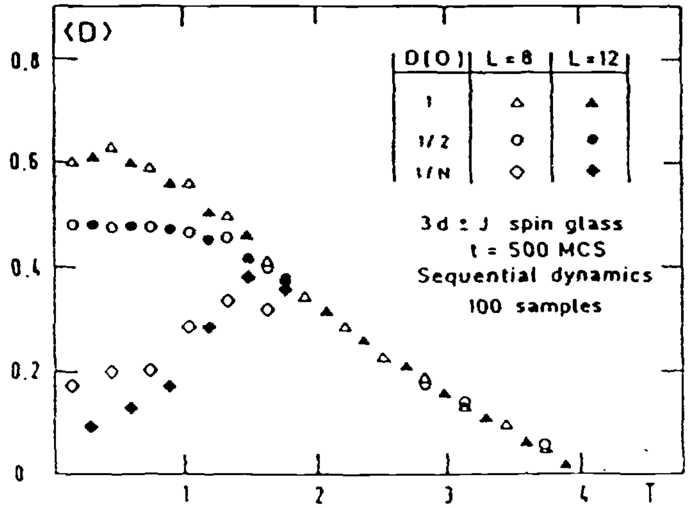


FIGURE 13
The distance $D(T)$ versus temperature (as in Fig.12) for the 3d $\pm J$ spin glass ($N = L^3$).

the dynamical phase transitions associated to the distance (between two configurations subjected to the same noise) should coincide with equilibrium phase transitions. So one can only compare the numerical data.

In his extensive numerical simulations on the 3d $\pm J$ spin glass[35], Ogielsky obtained that there are 3 regimes for 3d $\pm$ spin glass: a high temperature region $T > T_1^*$ where the spin autocorrelation function $q(t)$ decreases exponentially with time. An intermediate region $T_2^* < T < T_1^*$ where $q(t)$ has a slow decay (stretched exponential). A low temperature phase $T < T_1^*$ where $q(t)$ does not decay to zero. His estimates for T_1^* and T_2^* were $T_1^* \simeq 4.5$ (because T_1^* is probably the transition temperature of the ferromagnet) and $T_2^* \simeq 1.2$.

We see that the values of $T_1 \simeq 4.$ and $T_2 \simeq 1.5$ are rather close to $T_1^* \simeq 4.5$ and $T_2^* \simeq 1.2$ by Ogielsky. The quality of the data for the determination of T_1 and T_2 is however not good enough at the moment to be certain that they are the same.

Since the lower temperature T_2 seems to be close (in the 3d case and for the SK model) to the equilibrium phase transition, it was tempting to study the effect of a magnetic field h. The numerical data obtained so far indicate that $T_2(h)$ decreases[36] with h for the SK model whereas it increases in the 3d case[37].

4. CONCLUSION

These lectures describe a few models and methods which have been developed recently in the study of the dynamics of spin models and automata. There is a special emphasis on the distance method which consists of comparing the time evolution of two configurations.

For deterministic models, this method gives a good way of finding the phase boundary between a frozen and a chaotic phase (section 2).

For stochastic systems, one finds that this method gives dynamical phase transitions (section 3) but, except in the ferromagnetic case, one does not know yet how these dynamical phase transitions are related to the equilibrium phase transitions[38,39]. It would be, of course, very interesting to understand better the relation between the dynamical phase transitions (section 3) and the equilibrium ones.

Another challenging question is the calculation of the periods in deterministic models. The models discussed in sections 1.C and 2. are exactly soluble mean field models[40] in the sense that in the limit $N \longrightarrow \infty$, there are some quantities (local magnetizations, distances,...) which can be calculated exactly. Much less is known on these models for finite N. In particular, there does not exist at present any analytic way of computing the period of these mean field models.

To conclude these lectures, I would like to mention an attempt in that direction. The formula I obtained for the period of a Kauffman model is in the limit $N \longrightarrow \infty$

$$\frac{\log\,(\text{Period})}{N} = \frac{1}{2} \int P(m) dm \log\left(\frac{2}{1+m^2}\right) \tag{100}$$

where $P(m)$ is the distribution of activities discussed in section 2.2. Let me explain briefly how this formula was obtained and what assumption was made to derive it. Consider a random map[41]

$$C' = F(C) \tag{101}$$

in a phase space of 2^N configurations. For each C, one chooses its image $C' = F(C)$ at random according to a given probability $p(C)$. Then one can show[42] that the typical period of this random map is

$$\text{Period} \sim \left(\sum_C (p(C))^2 \right)^{-1/2} \tag{102}$$

Now let us identify a configuration C in phase space with a spin configuration $\{S_i\}$ and choose for $p(C)$

$$p(C) = \prod_{i=1}^{N} \left(\frac{1 + m_i S_i}{2} \right) \tag{103}$$

Then from (102) one gets

$$\text{Period} \sim \left[\frac{1}{2^N} \prod_{i=1}^{N} (1 + m_i^2) \right]^{-1/2} \tag{104}$$

which gives (100) in the limit $N \longrightarrow \infty$.

So (100) is exact for the random map model defined by (103). This random map model has the same local magnetizations as the Kauffman model. If one assumes that the correlations between the spins do not change the period, then one concludes that the exponential growth of the period in the Kauffman model would be given by (100). At present, I do not know whether this assumption (which neglects the correlations as one usually does in mean field models) is justified for the mean field models discussed in section 1.C and 2.

These lectures have for main content the result of collaborations which came out from very pleasant and stimulating discussions with H. Flyvbjerg, O. Golinelli, A.U. Neumann, Y. Pomeau, D. Stauffer and G. Weisbuch.

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